

quiz 2

EMA 545
Mechanical Vibrations

Spring 2013
University of Wisconsin, Madison

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Spring 2013

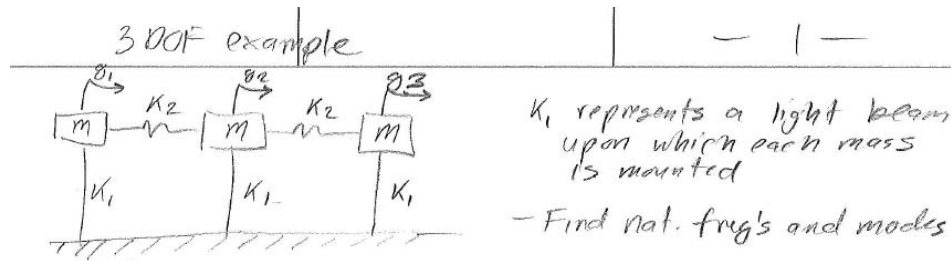
Compiled on April 19, 2021 at 3:04am [public]

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1 Problem description

Consider this 3 DOF system



Suppose a harmonic force $f(t) = A \cos(\omega t)$ is applied to the mass in the center. Use modal analysis to do the following:

1. Find the uncoupled modal equations of motion. Consider the steady state solution for each of these equations. Sketch the modal amplitude (X_j in the book on page 275) for each mode versus frequency. A hand sketch is sufficient.
2. Use that result to sketch the frequency response of each of the masses, in other words the complex amplitude Y_n versus ω

2 Answer part (1)

A summary of the steps needed for full modal analysis is first given. In these steps, a column vector is shown as bold letter \mathbf{Y} and a matrix is shown as $[M]$. In this summary, the system is assumed to have n degree of freedom.

The steps are

1. Determine the system of equations of motion and set up $[M]\mathbf{Y}'' + [C]\mathbf{Y}' + [K]\mathbf{Y} = \mathbf{F}$ in matrix form.
2. Solve the eigenvalue problem $\det([K] - \omega^2[M]) = 0$ in order to determine the n natural frequencies.
3. For each natural frequency ω_j determine the corresponding j^{th} eigenvector \mathbf{j} by solving $([K] - \omega_j^2[M])\mathbf{j} = \mathbf{0}$. In this step, the first component of \mathbf{j} is set to 1 and the other components are solved relative to it.
4. Obtain the normalized eigenvectors Φ_j for each j using $\Phi_j = \frac{\mathbf{j}}{\sqrt{u_j}}$ where $u_j = \mathbf{j}^T [M] \mathbf{j}$. Each u_j will be a scalar.
5. Set up the modal transformation matrix $[\Phi] = [\Phi_1 \Phi_2 \cdots \Phi_n]$. This will be an $n \times n$ matrix.

6. The transformation from normal solution $y(t)$ to modal $\eta(t)$ will be $\mathbf{Y} = [\Phi]$ and $= [\Phi]^{-1}\mathbf{Y} = [\Phi]^T[M]\mathbf{Y}$
7. Apply the above transformation on the original equations of motions in matrix form to obtain the equations of motion in modal coordinates $[\Phi]^T[M][\Phi]\mathbf{Y}'' + [\Phi]^T[C][\Phi]\mathbf{Y}' + [\Phi]^T[K][\Phi]\mathbf{Y} = [\Phi]^T\mathbf{F}$. This becomes $\mathbf{I}''(t) + [\tilde{C}]\mathbf{Y}'(t) + [\tilde{K}]\mathbf{Y}(t) = [\Phi]^T\mathbf{F}$ where \mathbf{I} is the identity matrix, $[\tilde{C}]$ is a diagonal damping matrix obtained using a method such as weak damping approximation and $[\tilde{K}]$ is diagonal matrix with diagonal that contains the natural frequencies squared ω_j^2 in each of entries.
8. For steady state solution in modal coordinates, the loading vector $[\Phi]^T\mathbf{F}$ is assumed to be $\mathbf{Q} = [\Phi]^T\mathbf{F} = \text{Re}(\hat{\mathbf{Q}}e^{i\omega t})$ where $\hat{\mathbf{Q}}$ is the complex amplitude of the loading vector in modal coordinates. Therefore, the steady state solution is $ss(t) = \text{Re}(\hat{\mathbf{X}}e^{i\omega t})$ where $\hat{\mathbf{X}}$ is the complex amplitude of each modal response is $\hat{X}_j = \frac{j^T\mathbf{F}}{-\omega^2 + i2\zeta_j\omega_j\omega + \omega_j^2}$. For a system with no damping this simplifies to $\hat{X}_j = \frac{j^T\mathbf{F}}{-\omega^2 + \omega_j^2}$. In here, j^T represents the transpose of the j^{th} column of the modal transformation matrix $[\Phi]$, or the transpose of the j^{th} mass normalized eigenvector, and ω_j is the j^{th} natural frequency.
9. Now the steady state solution in modal coordinate is used to obtain the solution in normal coordinates since $\mathbf{Y} = [\Phi]$. Therefore $\mathbf{Y}_{ss} = \text{Re}(\hat{\mathbf{X}}e^{i\omega t}) = \text{Re}([\Phi]\hat{\mathbf{X}}e^{i\omega t}) = \text{Re}(\hat{\mathbf{Y}}e^{i\omega t})$. In component form $\mathbf{Y}_{ss} = \text{Re}\left(\left(\sum_{j=1}^n \hat{X}_j\right)e^{i\omega t}\right)$

