University Course

EMA 545
Mechanical Vibrations

University of Wisconsin, Madison
spring 2013

My Class Notes

Nasser M. Abbasi

spring 2013

Contents

Chapter 1

introduction

Local contents

I took this course in Spring 2013 part of masters degree in engineering mechanics. University of Wisconsin, Madison.

Instructor is professor Matt Allen

External class web page `http://courses.engr.wisc.edu/ema/ema545.html`

Text book: Mechanical and Structural Vibrations: Theory and Applications, Jerry H.

Ginsberg, 1st Edition, Wiley, 2001.

1.1 Syllubus

EP/EMA 545 – Spring 2013
Mechanical Vibrations

Course Home Page: Engineering Moodle Courses: https://courses.moodle.wisc.edu
Lecture notes, homework and solutions will be posted on course web site.

Instructor:
Prof. Matt Allen
Department of Engineering Physics
535 Engineering Research Building
Email: msallen@engr.wisc.edu, Office Phone: 608-890-1619
Office Hours: T 9:15-10AM, 2-3PM, W 2-3PM, or by appointment

Grader: (available by appointment if needed)
Samuel Fedenia, sfedenia@wisc.edu

Prerequisites:
EMA 202 or 221, EMA 304 or 306, Math 223

Required Textbook: Mechanical and Structural Vibrations: Theory and Applications, Jerry H. Ginsberg, 1st Edition, Wiley, 2001.

Evaluation:

- **Weekly Homework sets** – Problems assigned weekly and typically due on Thursday. Late homework will not be accepted unless prior arrangements have been made with the instructor. Consulting with your peers is allowed so long as it is done responsibly.
  - If you are ill or otherwise unable to turn in an assignment, contact Prof. Allen immediately by phone or email to make arrangements to turn the assignment. Late homework will not be accepted unless prior arrangements have been made.
- **Exams (2)**
  - Two in-class exams, each approximately one hour long.
  - The instructor will also give occasional 5-min pop-quizzes focusing on very fundamental concepts, which will be graded for homework credit.
- **Design Project**
  - Work in groups of two and turn in a short but high-quality written report with handwritten calculations in the appendix.
- **Final Exam**
- Grades in the course will be based on the following weighting:
  - Homework Sets 30%
  - Exams 40%
  - Design Project 10%
  - Final Exam 20%

Academic Misconduct:
The instructor takes dishonesty very seriously. Cheating will not be tolerated, whether on exams, quizzes or homework. If there is reasonable evidence that you have cheated on a homework assignment, the instructor reserves the right to give you negative credit for the assignment up to three times the value of the assignment. (Score = –3*value of assignment.) Serious infractions will be handled through the designated university channels.

Online course description:
General theory of free, forced, and transient vibrations; vibration transmission, isolation, and measurement; normal modes and generalized coordinates; method of matrix equation formulation and

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solution. The application of theory and methods to the analysis, measurement and design of dynamic systems.

General Topic Areas Covered:

- Equations of Motion for Discrete Vibratory Systems
- Transient Response of Single-Degree of Freedom (SDOF) Systems
- Steady State Response to Harmonic Excitation
- Modal Analysis of Multi-Degree-of-Freedom (MDOF) Systems
- Harmonic Excitation of MDOF Systems
- Vibration of Continuous Systems: The Ritz Method

Miscellaneous
Please inform me within the first two weeks of class of any specific days during the semester that may conflict with your religious observances, so I can make alternate arrangements for you.

Matlab
Many of the homework assignments require a computer package such as Matlab to complete. Matlab is available in all CAE computer labs. You can also purchase a student version for around $100 to install on your personal computer. Alternatively, there are a few clones of Matlab which may provide enough functionality to meet the needs of this course. These are compared in the following and in many blogs and websites:
http://www.webcitation.org/6BbWqerg3
The most notable for laptop/desktop computers seem to be:
Octave – www.octave.org
Freemat – www.freemat.org
Python with SciPy - http://en.wikipedia.org/wiki/SciPy
For Android:
Addi – https://play.google.com/store/apps/details?id=com.addi
Octave - https://play.google.com/store/apps/details?id=com.octave&hl=en

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TENTATIVE **Semester Schedule:**

*** NOTE: All of the dates below are approximate.** The instructor reserves the right to adjust the schedule!

Date*	**Week**	**Topic** (*Book sections*)	**Due Dates**
1/22	1	Introduction, Review of Newton-Euler EOM (*1.1-1.4*)	
1/24		Numerical solution methods, Harmonic Functions (*2.1*)	
1/29	2	Intro – Solutions to EOM, Beating	
1/31		Free Response: Underdamped/Overdamped (*2.2*)	**HW #1 due**
2/5	3	Forced Response (*2.3*)	
2/7		Finish Forced Response (2.3)	**HW #2 due**
2/12	4	MSA OUT (IMAC) Video Lecture RF Switch Example, Aircraft Engine Example	
2/14		MSA OUT (IMAC) – Lab Demo (John Dreger)	**HW #3 due**
2/19	5	Frequency Response (*3.1, 3.2.1, 3.3*)	
2/21		**Exam #1** (Ch. 2, SDOF Time Response)	
2/26	6	Base Excitation, examples	*(Assign Design Project)*
2/28		Damping (3.2) Resonance (3.3) Stress in Springs (notes)	**HW #4 due**
3/5	7	Rotating Imbalance (*3.4*) Force Transmissibility (*3.6*)	
3/7		Intro to Fourier Series (3.7)	**HW#5 due**
3/12	8	FFT for Periodic Excitation (3.7.5) Fourier Transform, DFT/FFT, Aliasing	
3/14		FFT for Transient Excitation (3.8.2), Accelerometer (3.7.6), Examples	**HW#6 due**
3/19	9	MDOF EOM, Large Deformation & Linearization, Gravity Stiffness & Stiff Spring Approx	**DP analysis due**
3/21		Additional examples	**HW #7 due**
3/26-28		**Spring Recess** (Mar 23-31)	
4/2	10	MDOF Solution, Shuttle Example	
4/4		**Exam #2** (SDOF System Response)	**HW #8 due**
4/9	11	MDOF Systems – Intro to Eigenproblem (4.1-4.2)	**DP due** (Monday 5PM)
4/11		MDOF Systems – Orthogonality, Normalization (4.2.2)	**HW#9 due**
4/16	12	Examples: Free Response, Transient Response, Damping	
4/18		Modal Transformation – (4.3), 3DOF Example and Matlab, (done by video) (do after close fn's next time)	**HW #10 due**
4/23	13	Close Nat. Freqs. (4.2.5-6), Rigid Body Modes Frequency Response using Modal Coordinates	
4/25		Freq. Domain TF (5.1), Vibration Absorber (5.3)	
4/30	14	Introduction to Power Balance & Lagrange Methods (1.5)	
5/2		Ritz Method (6.1), Sound from a rod excited axially	**HW #11 due**
5/7	15	Examples Continued	
5/9		Aircraft Modes using Ritz, Aeroelasticity Example, Review for Final, Nonlinear Vibration	**HW #12 due**
5/16		**Final Exam: Thurs. May 16, 10:05AM - 12:05PM**	

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1.2 course related links

1. final exam schedule
2. Syllabus
3. public course web page
4. internal course web page
5. Lectures download

Chapter 2

HW's

Local contents

2.1 HW lookup Table

HW	grade	about
1	95%	series/parallel stiffness, How to use $x = \mathrm{Re}\left\{Xe^{i\omega t}\right\}$ to analyze systems earliest time to reach maximum value/speed, complex exponential
2	95.70%	eq. of motion cube in water, more use of complex exponential analyzing in complex plane. Logarithmic decrement from graph. Impulse problem
3	100%	small lab
4	98.75%	force applied for small period, find response. Impulse force on system. analyse in complex plane. resonance problem, students on bridge.
5	88%	2 DOF system, shock observer on spring.Force transmission to base. off center motion, find EQM. 2 counter-rotating masses
6	97.50%	Find complex Fourier series.Verify using fft. Simple model of car moving on ground, find EQM. Fourier series. rectangle force, Fourier series.

7	95%	Using FFT to find response. Transfer functions Compare to analytical. Using Lagrange to find EQM, 2 DOF.
8	99%	non-linear EQM, spring stiff approximation. Lagrangian. Model of wing. cart on spring with sliding mass on it with spring. Lagrangian. Finding ω_n for 2DOF
A1	100%	more spring stiff approximation. manipulation of complex form of solution. half-power point, finding phase lag, solving step response using appendix B method
9	95%	Full solution in modal coordinates. Mass normalized. Initial conditions in modal coordinates. All problems done in power balance method. Double physical pendulum
10	92.5%	Full solution in modal coordinates. 3 DOF problem
11	93.3%	modal analysis, with damping using specific modal damping. Structual damping Compare transfer functions for each damping method used. Ritz method, shape functions. plot mode shapes.

Table 2.1: Homeworks summary table

2.2 HW1

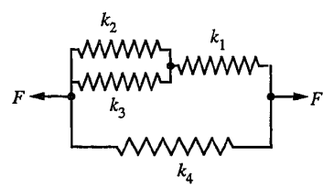
Local contents

2.2.1 problem description

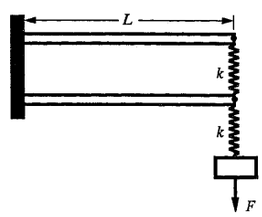
Homework #1
EMA 545, Spring 2013

Problem 1: 1.1 from Ginsberg:

1.1 Determine the spring stiffness that is equivalent to the action of the four springs in the sketch.



Problem 2: Find the equation of motion of the system pictured below. The mass of the block is *m* and the mass of the beams and springs is negligible. Assume that all of the displacements are very small. (Recall that the displacement of the tip of a cantilever beam, Δ_{tip}, is related to the force at the tip by: $F_{tip}=(3EI/L^3)\Delta_{tip}$)



USE COMPLEX EXPONENTIALS to derive the solution to problems 3-6 (i.e. do not simply look up a trig identity).

Problem 3: 2.3 from Ginsberg

Problem 4: 2.5 from Ginsberg. Note that "this quantity" in the last sentence is referring to "the complex amplitude of d*v*/dt."

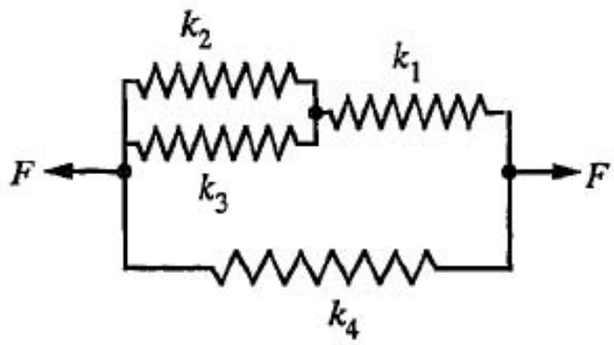
Problem 5: 2.8 from Ginsberg.

Problem 6: 2.10 from Ginsberg.

Extra: (this problem will not be graded) If you are not already familiar with Matlab, review the Matlab® tutorial on the EMA 545 course website (created by Prof. Negrut).

2.2.2 Problem 1 (1.1 book)

1.1 Determine the spring stiffness that is equivalent to the action of the four springs in the sketch.



k_3 and k_2 are in parallel, hence the effective stiffness is

$$k_{23} = k_2 + k_3$$

k_{23} and k_1 are now in series, hence the effective stiffness is

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_{23}} = \frac{k_{23} + k_1}{k_1 k_{23}} = \frac{k_2 + k_3 + k_1}{k_1(k_2 + k_3)} = \frac{k_2 + k_3 + k_1}{k_1 k_2 + k_1 k_3}$$

Therefore

$$k_{123} = \frac{k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1}$$

k_{123} and k_4 are now in parallel, hence the effective stiffness is

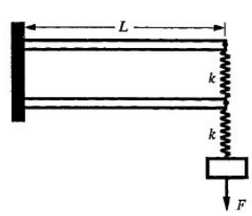
$$\begin{aligned} k_{1234} &= k_4 + k_{123} \\ &= k_4 + \frac{k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1} \end{aligned}$$

Hence the final effective stiffness is

$$k_{eq} = \frac{k_4(k_2 + k_3 + k_1) + k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1}$$

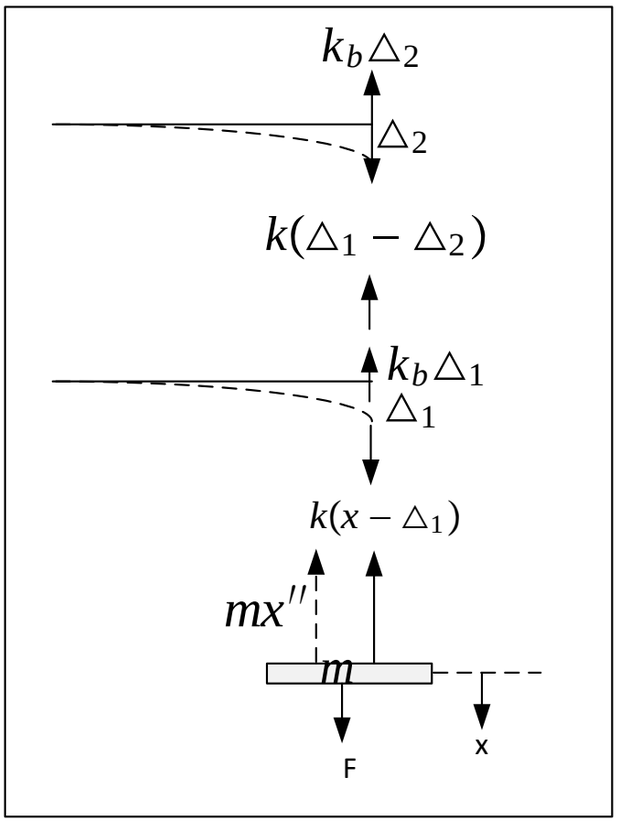
2.2.3 Problem 2

Problem 2: Find the equation of motion of the system pictured below. The mass of the block is m and the mass of the beams and springs is negligible. Assume that all of the displacements are very small. (Recall that the displacement of the tip of a cantilever beam, Δ_{tip}, is related to the force at the tip by: $F_{tip}=(3EI/L^3)\Delta_{tip}$)



We start by drawing a free body diagram and taking displacement of mass from the static equilibrium position. Let the displacement of the mass be x and positive pointing upwards.

Let $\triangle_1$ be the downward deflection at right end of the bottom beam. Let $\triangle_2$ be the downward deflection at right end of top beam. The free body diagram is



Applying equilibrium of vertical forces $\sum F_v = 0$ for mass m and noting that inertial forces opposes motion, results in the equation of motion

$$mx'' + k(x - \triangle_1) = F \tag{2.1}$$

To find an expression for $\triangle_1$ in terms of x, we apply equilibrium of vertical forces at the right end of the lower beam[1]

$$k(x - \triangle_1) = k_b\triangle_1 + k(\triangle_1 - \triangle_2) \tag{2.2}$$

Similarly, applying equilibrium of vertical forces at the right end of the top beam

$$k(\triangle_1 - \triangle_2) = k_b\triangle_2 \tag{2.3}$$

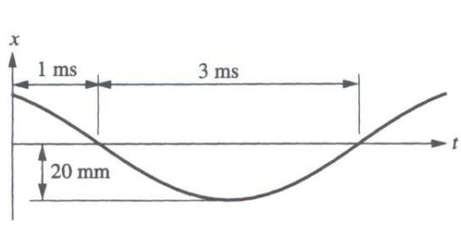
Solving for $\triangle_1, \triangle_2$ from Eqs 2.111,2.96 (2 equations, 2 unknowns) gives

$$\triangle_1 = \frac{k(k + k_b)}{k^2 + 3kk_b + k_b^2}x$$

Substituting the above value into Eq 2.110 results in the equation of motion

$$mx'' + kx\left(1 - \frac{k(k + k_b)}{k^2 + 3kk_b + k_b^2}\right) = F$$

2.2.4 Problem 3



EXERCISE 2.3

Use the properties in this trace to determine

(a) The initial values of x and $\dot{x}$.

(b) The value of t at which the minimum value of x first occurs.

(c) The maximum positive value of $\dot{x}$ and the earliest value of t at which this maximum occurs.

(d) The maximum positive value of $\ddot{x}$ and the earliest value of t at which this maximum occurs.

[1] k_b is beam stiffness against vertical displacement at the end and is given as $k_b = \frac{3EI}{L^3}$

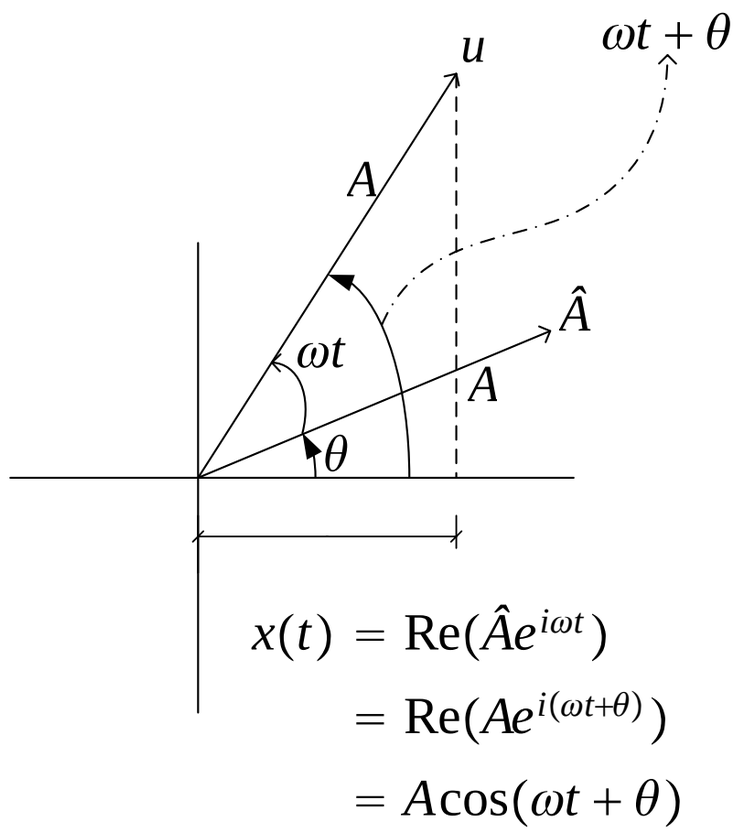
Assuming periodic motion, the period is $T = 6$ ms, or 6×10^{-3} sec. Hence $\omega = \frac{\pi}{3}$ rad/ms Representing this as a cosine signal with phase gives

$$x(t) = A\cos(\omega t + \theta)$$

Then

$$\begin{aligned} x(t) &= \mathrm{Re}[A + \cos(\omega t + \theta)] \\ &= \mathrm{Re}[Ae^{i\theta}e^{i\omega t}] \\ &= \mathrm{Re}[\bar{A}e^{i\omega t}] \end{aligned} \tag{2.4}$$

Where now $\hat{A} = Ae^{i\theta}$.Using phasor diagram



Hence from the diagram we see that for $x(t_0)$ to be zero when $t_0 = 1$ ms, we need to have

$$\omega t_0 + \theta = \frac{\pi}{2}$$

But $\omega = \frac{\pi}{3}$ rad/ms, hence

$$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

To find A we see that the maximum absolute value of $x(t)$ is 20 mm hence $A = 20$ mm or 20×10^{-3} meter. The equation of $x(t)$ when substituting all numerical values becomes