The EOM are derived in the hand out given. The force $f(t)$ acting on the second mass is now added, resulting in the following equations of motion for the system

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_1 + 2k_2 & -k_2 \\ 0 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ A \cos(\omega t) \\ 0 \end{Bmatrix}$$

The first step is to obtain the natural frequencies of the system. This is done by solving the eigenvalue problem $\det([K] - \omega^2[M]) = 0$. The solutions are also given in handout. They are $\omega_1^2 = \frac{k_1}{m}$, $\omega_2^2 = \frac{k_1+k_2}{m}$, $\omega_3^2 = \frac{k_1+3k_2}{m}$. The non mass normalized eigenvectors associated with these eigenvalues are found as

$$1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, 2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, 3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

The next step is to mass normalize the eigenvectors as follows

$$\begin{aligned}\mu_1 &= {}_1^T[M]_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3m \\ \mu_2 &= {}_2^T[M]_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2m \\ \mu_3 &= {}_3^T[M]_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6m\end{aligned}$$

Hence the mass normalized eigenvectors are

$$\begin{aligned}{}_1 &= \frac{1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{3m}} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \\ {}_2 &= \frac{2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{2m}} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \\ {}_3 &= \frac{3}{\sqrt{\mu_3}} = \frac{1}{\sqrt{6m}} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}\end{aligned}$$

Hence the modal transformation matrix $[\Phi]$ is

$$[\Phi] = [{}_{123}] = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}$$

The modal EOM's are now found using the modal transformation matrix $[\Phi]$

$$[\Phi]^T[M][\Phi]\{\eta''\} + [\Phi]^T[K][\Phi]\{\eta\} = [\Phi]^T\mathbf{Q}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 0 \\ A \cos(\omega t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \frac{1}{m} \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_1 + k_2 & 0 \\ 0 & 0 & k_1 + 3k_2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}^T \begin{Bmatrix} 0 \\ A \cos(\omega t) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \frac{1}{m} \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_1 + k_2 & 0 \\ 0 & 0 & k_1 + 3k_2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{Bmatrix} 0.577 A \cos(\omega t) \\ 0 \\ -0.816 A \cos(\omega t) \end{Bmatrix}$$

Therefore, the 3 uncoupled modal EOM's are

$$\eta_1''(t) + \frac{k_1}{m} \eta_1(t) = \frac{0.577 A}{\sqrt{m}} \cos(\omega t)$$

$$\eta_2''(t) + \frac{k_1 + k_2}{m} \eta_2(t) = 0$$

$$\eta_3''(t) + \frac{k_1 + 3k_2}{m} \eta_3(t) = -\frac{0.816 A}{\sqrt{m}} \cos(\omega t)$$

To complete the solution, the above EOM are written as follows by using complex form for the loading vector

$$\eta_1''(t) + \frac{k_1}{m} \eta_1(t) = \text{Re} \left(\frac{0.577 A}{\sqrt{m}} e^{i\omega t} \right)$$

$$\eta_2''(t) + \frac{k_1 + k_2}{m} \eta_2(t) = 0$$

$$\eta_3''(t) + \frac{k_1 + 3k_2}{m} \eta_3(t) = \text{Re} \left(\frac{-0.816 A}{\sqrt{m}} e^{i\omega t} \right)$$

Assuming the steady state solution is

$$= \text{Re}(\hat{\mathbf{X}} e^{i\omega t})$$

or in expanded form

$$\eta_1(t) = \text{Re}(\hat{X}_1 e^{i\omega t})$$

$$\eta_2(t) = \text{Re}(\hat{X}_2 e^{i\omega t})$$

$$\eta_3(t) = \text{Re}(\hat{X}_3 e^{i\omega t})$$

Where

$$\hat{X}_1 = \frac{\frac{0.577 A}{\sqrt{m}}}{\omega_1^2 + 2i\zeta_1\omega_1\omega - \omega^2}$$

$$\hat{X}_2 = 0$$

$$\hat{X}_3 = \frac{\frac{-0.816 A}{\sqrt{m}}}{\omega_3^2 + 2i\zeta_3\omega_3\omega - \omega^2}$$

Dividing the numerator and the denominator by ω_i^2 where $i = 1, 2, 3$ and using $r_i = \frac{\omega}{\omega_i}$ and letting $\zeta = 0$ since no damping exists, results in

$$\hat{X}_1 = \frac{A\sqrt{m}}{k_1} \left(\frac{0.577}{1 - m\frac{\omega^2}{k_1}} \right)$$

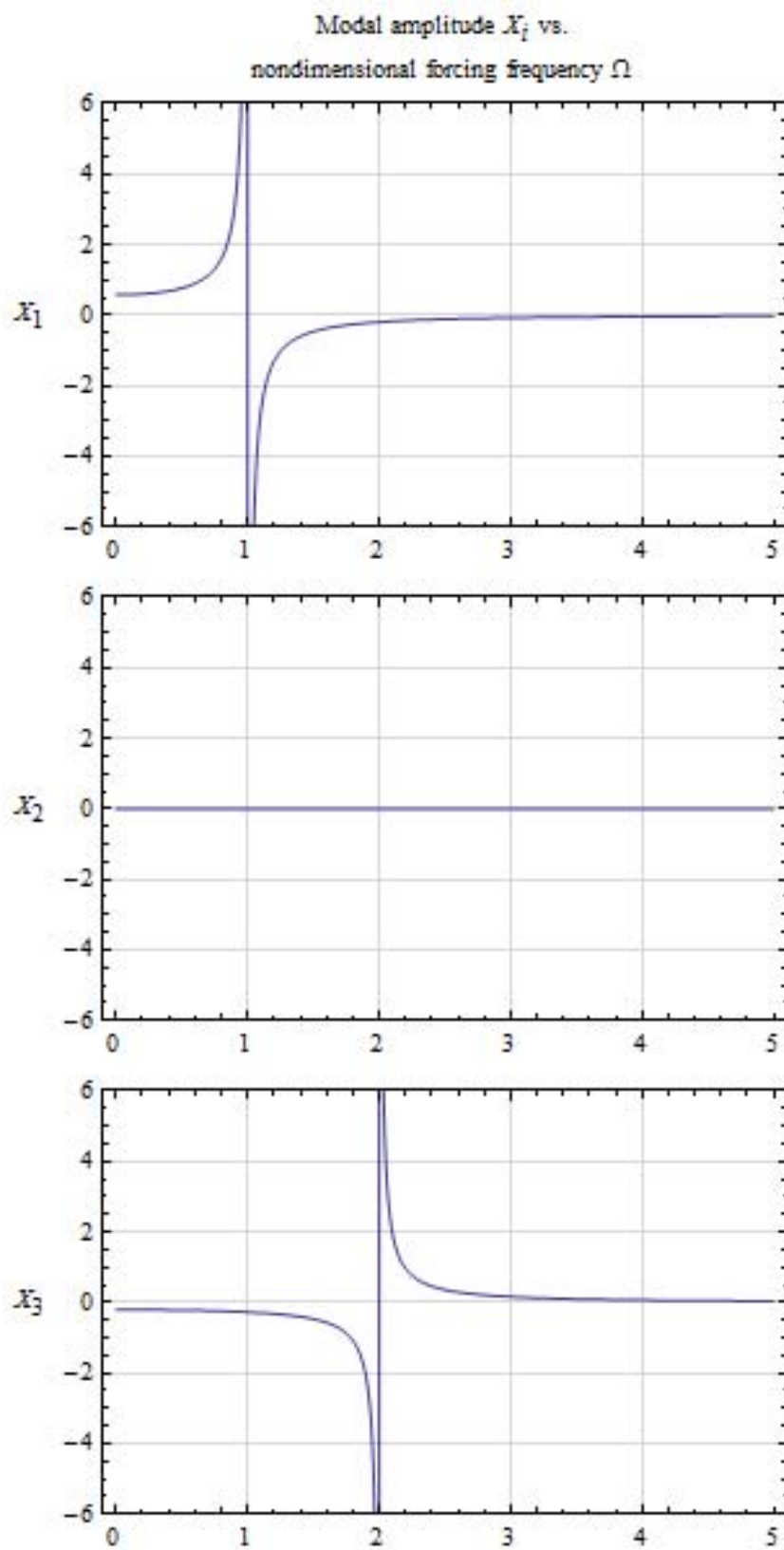
$$\hat{X}_2 = 0$$

$$\hat{X}_3 = \frac{A\sqrt{m}}{k_1 + 3k_2} \left(\frac{-0.816}{1 - m\frac{\omega^2}{k_1 + 3k_2}} \right)$$