$$x(t) = 20\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \tag{2.5}$$

Where units used are radians, milliseconds and mm. This is a plot of the above function

```
parms = f -> 1/(6 10^-3);
Plot[0.02 Cos[2 Pi f t + (Pi/6)] /. parms, {t,0,0.005},
 AxesLabel -> {t,x[t]}, ImageSize -> 300]
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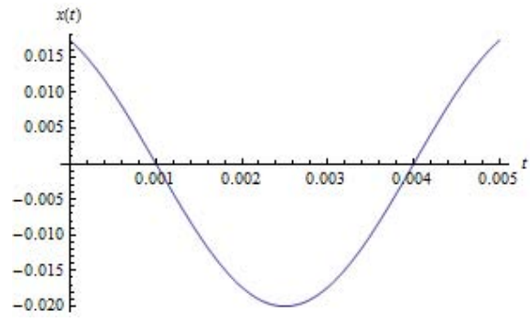


2.2.4.1 part(a)

At $t = 0$, from 2.117 $x(0) = \text{Re}[\hat{A}] = A\cos(\theta) = 20\cos(\frac{\pi}{6})$ hence

$$x(0) = 17.321 \text{ mm}$$

From 2.117 $x'(t) = \text{Re}[\omega\hat{A}e^{i\omega t}]$ hence $x'(0) = \text{Re}[\omega\hat{A}] = \omega A\cos(\theta) = 20\frac{\pi}{3}\cos(\frac{\pi}{6})$ giving

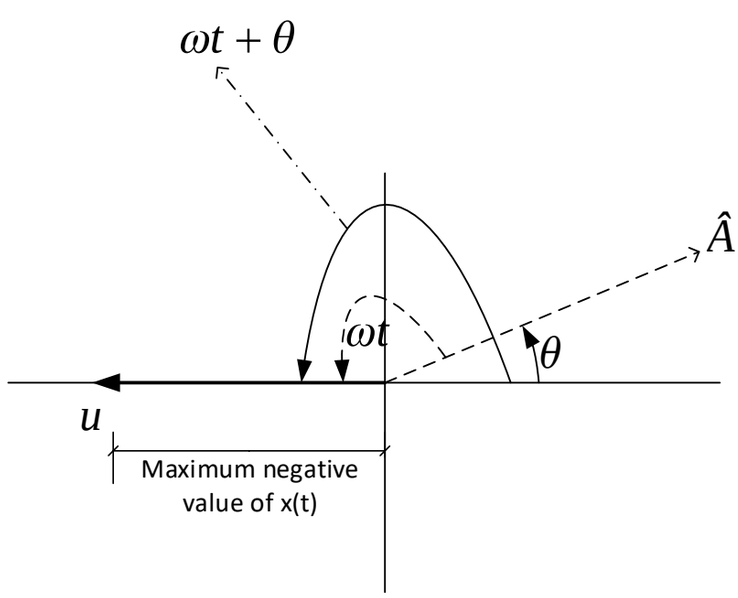
$$x'(0) = 18.138 \text{ m/sec}$$

2.2.4.2 part(b)

This can be solved using calculus[2]

$$\begin{aligned}
x'(t) &= -2\pi f A\sin\left(2\pi f t + \theta\right)\\
0 &= -2\pi f A\sin\left(2\pi f t + \frac{\pi}{6}\right)\\
&= -\left(\frac{2\pi}{6\times 10^{-3}}\right)\left(20\times 10^{-3}\right)\sin\left(\frac{2\pi}{6\times 10^{-3}}t + \frac{\pi}{6}\right)\\
0 &= \sin\left(\frac{2\pi}{6\times 10^{-3}}t + \frac{\pi}{6}\right)
\end{aligned}$$

We solve for t and find t=2.5 ms. But this can be solved more easily by looking at the phasor diagram



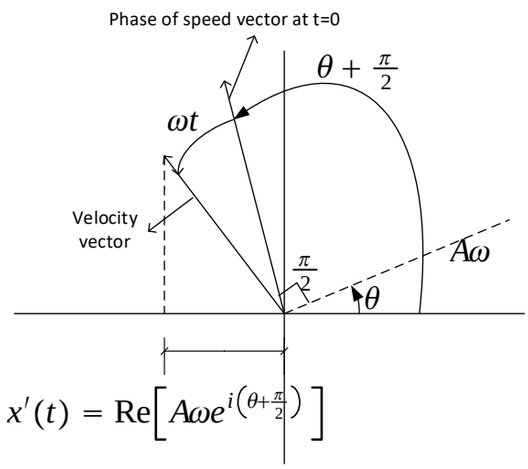
The minimum $x(t)$ (in negative sense and not in absolute value sense) occurs when $\omega t_{min} + \theta = \pi$, hence $t_{min} = \frac{\pi-\theta}{\omega}$, therefore

$$t_{min} = 2.5$$

[2]Taking derivative of $x(t)$ and setting the result to zero and solving for t

2.2.4.3 part(c)

This is solved in a similar way by treating the speed as the rotating vector in complex plan. Since $x'(t) = \text{Re}\left(A\omega e^{i\left(\omega t+\theta+\frac{\pi}{2}\right)}\right)$ then in complex plan as follows



The difference is that the velocity vector has phase of $\theta + \frac{\pi}{2}$ instead of θ as was the case with the position vector, and the amplitude is $A\omega$ instead of A. Hence the first time the speed vector will have the maximum value is when

$$\theta + \frac{\pi}{2} + \omega t = 2\pi$$

Hence

$$\begin{aligned} t &= \frac{2\pi - \frac{\pi}{2} - \theta}{\omega} \\ &= \frac{2\pi - \frac{\pi}{2} - \frac{\pi}{6}}{\frac{\pi}{3}} \end{aligned}$$

Hence $t = 4$ ms and the amplitude is given by $A\omega = 20\frac{\pi}{3}$ hence $A\omega = 20.944$ meter/sec

2.2.4.4 part(d)

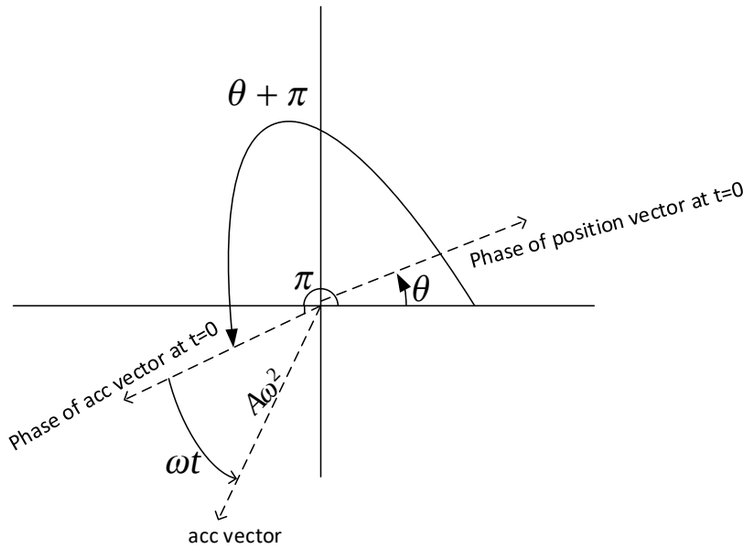
Now treating the acceleration as the rotating vector in complex plan

$$\begin{aligned} x(t) &= \text{Re}\left(Ae^{i(\theta+\omega t)}\right) \\ x'(t) &= \text{Re}\left(iA\omega e^{i(\theta+\omega t)}\right) \\ x''(t) &= \text{Re}\left(-A\omega^2 e^{i(\theta+\omega t)}\right) \end{aligned}$$

But $-1 = e^{i\pi}$ This adds a π to the phase resulting in

$$x''(t) = \text{Re}\left(A\omega^2 e^{i(\theta+\omega t+\pi)}\right)$$

Representing $x''(t)$ in complex plan gives



The first time the $x''(t)$ vector will have the maximum value is when

$$\theta + \pi + \omega t = 2\pi$$

Hence

$$\begin{aligned} t &= \frac{2\pi - \pi - \theta}{\omega} \\ &= \frac{\pi - \frac{\pi}{6}}{\frac{\pi}{6}} \end{aligned}$$

Hence $t = 2.5$ ms and the amplitude is

$$\begin{aligned} A\omega^2 &= 20 \text{ mm}\left(\frac{\pi}{3} \text{ rad/msec}\right)^2 \\ &= 21.933 \times 10^3 \text{ meter/sec}^2 \end{aligned}$$

2.2.5 Problem 4 (2.5 book)

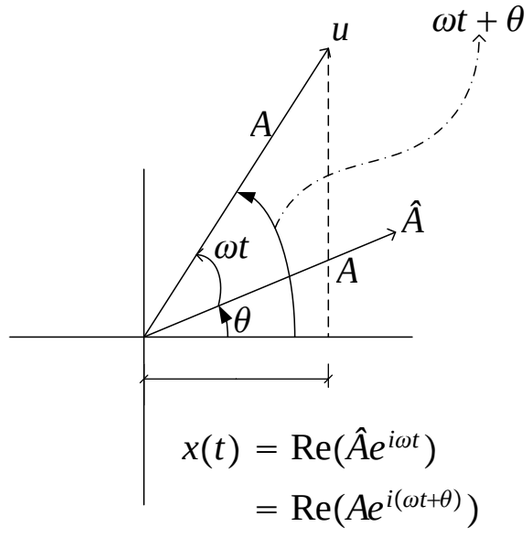
2.5 An oscilloscope trace indicates that the voltage output v from a sensor varies harmonically, with zeroes occurring every 8 ms. The first zero of v occurs at $t = 5.5$ ms, the amplitude of the signal is 1.2 V, and $v > 0$ at $t = 0$.

(a) Express this signal as a complex exponential. Write the complex amplitude in polar and rectangular forms.

(b) Express the time rate of change of the voltage as a complex exponential. Write this quantity in polar and rectangular forms.

2.2.5.1 part(a)

The function of the signal is converted to complex exponential. A sin or cos can be used to represent the signal as long as we are consistent. Assuming the signal is $x(t) = A\cos(\omega t + \theta)$, plotting the general representation of the position vector in complex plan gives



$$\begin{aligned} x(t) &= \mathrm{Re}(\hat{A}e^{i\omega t}) \\ &= \mathrm{Re}(Ae^{i(\omega t+\theta)}) \\ &= A\cos(\omega t+\theta) \end{aligned}$$

The complex representation of the position vector is

$$x(t) = \mathrm{Re}\left[Ae^{i(\omega t+\theta)}\right]$$

We are given that $\omega = \frac{2\pi}{T} = \frac{2\pi}{16}$, and since $x(t_0)$ has first zero at $t_0 = 5.5$ ms this means from looking at the above diagram that

$$\theta + \omega t_0 = \frac{\pi}{2}$$

Hence $\theta = \frac{\pi}{2} - (\omega t_0) = \frac{\pi}{2} - (\frac{\pi}{8}\frac{55}{10})$ which gives

$$\theta = \frac{-3\pi}{16}\text{radians}$$

Hence the signal is

$$\begin{aligned} x(t) &= \mathrm{Re}\left[Ae^{i(\omega t+\theta)}\right] \\ &= \mathrm{Re}\left[1.2e^{i\left(\frac{\pi}{8}t-\frac{3\pi}{16}\right)}\right] \\ &= \mathrm{Re}\left[1.2e^{-i\frac{3\pi}{16}}e^{i\frac{\pi}{8}t}\right] \\ &= \mathrm{Re}\left[\hat{A}e^{i\frac{\pi}{8}t}\right] \end{aligned}$$

Where $\hat{A} = 1.2e^{-i\frac{3\pi}{16}}$ is the complex amplitude in polar coordinates. In rectangular coordinates it becomes

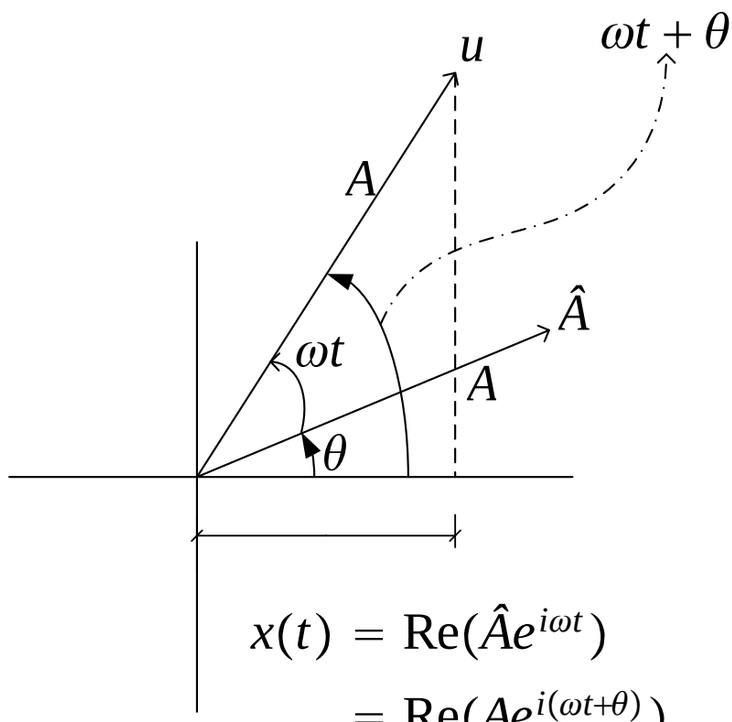
$$\begin{aligned} \hat{A} &= 1.2e^{-i\frac{3\pi}{16}} \\ &= 1.2\left(\cos\left(\frac{3\pi}{16}\right) - i\sin\left(\frac{3\pi}{16}\right)\right) \\ &= 1.2(0.831 - i0.5556) \\ &= \boxed{0.9977 - i0.6667} \end{aligned}$$

Hence

$$\begin{aligned} x(t) &= \mathrm{Re}\left[(0.998 - i0.668)\left(\cos\frac{\pi}{8}t + i\sin\frac{\pi}{8}t\right)\right] \\ &= \mathrm{Re}\left[\left(0.998\cos\frac{\pi}{8}t + 0.668\sin\frac{\pi}{8}t\right)+\right. \\ &\left. i\left(0.998\sin\frac{\pi}{8}t - 0.668\cos\frac{\pi}{8}t\right)\right] \end{aligned}$$

Here is a plot of the signal for 20 ms

```
w = Pi/8;
f = 1.2 Cos[w t - 3 Pi/16];
Plot[f, {t, 0, 20}, AxesLabel -> {t, x[t]},
                    ImageSize -> 300,
                    GridLines -> Automatic,
     GridLinesStyle->{{Dashed,Gray},{Dashed,Gray}},
     PlotStyle -> Red]
```



$$
\begin{aligned}
x(t) &= \mathrm{Re}(\hat{A}e^{i\omega t}) \\
&= \mathrm{Re}(Ae^{i(\omega t+\theta)}) \\
&= A\cos(\omega t+\theta)
\end{aligned}
$$

2.2.5.2 part(b)

From above it was found that

$$x(t) = \mathrm{Re}\left[Ae^{i(\omega t+\theta)}\right]$$

Hence

$$
\begin{aligned}
x'(t) &= \mathrm{Re}\left[i\omega Ae^{i(\omega t+\theta)}\right] \\
&= \mathrm{Re}\left[e^{i\frac{\pi}{2}}\omega Ae^{i\theta}e^{i\omega t}\right] \\
&= \mathrm{Re}\left[\omega Ae^{i(\frac{\pi}{2}+\theta)}e^{i\omega t}\right] \\
&= \mathrm{Re}\left[\hat{A}e^{i\omega t}\right]
\end{aligned}
$$

Where $\hat{A} = \omega Ae^{i(\frac{\pi}{2}+\theta)}$ Replacing numerical values gives $\hat{A} = \frac{\pi}{8}(1.2)e^{i(\frac{\pi}{2}-\frac{3}{16}\pi)} = 0.471e^{i0.983}$ and

$$
\begin{aligned}
x'(t) &= \mathrm{Re}\left[0.471e^{i0.983}e^{i\omega t}\right] \\
&= \mathrm{Re}\left[0.471e^{i0.983}e^{i\frac{\pi}{8}t}\right] \\
&= \mathrm{Re}\left[0.471e^{i0.983}e^{i0.3923t}\right]
\end{aligned}
$$

In rectangular coordinates, the above becomes

$$\begin{aligned}
x'(t) &= \mathrm{Re}\Big[0.471(\cos 0.983 + i\sin 0.983) \\
&\quad (\cos 0.3923t + i\sin 0.3923t)\Big] \\
&= \mathrm{Re}[(0.261 + 0.392i)(\cos 0.392t + i\sin 0.392t)] \\
&= \mathrm{Re}\Big[(0.261\cos 0.392t - 0.392\sin 0.392t) \\
&\quad +i(0.261\sin 0.392t + 0.392\cos 0.392t)\Big]
\end{aligned}$$

2.2.5.3 part(c)

To find the maximum rate of the signal

$$x'(t) = \mathrm{Re}\Big[\hat{A}e^{i\omega t}\Big]$$

Then the maximum $x'(t)$ is $\left|\hat{A}\right|$ which is

$$\begin{aligned}
\left|\hat{A}\right| &= |0.261 + 0.392i| \\
&= \sqrt{0.261^2 + 0.392^2} \\
&= 0.471
\end{aligned}$$

Hence maximum $x'(t)$ is 0.471 v/ms or 471 volt/sec.

Maximum velocity in simple harmonic motion occurs when $x(t) = 0$. This occurs at $t = 5.5$ ms and at 8 ms henceforth. Hence maximum speed occurs at

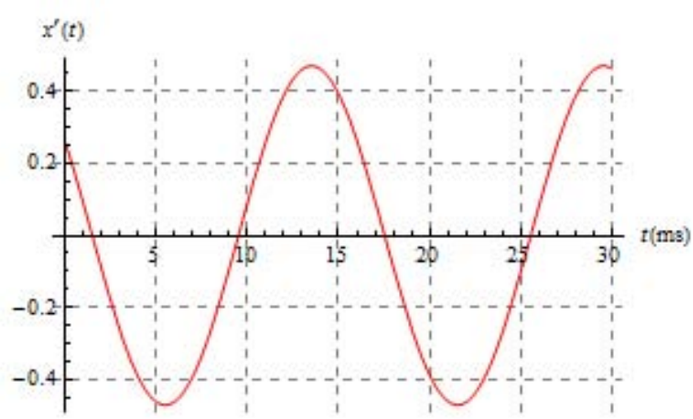
$$t = 5.5 + n(8)$$

for $n = 0, 1, 2, \cdots$ this results in

$$t = 5.5, 13, 5, 21.5, \cdots \text{ ms}$$

Here is a plot of $x'(t)$ in units of volt/ms

```
f = 0.261 Cos[0.392 t] - 0.392 Sin[0.392 t];
Plot[f, {t, 0, 30},
 AxesLabel -> {Row[{t, "(ms)"}], x'[t]},
 ImageSize -> 300, GridLines -> Automatic,
 GridLinesStyle -> {{Dashed, Gray}, {Dashed, Gray}},
 PlotStyle -> Red]
```



2.2.6 Problem 5 (2.8 book)

2.8 Suppose $q = 0.01\sin(50t) - 0.02\cos(50t - 0.3\pi)$.

(a) Write q in complex exponential form. What is the complex amplitude?

(b) What is the time interval separating instants at which $q = 0$?

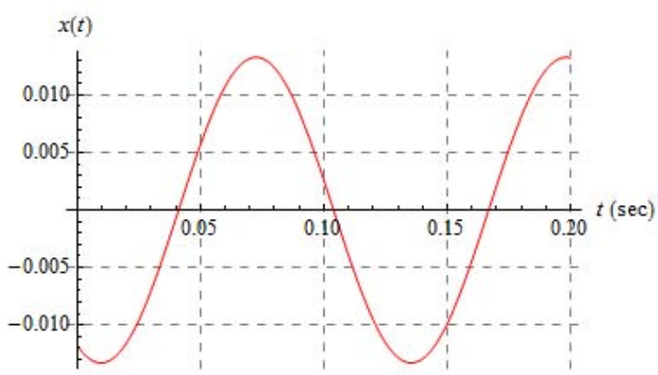
(c) What is the earliest positive t at which $q = 0$?

(d) What is the largest value of q that will occur, and what is the earliest positive t at which this maximum occurs?

2.2.6.1 part(a)

This is a plot of the signal