To sketch these amplitudes, the equations are normalized. This is in effect the same as setting $m = 1, k_1 = k_2 = 1, A = 1$ resulting in

$$\hat{\mathbf{X}} = \begin{Bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \end{Bmatrix} = \begin{Bmatrix} \frac{0.577}{1 - \omega^2} \\ 0 \\ \frac{1}{4} \left(\frac{-0.816}{1 - \frac{\omega^2}{4}} \right) \end{Bmatrix}$$

Here is a plot of each X_i vs ω . The x-axis is the nondimensional forcing frequency Ω



Since there is no damping, resonance will occur at $\Omega = 1$ in first mode and at $\Omega = 2$ for mode 3.

3 Answer part (2)

The transformation from modal coordinates to normal coordinates is

$$\mathbf{q} = [\Phi]$$

In expanded form

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \Phi_1^T \{\eta\} \\ \Phi_2^T \{\eta\} \\ \Phi_3^T \{\eta\} \end{Bmatrix}$$

But $[\Phi] = \frac{1}{\sqrt{m}} \begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}$ and $\mathbf{q} = \text{Re}(\hat{\mathbf{X}}e^{i\omega t})$ hence the above becomes

$$\begin{aligned} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} &= \begin{pmatrix} \begin{pmatrix} 0.577 \\ 0.577 \\ 0.577 \end{pmatrix}^T \begin{pmatrix} \text{Re}(\hat{X}_1 e^{i\omega t}) \\ \text{Re}(\hat{X}_2 e^{i\omega t}) \\ \text{Re}(\hat{X}_3 e^{i\omega t}) \end{pmatrix} \\ \begin{pmatrix} 0.707 \\ 0 \\ -0.707 \end{pmatrix}^T \begin{pmatrix} \text{Re}(\hat{X}_1 e^{i\omega t}) \\ \text{Re}(\hat{X}_2 e^{i\omega t}) \\ \text{Re}(\hat{X}_3 e^{i\omega t}) \end{pmatrix} \\ \begin{pmatrix} 0.408 \\ -0.816 \\ 0.408 \end{pmatrix}^T \begin{pmatrix} \text{Re}(\hat{X}_1 e^{i\omega t}) \\ \text{Re}(\hat{X}_2 e^{i\omega t}) \\ \text{Re}(\hat{X}_3 e^{i\omega t}) \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 0.577 \text{Re}(X_1 e^{i\omega t}) + 0.577 \text{Re}(X_2 e^{i\omega t}) + 0.577 \text{Re}(X_3 e^{i\omega t}) \\ 0.707 \text{Re}(X_1 e^{i\omega t}) - 0.707 \text{Re}(X_3 e^{i\omega t}) \\ 0.408 \text{Re}(X_1 e^{i\omega t}) - 0.816 \text{Re}(X_2 e^{i\omega t}) + 0.408 \text{Re}(X_3 e^{i\omega t}) \end{pmatrix} \\ &= \text{Re} \left(\begin{pmatrix} 0.577 X_1 + 0.577 X_2 + 0.577 X_3 \\ 0.707 X_1 - 0.707 X_3 \\ 0.408 X_1 - 0.816 X_2 + 0.408 X_3 \end{pmatrix} e^{i\omega t} \right) \end{aligned}$$