```
f = 0.01 Sin[50 t] - 0.02 Cos[50 t - 0.3 Pi];
Plot[f, {t, 0, 0.2},
        AxesLabel -> {Row[{t , " (sec)"}], x[t]},
        ImageSize -> 300,
        GridLines -> Automatic,
        GridLinesStyle->{{Dashed,Gray},{Dashed,Gray}},
        PlotStyle -> Red]
```



$$
\begin{aligned}
q &= 0.01\sin(50t) - 0.02\cos(50t - 0.3\pi) \\
&= \mathrm{Re}\left[\frac{0.01}{i}e^{i50t} - 0.02e^{i(50t-0.3\pi)}\right] \\
&= \mathrm{Re}\left[0.01e^{-i\frac{\pi}{2}}e^{i50t} - 0.02e^{i50t}e^{-i0.3\pi}\right] \\
&= \mathrm{Re}\left[\left(0.01e^{-i\frac{\pi}{2}} - 0.02e^{-i0.3\pi}\right)e^{i50t}\right] \\
&= \mathrm{Re}\left[\hat{A}e^{i50t}\right]
\end{aligned}
$$

Hence the complex amplitude is

$$\hat{A} = 0.01e^{-i\frac{\pi}{2}} - 0.02e^{-i0.3\pi}$$

2.2.6.2 part(b)

From above, we see that

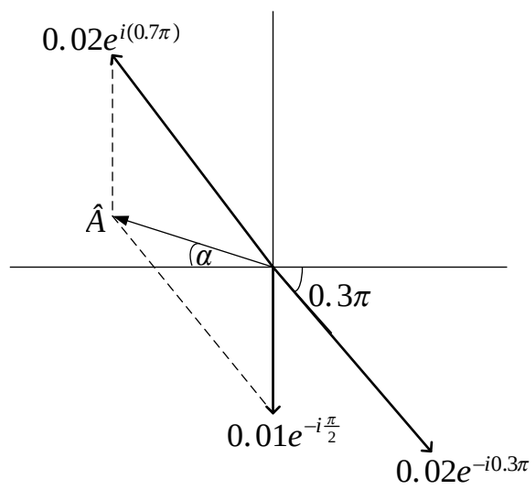
$$\omega = 50 \text{ rad/sec}$$

Hence $f = \frac{50}{2\pi}$Hz, or the period $T = \frac{2\pi}{50} = 0.126$ sec, therefore the time period separating the zeros is $\frac{0.126}{2} = 0.063$ sec or 63 ms

2.2.6.3 part(c)

The complex phase $\hat{A}$ can be found by adding the vector $0.01e^{-i\frac{\pi}{2}}$ and $-0.02e^{-i\frac{3\pi}{10}}$ by completing the parallelogram as shown in this diagram. $\hat{A} = -0.02\cos 0.7\pi + i(-0.01 + 0.02\sin 0.7\pi)$, hence the angle α that $\hat{A}$ makes with the horizontal is

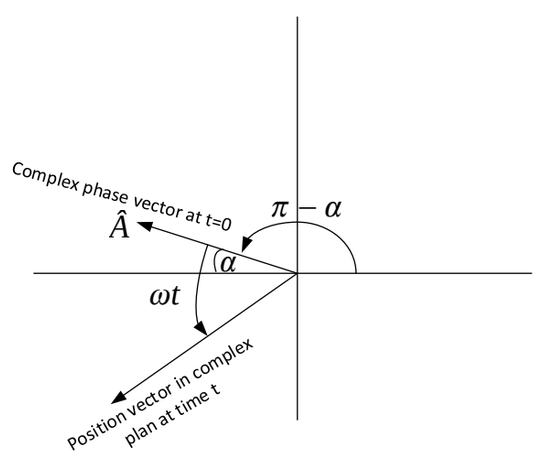
$$\begin{aligned}\tan^{-1}\left(\frac{-0.01 + 0.02\sin 0.7\pi}{-0.02\cos 0.7\pi}\right) &= \arctan(0.526)\\ &= 0.484 \text{ radian}\\ &= 27.73 \text{ degree}\end{aligned}$$



and the amplitude is

$$\sqrt{(-0.01 + 0.02\sin 0.7\pi)^2 + (0.02\cos 0.7\pi)^2} = 0.0133\text{V}$$

To find the earliest time q will be zero, we need to find the time the complex position vector will take to rotate and reach the imaginary axis.



Hence we need to solve

$$\begin{aligned}\pi - \alpha + \omega t_0 &= \frac{3}{2}\pi \\ t_0 &= \frac{\frac{3}{2}\pi - \pi + 0.48402}{50} \\ &= 0.0411 \text{ s}\end{aligned}$$

Therefore

$$t = 41.1 \text{ ms}$$

2.2.6.4 part(d)

The largest value of q is the absolute value of its complex amplitude. We found this above as

$$\left|\hat{A}\right| = 0.0133 \text{ Volt}$$

To find when this occur first time, the time the position vector will align with the real axis in the positive direction is found. Hence solving for t_0 from

$$\begin{aligned}\pi - \alpha + \omega t_0 &= 2\pi \\ t_0 &= \frac{2\pi - \pi + 0.484}{50}\end{aligned}$$

Gives $t = 72.5$ ms. Another way would be to take derivative of qt) and set that to zero and solve for first t which satisfy the equation.

2.2.7 Problem 6 (2.10 book)

2.10 Two parts of the harmonic motion of a system are $x_1 = 8\sin(10t - 5\pi/6)$, $x_2 = 12\cos(10t + \phi)$. Find the phase angle ϕ for which $x = x_1 + x_2$ is a pure sine function. What is the amplitude of x in that case?

$$\begin{aligned}x_1 &= 8\sin\left(10t - \frac{5}{6}\pi\right) \\ x_2 &= 12\cos\left(10t + \phi\right)\end{aligned}$$

Let $\omega = 10$, hence

$$\begin{aligned}x &= x_1 + x_2 \\ &= \text{Re}\left[\frac{8}{i}e^{i\left(\omega t - \frac{5}{6}\pi\right)}\right] + \text{Re}\left[12e^{i\left(\omega t + \phi\right)}\right] \\ &= \text{Re}\left[\frac{8}{i}e^{i\left(\omega t - \frac{5}{6}\pi\right)} + 12e^{i\left(\omega t + \phi\right)}\right] \\ &= \text{Re}\left[8e^{-i\frac{\pi}{2}}e^{i\left(\omega t - \frac{5}{6}\pi\right)} + 12e^{i\left(\omega t + \phi\right)}\right] \\ &= \text{Re}\left[8e^{-i\frac{\pi}{2}}e^{i\omega t}e^{-i\frac{5}{6}\pi} + 12e^{i\omega t}e^{i\phi}\right] \\ &= \text{Re}\left[\left(8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i\phi}\right)e^{i\omega t}\right] \\ &= \text{Re}\left[\hat{A}e^{i\omega t}\right] \qquad (2.6)\end{aligned}$$

Where

$$\begin{aligned}\hat{A} &= 8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i\phi}\\ &= (-4 + 6.928i) + 12\left(\cos\phi + i\sin\phi\right)\\ &= \left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\end{aligned}$$

Hence Eq 2.6 becomes

$$x = \mathrm{Re}\left[\left\{\left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\right\}e^{i\omega t}\right]$$

To convert to sin we multiply and divide by i hence

$$\begin{aligned}x &= \mathrm{Re}\left[\left\{\left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\right\}i\frac{e^{i\omega t}}{i}\right]\\ &= \mathrm{Re}\left[\left\{-\left(6.928 + \sin\phi\right) + i\left(-4 + 12\cos\phi\right)\right\}\frac{e^{i\omega t}}{i}\right] \qquad (2.7)\end{aligned}$$

The complex number $-\left(6.928 + \sin\phi\right) + i\left(-4 + 12\cos\phi\right)$ can be written in polar form as $ke^{i\beta}$ where $K = \sqrt{\left(6.928 + \sin\phi\right)^2 + \left(-4 + 12\cos\phi\right)^2}$ and $\beta = \tan^{-1}\left(\frac{-4+12\cos\phi}{-\left(6.928+\sin\phi\right)}\right)$, hence Eq 2.7 becomes

$$\begin{aligned}x &= \mathrm{Re}\left[ke^{i\beta}\frac{e^{i\omega t}}{i}\right]\\ &= \mathrm{Re}\left[k\frac{e^{i\left(\omega t+\beta\right)}}{i}\right]\\ &= k\sin\left(\omega t + \beta\right)\end{aligned}$$

or in full form

$$x = \sqrt{\left(6.928 + \sin\phi\right)^2 + \left(-4 + 12\cos\phi\right)^2}\sin\left(\omega t + \tan^{-1}\left(\frac{-4 + 12\cos\phi}{-\left(6.928 + \sin\phi\right)}\right)\right)$$

For pure sine function we need $\frac{-4+12\cos\phi}{-\left(6.928+\sin\phi\right)} = 0$ or $12\cos\phi = 4$ or $\cos\phi = \frac{1}{3}$, hence

$$\begin{aligned}\phi &= 1.23096 \text{ radian}\\ &= 70.529^\circ\end{aligned}$$

The amplitude can also be found from the complex amplitude above when $\phi = 1.23096$ as follows

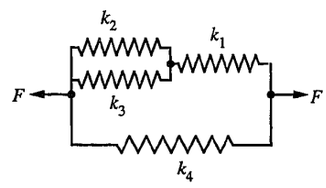
$$\begin{aligned}\left|8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i1.23096}\right| &= \left|-6.592\times10^{-6} + 18.242i\right|\\ &= \sqrt{\left(-6.592\times10^{-6}\right)^2 + (18.242)^2}\\ &= 18.242\end{aligned}$$

2.2.8 Key solution for HW 1

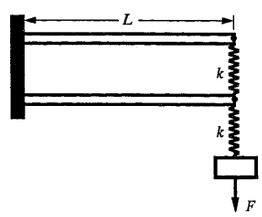
Homework #1
EMA 545, Spring 2013

Problem 1: 1.1 from Ginsberg:

1.1 Determine the spring stiffness that is equivalent to the action of the four springs in the sketch.



Problem 2: Find the equation of motion of the system pictured below. The mass of the block is *m* and the mass of the beams and springs is negligible. Assume that all of the displacements are very small. (Recall that the displacement of the tip of a cantilever beam, Δ_{tip}, is related to the force at the tip by: $F_{tip}=(3EI/L^3)\Delta_{tip}$)



USE COMPLEX EXPONENTIALS to derive the solution to problems 3-6 (i.e. do not simply look up a trig identity).

Problem 3: 2.3 from Ginsberg

Problem 4: 2.5 from Ginsberg. Note that "this quantity" in the last sentence is referring to "the complex amplitude of d*v*/dt."

Problem 5: 2.8 from Ginsberg.

Problem 6: 2.10 from Ginsberg.

Extra: (this problem will not be graded) If you are not already familiar with Matlab, review the Matlab® tutorial on the EMA 545 course website (created by Prof. Negrut).

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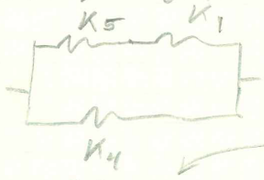
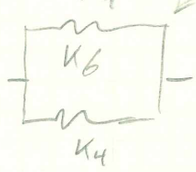
P1 (1.1) eq. to



$K_5 = K_2 + K_3$ (parallel)

K_1, K_5 in Series



$$K_6 = \frac{K_5 K_1}{K_5 + K_1}$$

$$\Rightarrow \boxed{K_{eq} = \frac{(K_2 + K_3) K_1}{K_1 + K_2 + K_3} + K_4}$$

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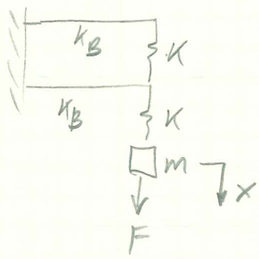
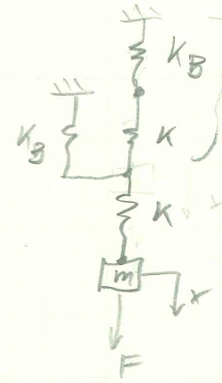
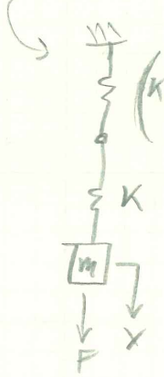
P3 $F_{tip} = \frac{3EI}{L^3}\Delta_{tip} = K_b \Delta_{tip} \rightarrow K_b = \frac{3EI}{L^3}$



equivalent to:



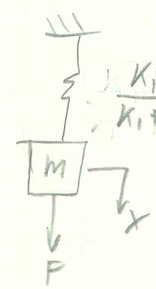
$\frac{KK_B}{K+K_B}$ series



Parallel

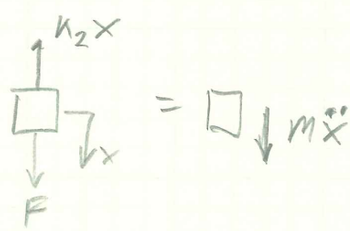
$\left(K_B + \frac{KK_B}{K+K_B}\right) = K_1$

series $\rightarrow$



$\frac{K_1 K}{K_1 + K} = K_2$

FBD:



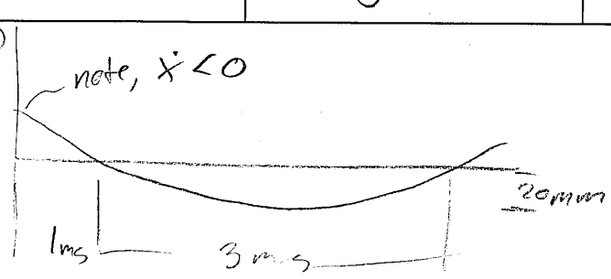
ΣF in x dir:

$m\ddot{x} = F - K_2 x$

$$\boxed{m\ddot{x} + \left(\frac{K_1 K}{K_1 + K}\right)x = F \qquad K_1 = K_B + \frac{KK_B}{K+K_B}}$$

Ginsberg - M+SV | M.S. Allen

(2.3)



$\dot{x}(0) < 0$,
$x(0) > 0$

a) find $x, \dot{x}$

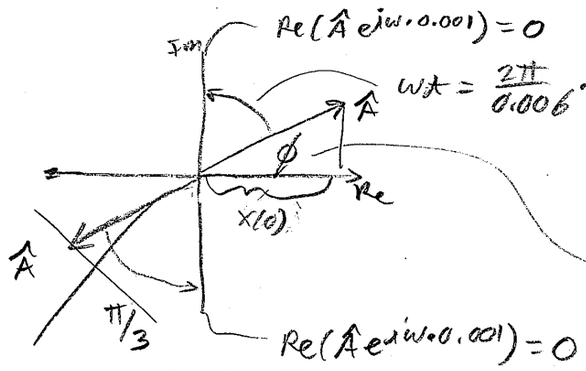
Using complex exponentials, write $x(t)$ as $\boxed{x(t) = Re(\hat{A} e^{i\omega t})}$

$\omega = ?$ $T = 6ms \rightarrow \omega = \frac{2\pi}{T} \rightarrow \boxed{\omega = \frac{2\pi}{0.006}}$

→ From plot $|\hat{A}| = 20mm$, so write $\hat{A} = A e^{i\phi}$, $A = 20mm$

→ Harmonic function reaches first maximum 1ms after $t = 0$, and since $T = 6$, this is 1/6th a cycle or $2\pi/6$ rad,

→ so, $\hat{A}$ must be in the 1st or 3rd quadrants



* 3rd quad. gives
$x(0) = Re(\hat{A}) < 0$ - Wrong

$\phi = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$
(from geometry)

so, $\boxed{\hat{A} = 20mm\, e^{i\pi/6}} \rightarrow \boxed{x(t) = Re(20mm\, e^{i\pi/6} e^{i\omega t})}$

a) $x(0) = Re(\hat{A}) = 20mm \cdot \cos(\pi/6) = 17.321$ $\boxed{x(0) = 17.3}$

$\dot{x}(t) = Re(i\omega \cdot 20mm\, e^{i\pi/6} e^{i\omega t})$

$\dot{x}(0) = Re(i \cdot \frac{2\pi}{0.006s} \cdot 0.020m \cdot e^{i\pi/6} \cdot 1) \rightarrow \boxed{\dot{x}(0) = -10.47\ m/s}$

b) Minimum at $t = 1ms + 3ms/2 = \boxed{min(x) \text{ at } 2.5\ ms}$

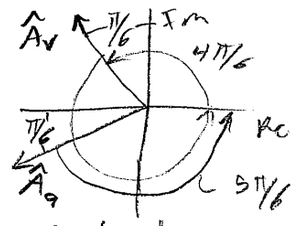
(2.3) max $\dot{x}$ and when? ~ From plot, point where slope is maximum positive $\boxed{t = 4\,ms}$

c) $\max \dot{x} = \left| i \cdot \frac{2\pi}{0.006\,s} \cdot 0.020\,m \cdot e^{i\pi/6} \right|$ $\boxed{\max \dot{x} = 20.944}$

or:

$\dot{x} = Re\left(e^{i\pi/2} \cdot \frac{2\pi \cdot 20}{6} e^{i\pi/6} e^{i\omega t}\right)$

$\dot{x}(t) = Re\left(\underbrace{\frac{2\pi \cdot 20}{6} \cdot e^{i4\pi/6}}_{\hat{A}_v} e^{i\omega t}\right)$



Velocity

max pos. when $\omega t = 2\pi - \frac{4\pi}{6}$

$\omega t = \frac{12\pi - 4\pi}{6} = \frac{8\pi}{6}$

$t = \frac{4\cancel{8\pi}}{6} \cdot \frac{0.006}{\cancel{2\pi}} = \boxed{0.004\,s}$ ✓ $\boxed{\max(\dot{x}) = |\hat{A}_v| = \frac{2\pi \cdot 20}{6}}$

d) $\ddot{x} = Re\left(\underbrace{i\left(\frac{2\pi}{0.006\,s}\right) \cdot \frac{2\pi \cdot 20}{6}\left(\frac{m}{s}\right) e^{i4\pi/6}}_{\hat{A}_a} e^{i\omega t}\right)$

$\boxed{\max(\ddot{x}) = |\hat{A}_a| = \frac{4\pi^2 \cdot 20}{0.036} \approx 21{,}932\ m/s^2}$

$\omega t = \frac{5\pi}{6}$ (from picture above) $\rightarrow \boxed{t = \frac{5\cancel{\pi}}{6} \cdot \frac{0.006}{2\cancel{\pi}} = 2.5\,ms}$

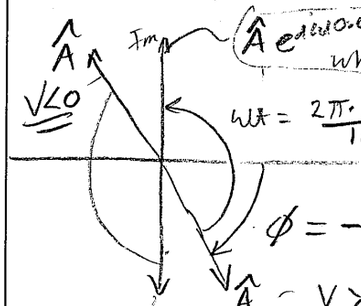
M. S. Allen

(2.5) Zeros ~ T/2, so $\boxed{\omega = \frac{2\pi}{0.016\,s} \approx 393 \text{ rad/s}}$

$x(5.5) = 0$, $|\hat{A}| = 1.2\,V$, $V > 0$ @ $t = 0$ Use polar form $\hat{A} = 1.2V e^{i\phi}$

let $V(t) = Re(1.2V \cdot e^{i\phi} e^{i\omega t})$

$V(5.5) = Re(1.2V \cdot e^{i\phi} \underbrace{e^{i \cdot \frac{2\pi}{0.016} \cdot 0.055}}_{\gamma = \frac{2\pi \cdot 5.5}{16} \approx 0.7\pi}) = 0$



$\phi = -\left(\frac{2\pi \cdot 5.5}{16} - \frac{\pi}{2}\right) \approx -0.589$ $\left(\phi = -\frac{3\pi}{16}\right)$

$\hat{A}$ - $V > 0$ for $\hat{A}$ here,

or $\hat{A} e^{i\omega \cdot 0.055}$.

Matlab A_hat = 1.2*exp(-i*0.589)

so,

a.) $\boxed{\hat{A} = 1.2V \cdot e^{-i\,0.589} = 1.2\cos 0.589 - i\,1.2\sin 0.589}$

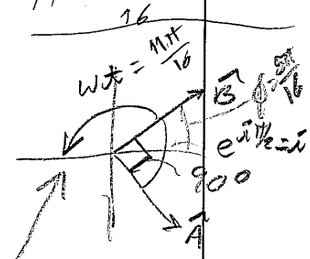
$\boxed{\hat{A} = 0.998 - 0.667\hat{i} \qquad V(t) = Re(\hat{A} e^{i393t})}$

b.) $\dot{V}(t) = Re(\underbrace{i\omega \cdot \hat{A}}_{\hat{B}} e^{i\omega t}) = Re(\hat{B} e^{i\omega t})$

$\hat{B} = e^{i\pi/2} \cdot \frac{2\pi}{0.016\,s} \cdot 1.2V\, e^{i\left(\frac{\pi}{2} - \frac{11\pi}{16}\right)}$ ~ $\pi - \frac{11\pi}{16}$

$\boxed{\hat{B} = \frac{\pi \cdot 2.4}{0.016}(V/s) \cdot e^{i5\pi/16}}$ polar

$\boxed{\hat{B} = \frac{\pi \cdot 2.4}{0.016}\left(\cos\left(\frac{5\pi}{16}\right) + i\sin\left(\frac{5\pi}{16}\right)\right) \quad \hat{B} \approx 261.8 + 391.8i}$ rect.



c) $\boxed{\max(\dot{x}) = |\hat{B}| = \frac{\pi \cdot 2.4}{0.016} V/s \approx 471.2\ V/s}$

Occurs when $\omega t = \frac{11\pi}{16} + n\pi$ considering $\max(|\dot{x}|)$

$t = \left(\frac{11\cancel{\pi}}{16} + n\cancel{\pi}\right) \cdot \frac{0.016\,s}{2\cancel{\pi}}$

$t = (5.5\,ms, 13.5\,ms, 21.5\,ms, 29.5\,ms)$

positive $\dot{x}$ only

either answer OK if done correctly

2/17/2011

(2.8) $g = 0.01\sin(50t) - 0.02\cos(50t - 0.3\pi)$

a) $g = Re(-i \cdot 0.01\, e^{i50t} - 0.02\, e^{i(50t - 0.3\pi)})$

$g = Re(-0.01i\, e^{i50t} - 0.02 e^{-i0.3\pi} e^{i50t})$

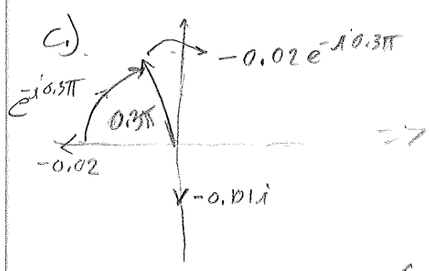
$\boxed{g = Re(A e^{i50t}) \qquad A = -0.01i - 0.02 e^{-i0.3\pi}}$

or could write $A = -0.01i - 0.02(\cos(-0.3\pi) + i\sin(-0.3\pi))$
$A = -0.02\cos(-0.3\pi) + i(-0.02\sin(-0.3\pi) - 0.01)$

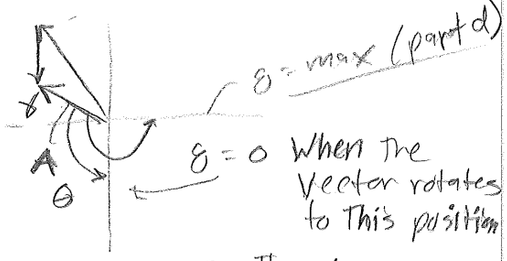
b.) frequency $\omega = 50$, so $T = \frac{2\pi}{50}$ $\quad (e^{i50t} \to e^{i2\pi} = 1 \text{ cycle})$

zeros every half cycle, $\boxed{\Delta t = T/2 = \pi/50}$

c.)



=> sum



When $50t = \theta$, $g = Re(Ae^{i\theta}) = 0$

$\theta = \frac{\pi}{2} + \gamma$

$\tan\gamma = \frac{|Im(A)|}{|Re(A)|}$

$$\boxed{\begin{array}{l} \gamma = \tan^{-1}\left|\frac{-0.02\sin(-0.3\pi) - 0.01}{-0.02\cos(-0.3\pi)}\right| \\ t = \left(\frac{\frac{\pi}{2} + \gamma}{50}\right) \qquad t \approx 41.1\text{ms} \text{ at first zero} \end{array}}$$

d.) $\boxed{\begin{array}{l} \max(g) = |A| = 0.0133 \\ \text{occurs when } 50t = \gamma + \pi \longrightarrow t \approx 72.5\text{ ms} \end{array}}$ see picture above

2/23/2011

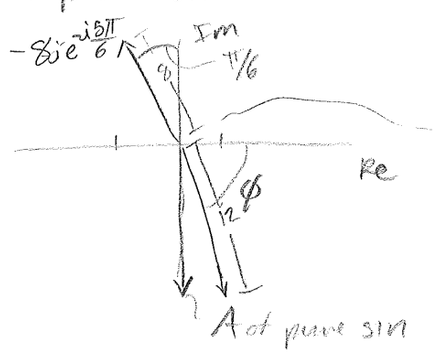
(2.10) $x = x_1 + x_2 = 8\sin(10t - 5\frac{\pi}{6}) + 12\cos(10t + \phi)$

Find ϕ s.t. x is a pure sine. I'll assume that the problem means "pure sine" = "positive sine", although it is, perhaps, a bit ambiguous.

Using definitions $\sin(\omega t) = \mathrm{Re}(-iAe^{i\omega t})$, etc...

$$x = \mathrm{Re}\Big[\underbrace{(-8ie^{-i5\pi/6} + 12e^{i\phi})}_{A}\, e^{i10t}\Big]$$

Need A to be purely imaginary and negative for a pure sin



So, ϕ must put the comp. Amp at the cos. term in the IV quad. to get A purely imaginary