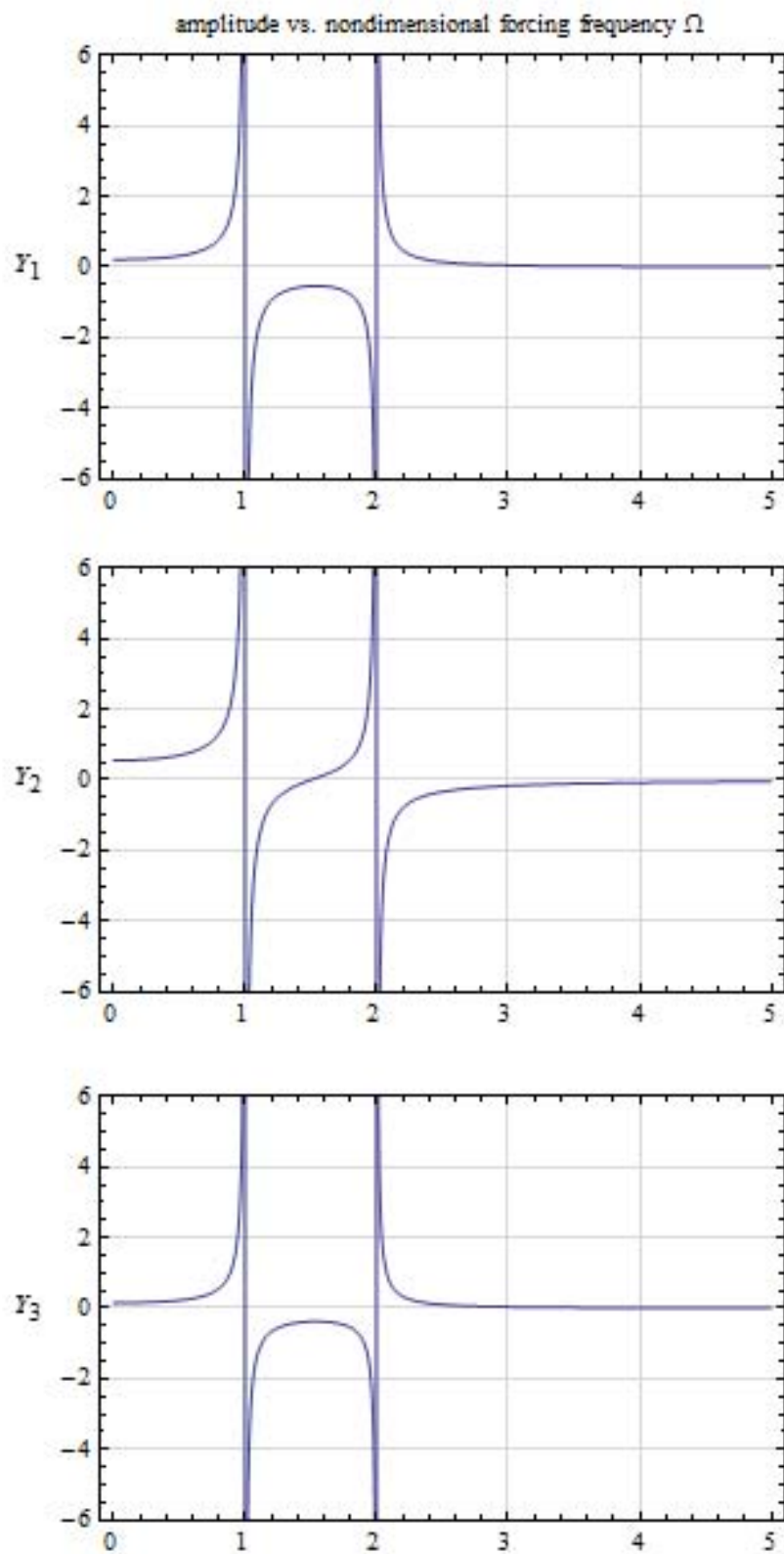
Comparing the above to $\mathbf{q}_{ss} = \text{Re}(\mathbf{Y}e^{i\omega t})$ shows that

$$\mathbf{Y} = \begin{pmatrix} 0.577 X_1 + 0.577 X_2 + 0.577 X_3 \\ 0.707 X_1 - 0.707 X_3 \\ 0.408 X_1 - 0.816 X_2 + 0.408 X_3 \end{pmatrix}$$

To plot each Y_i , let $m = 1, k_1 = 1, k_2 = 1, A = 1$, and letting $X_2 = 0$ as found earlier, results

in

$$\mathbf{Y} = \begin{pmatrix} 0.577 \frac{0.577}{1-\omega^2} + \frac{0.577}{4} \left(\frac{-0.816}{1-\frac{\omega^2}{4}} \right) \\ 0.707 \frac{0.577}{1-\omega^2} - \frac{0.707}{4} \left(\frac{-0.816}{1-\frac{\omega^2}{4}} \right) \\ 0.408 \frac{0.577}{1-\omega^2} + \frac{0.408}{4} \left(\frac{-0.816}{1-\frac{\omega^2}{4}} \right) \end{pmatrix}$$



The above shows that when the nondimensional frequency Ω is not close to a one of the nondimensional natural frequencies, then the Y values have comparable magnitudes. For nondimensional frequency Ω larger than 3 all amplitude are zero, which means the whole system does not oscillate any more in steady state.