real parts must cancel,

$$8\sin(\pi/6) = 12\cos|\phi|$$

$$|\phi| = \cos^{-1}\left(\frac{8}{12}\cos\left(\frac{\pi}{6}\right)\right)$$

$$\boxed{\phi = -1.231}$$

$$|x| = |A| = \left|(-8ie^{-i5\pi/6} + 12e^{i\phi})\right|$$

$$\boxed{|A| = 4.386} \qquad A = -4.386\hat{i}$$

✓ plotted in Matlab to verify that it gives a pure sine.

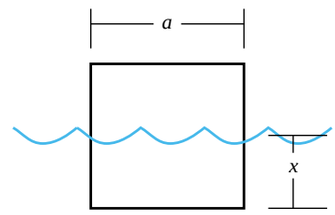
2.3 HW2

Local contents

2.3.1 problem description

Homework #2
EMA 545, Spring 2013

Problem 1:
A cube with density ρ and side length *a* is floating freely in a pool of water.



a.) Find the equation of motion of the cube when it is displaced in the vertical direction. (Recall that the buoyant force on a floating object is equal to the weight of the water displaced.) If necessary, re-define your vertical coordinate to eliminate any static forces.
b.) Derive an expression for the natural frequency of the cube.
c.) If the block is pine (ρ=400 kg/m^3) with a side length of 10cm, what is the natural frequency in Hz?

(Note that, while an analysis like this would be important when designing a boat or ocean vessel, in reality the rotational motions of the vessel would usually be more important and those require a more complicated analysis.)

Problem 2: 2.17 from Ginsberg

Problem 3: 2.19 from Ginsberg

Problem 4: Show that $x(t)=Be^{\lambda t}$ is a solution to $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=0$ and find λ for the following cases: 1.) Underdamped system, 2.) Overdamped system. Write the solution $x(t)$ for both cases for an arbitrary set of initial conditions and draw a sketch to illustrate how each response $x(t)$ would look. Show that $x(t)$ can be written as $x(t)=\mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{i\omega_d t}\right)$ in case (1).

Comment: I would encourage you to see if you can solve the following problems using only math and the fact that the general solution to an underdamped SDOF system

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x=0$$

is:

$$x(t)=\mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{i\omega_d t}\right)$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$ and A is a complex constant. If you're hunting through the book for equations to solve these problems then you might be making them more difficult than they need to be and perhaps failing to connect the concepts.

Problem 5: 2.29 from Ginsberg

Problem 6: 2.32 from Ginsberg. **(part a=5pts, b=5pts, c=10pts)**

2.3.2 problem 1

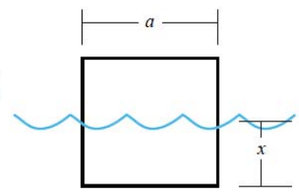
Problem 1:
A cube with density ρ and side length a is floating freely in a pool of water.

a.) Find the equation of motion of the cube when it is displaced in the vertical direction. (Recall that the buoyant force on a floating object is equal to the weight of the water displaced.) If necessary, re-define your vertical coordinate to eliminate any static forces.

b.) Derive an expression for the natural frequency of the cube.

c.) If the block is pine (ρ=400 kg/m^3) with a side length of 10cm, what is the natural frequency in Hz?

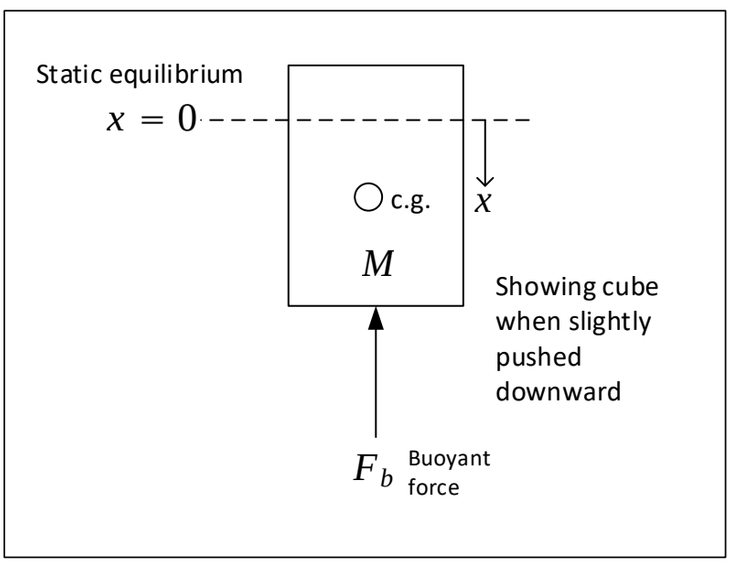
(Note that, while an analysis like this would be important when designing a boat or ocean vessel, in reality the rotational motions of the vessel would usually be more important and those require a more complicated analysis.)



2.3.2.1 Part(a)

We assume the cube is displaced downwards from its static equilibrium position and it is currently at distance x below the static position.

The buoyant force F_b will push the cube upwards. This force will equal the weight of water displaced which is $xa^2\rho_w g$ where ρ_w is density of water and g is the gravitational constant. The free body diagram is



Applying $F = mx''$ we obtain equation of motion

$$Mx'' = -F_b \tag{2.8}$$
$$Mx'' + F_b = 0 \tag{2.9}$$

$M = a^3\rho$ where ρ is density of pine. The above can be simplified to

$$a^3\rho x'' + xa^2\rho_w g = 0 \tag{2.10}$$
$$x'' + \frac{\rho_w g}{a\rho}x = 0 \tag{2.11}$$
$$x'' + \omega_n^2 x = 0 \tag{2.12}$$

2.3.2.2 Part(b)

Hence from the above equation

$$\omega_n = \sqrt{\frac{\rho_w g}{a\rho}}$$

2.3.2.3 Part(c)

Given $\rho = 400\ kg/m^3$ and $\rho_w = 1000\ kg/m^3$ and $a = 0.1m$ then

$$\omega_n = \sqrt{\frac{\rho_w g}{a\rho}} = \sqrt{\frac{1000 \times 9.81}{0.1 \times 400}} = 15.66\ \frac{rad}{\sec^2}$$

Hence frequency in Hz is

$$f = \frac{\omega_n}{2\pi} = \frac{15.66}{2\pi} = 2.492\ hz$$

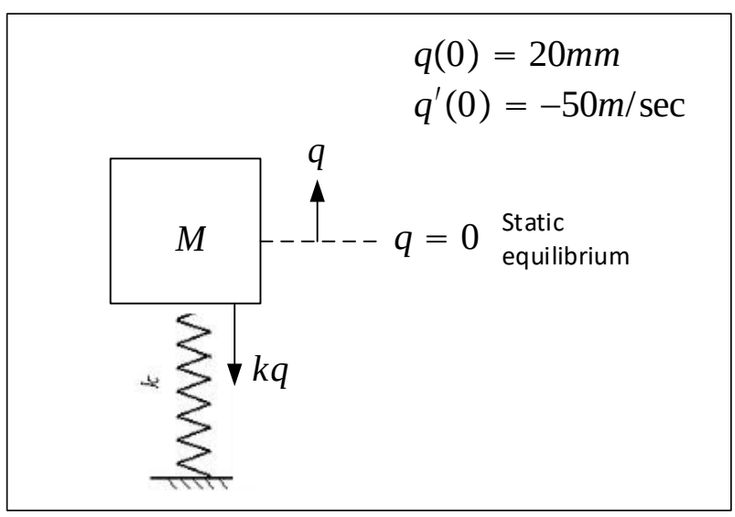
2.3.3 Problem 2

2.17 An undamped one-degree-of-freedom system has a mass coefficient of 50 kg and a natural frequency of 80 Hz. At $t = 0$ it is released from $q = 20$ mm with $\dot{q} = -50$ m/s.

(a) Determine the maximum positive value of q that occurs in the ensuing vibration, and the earliest instant at which it occurs.

(b) Determine the maximum positive value of $\dot{q}$ that occurs in the ensuing vibration, and the earliest instant at which it occurs.

2.3.3.1 Part(a)



applying $F = mq''$, we obtain equation of motion

$$Mq'' = -kq \tag{2.13}$$
$$Mq'' + kq = 0 \tag{2.14}$$
$$q'' + \frac{k}{M}q = 0 \tag{2.15}$$
$$q'' + \omega_n^2 q = 0 \tag{2.16}$$

Let solution be $q(t) = \text{Re}\left(\hat{A}e^{i\omega_n t}\right)$ where $\hat{A}$ is the complex amplitude, which is a complex number that can be written as $\hat{A} = a + ib$. We use initial conditions to determine $\hat{A}$. At

$t = 0$, let $q(0) = q_0$

$$\begin{aligned} q_0 &= \mathrm{Re}\left(\hat{A}e^{i\omega_n t}\right) && (2.17)\\ &= \mathrm{Re}\left(\hat{A}\right) && (2.18)\\ &= a && (2.19) \end{aligned}$$

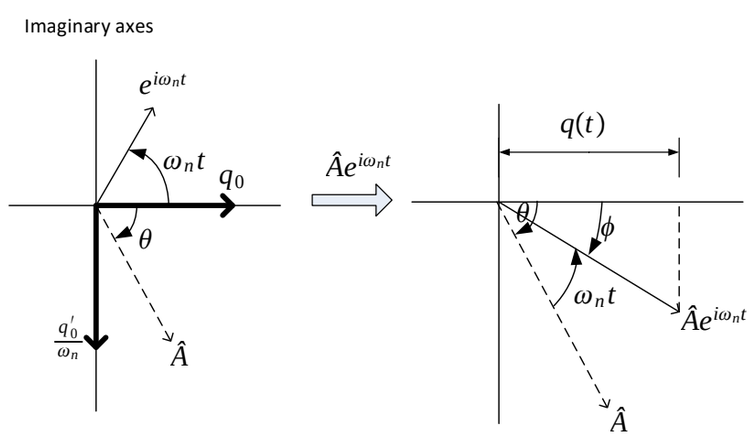
Hence $a = q_0$ And since $q'(t) = \mathrm{Re}\left(i\omega_n \hat{A}e^{i\omega_n t}\right)$, then $t = 0$ we have

$$\begin{aligned} q_0' &= \mathrm{Re}\left(i\omega_n \hat{A}\right) && (2.20)\\ &= \mathrm{Re}(i\omega_n(a + ib)) && (2.21)\\ &= \mathrm{Re}(i\omega_n a - \omega_n b) && (2.22)\\ &= -\omega_n b && (2.23) \end{aligned}$$

Hence $b = -\frac{q_0'}{\omega_n}$ therefore the general solution is

$$\begin{aligned} q(t) &= \mathrm{Re}\left(\hat{A}e^{i\omega_n t}\right) && (2.24)\\ &= \mathrm{Re}\left((a + ib)e^{i\omega_n t}\right) && (2.25)\\ &= \mathrm{Re}\left(\overbrace{\left(q_0 - i\frac{q_0'}{\omega_n}\right)}^{\hat{A}} e^{i\omega_n t}\right) && (2.26) \end{aligned}$$

Hence $\left|\hat{A}\right| = \sqrt{q_0^2 + \left(\frac{q_0'}{\omega_n}\right)^2}$ and $\arg\left(\hat{A}\right) = \theta = \tan^{-1}\left(\frac{\frac{q_0'}{\omega_n}}{q_0}\right)$. We have 2 complex quantities above being multiplied. The first is $\hat{A}$ and the second is $e^{i\omega_n t}$, therefore the result is obtained by adding the angles and by multiplied the magnitudes. The magnitude of $e^{i\omega_n t}$ is one. Hence on the complex plan, the above expression for $q(t)$ is represented as vector of length $\left|\hat{A}\right|$ and phase $\phi = \theta + \omega_n t$



From the above diagram we see that the maximum value of

$$q_{\max}(t) = \left|\hat{A}\right|$$

which occurs when

$$\phi = \theta + \omega_n t = 0$$

solving for t gives

$$t = \frac{-\theta}{\omega_n}$$

Notice that θ is negative, hence we will get positive value for t. Substituting the numerical values given we find thatAnd the earliest time this occurs is

$$t = \frac{1.3724}{2\pi(80)} = 2.7303 \times 10^{-3} = 2.73 \text{ ms}$$

We confirm this by noticing that the initial position vector was at about $\frac{1}{4}$*cycle* away from the positive x-axis (we found the phase of $\hat{A}$ above to be about −80 degrees), and the rotational speed is given as 80 cycles per second. Hence it takes 12.5 ms to make one cycle and $\frac{1}{4}$ of this is about $3ms$.

2.3.3.2 Part(b)

Since we found $q(t) = \text{Re}\left(\left(q_0 - i\frac{q_0'}{\omega_n}\right)e^{i\omega_n t}\right)$, then

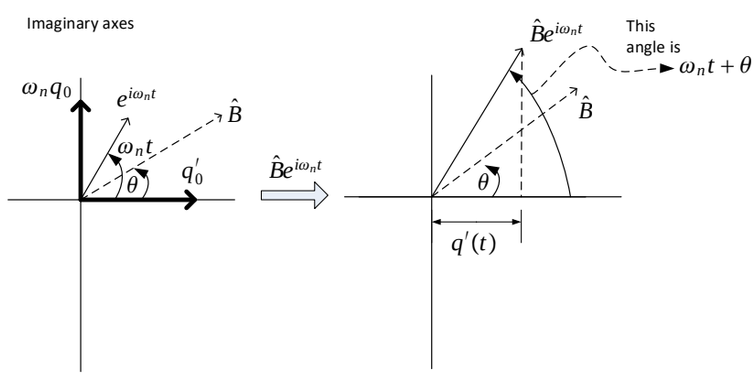
$$q'(t) = \text{Re}\left(i\omega_n\left(q_0 - i\frac{q_0'}{\omega_n}\right)e^{i\omega_n t}\right) \tag{2.27}$$

$$= \text{Re}\left(\left(\omega_n q_0 - e^{i\frac{\pi}{2}}q_0'\right)e^{i\frac{\pi}{2}}e^{i\omega_n t}\right) \tag{2.28}$$

$$= \text{Re}\left(\left(\omega_n q_0 e^{i\frac{\pi}{2}} - e^{i\pi}q_0'\right)e^{i\omega_n t}\right) \tag{2.29}$$

$$= \text{Re}\left(\overbrace{\left(\omega_n q_0 e^{i\frac{\pi}{2}} + q_0'\right)}^{\hat{B}} e^{i\omega_n t}\right) \tag{2.30}$$

Where now $\hat{B}$ is the complex amplitude of $q'(t)$. Hence $\left|\hat{B}\right| = \sqrt{\left(\omega_n q_0\right)^2 + \left(q_0'\right)^2}$ and its phase is $\arg\left(\hat{B}\right) = \tan^{-1}\frac{\omega_n q_0}{q_0'}$. The complex plane representation of $q'(t)$ is



From the above diagram we see that maximum magnitude of $q'(t)$ is $\left|\hat{B}\right|$ given by

$$\left|\hat{B}\right| = \sqrt{\left(\omega_n q_0\right)^2 + \left(q_0'\right)^2} \tag{2.31}$$

$$= \sqrt{\left(2\pi(80)\left(20\times10^{-3}\right)\right)^2 + (-50)^2} \tag{2.32}$$

$$= 51.001\text{m/s} \tag{2.33}$$

The earliest time it occurs is found by solving for t in

$$\omega_n t + \theta = 2\pi \tag{2.34}$$

$$t = \frac{2\pi - \theta}{\omega_n} \tag{2.35}$$

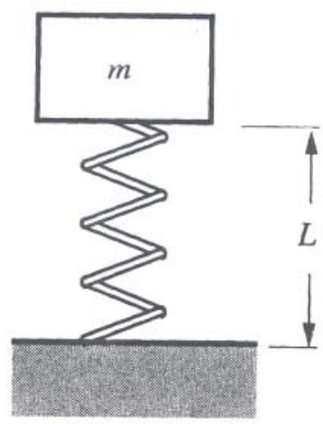
$$= \frac{2\pi - \tan^{-1}\frac{\omega_n q_0}{q_0'}}{2\pi(80)} = \frac{2\pi - \tan^{-1}\frac{2\pi(80)\left(20\times10^{-3}\right)}{-50}}{2\pi(80)} \tag{2.36}$$

$$= \frac{2\pi - \tan^{-1}(-0.20106)}{2\pi(80)} = 1.2895\times10^{-2} \tag{2.37}$$

$$= 0.129 \text{ ms} \tag{2.38}$$

2.3.4 Problem 3

2.19 A block of mass m is mounted on a spring having stiffness k. The block moves in the vertical direction. When the system is at rest, a 2 kg block is placed gently on the original block. It is observed that the static length of the spring after insertion of the additional block is 50 mm less than it was prior to the addition. It also is observed that the natural frequency with the additional mass is 5 Hz less than it was originally. Determine k and m.



EXERCISE 2.19

Adding 2 kg caused deflection of 50 mm, hence from $F = k\Delta$ we can find k as follows

$$k = \frac{F}{\Delta} = \frac{2g}{0.05} = \frac{2(9.81)}{0.05} = 392 \text{ N/m} \tag{2.39}$$

where g is the gravitational constant. We also told that $f_2 = f_1 - 5$ where f_2 is the natural frequency after adding the second mass and where $f_1 = \frac{1}{2\pi}\omega_1$ and $f_2 = \frac{1}{2\pi}\omega_2$, hence

$$f_2 = f_1 - 5 \tag{2.40}$$

$$\frac{1}{2\pi}\omega_2 = \frac{1}{2\pi}\omega_1 - 5 \tag{2.41}$$

$$\omega_2 = \omega_1 - 10\pi \tag{2.42}$$

But $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{k}{m+2}}$, hence

$$\sqrt{\frac{k}{m+2}} = \sqrt{\frac{k}{m}} - 10\pi$$

From Eq 2.117 the above becomes

$$\sqrt{\frac{392}{m+2}} = \sqrt{\frac{392}{m}} - 10\pi$$

Solving numerically gives $m = 0.1955$kg

2.3.5 Problem 4

Problem 4: Show that $x(t)=Be^{\lambda t}$ is a solution to $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ and find λ for the following cases: 1.) Underdamped system, 2.) Overdamped system. Write the solution $x(t)$ for both cases for an arbitrary set of initial conditions and draw a sketch to illustrate how each response $x(t)$ would look. Show that $x(t)$ can be written as $x(t) = \text{Re}\left(Ae^{-\zeta\omega_n t}e^{i\omega_d t}\right)$ in case (1).

To show that $x(t) = Be^{\lambda t}$ is solution to the differential equation, we substitute this solution into the LHS of the differential equation and see if we obtain zero.

$$x'(t) = \lambda Be^{\lambda t} = \lambda x(t) \tag{2.43}$$
$$x''(t) = \lambda^2 Be^{\lambda t} = \lambda^2 x(t) \tag{2.44}$$

Then

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0 \tag{2.45}$$
$$\lambda^2 x(t) + 2\zeta\omega_n \lambda x(t) + \omega_n^2 x(t) = 0 \tag{2.46}$$
$$\left(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\right)x(t) = 0 \tag{2.47}$$

Hence $x(t) = Be^{\lambda t}$ is a non-trivial solution to the differential equation provided $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ since then we obtain $0 = 0$.

Now we find λ for the different cases.

2.3.5.1 case 1

The roots of $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ are

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

For underdamped $\zeta < 1$, hence $\sqrt{\zeta^2 - 1} < 0$ and we write the above as

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \tag{2.48}$$
$$= -\zeta\omega_n \pm i\omega_d \tag{2.49}$$

where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Let $\lambda = -\zeta\omega_n + i\omega_d$ and its complex conjugate $\lambda^* = -\zeta\omega_n - i\omega_d$, hence the solution is

$$x(t) = B_1 e^{\lambda t} + B_2 e^{\lambda^* t}$$

To obtain a real solution we must have B_1 be complex say $\hat{B}$ and $B_2 = \hat{B}^*$. Hence the above can be written as

$$x(t) = \hat{B}e^{\lambda t} + \hat{B}^* e^{\lambda^* t} \tag{2.50}$$
$$= 2\,\mathrm{Re}\left(\hat{B}e^{\lambda t}\right) \tag{2.51}$$
$$= \mathrm{Re}\left(2\hat{B}e^{(-\zeta\omega_n + i\omega_d)t}\right) \tag{2.52}$$

Therefore

$$x(t) = \mathrm{Re}\left(\hat{A}e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{2.53}$$

Where $\hat{A} = 2\hat{B} = a + ib$. Hence

$$x(t) = \mathrm{Re}\left((a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right)$$

To find a, b we need to use initial conditions. Assuming $x(0) = x_0$ and $x'(0) = x_0'$ then from Eq 2.53 we obtain

$$x_0 = \mathrm{Re}(a + ib) = a$$

Hence

$$a = x_0$$

and taking derivative of 2.53

$$x(t) = \mathrm{Re}\left((a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{2.54}$$
$$x'(t) = \mathrm{Re}\left(-\zeta\omega_n(a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t} + i\omega_d(a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{2.55}$$
$$x'(0) = \mathrm{Re}(-\zeta\omega_n(a + ib) + i\omega_d(a + ib)) \tag{2.56}$$
$$= -\zeta\omega_n a - \omega_d b \tag{2.57}$$

Hence

$$b = \frac{x_0' + \zeta\omega_n a}{\omega_d}$$

But $a = x_0$, hence

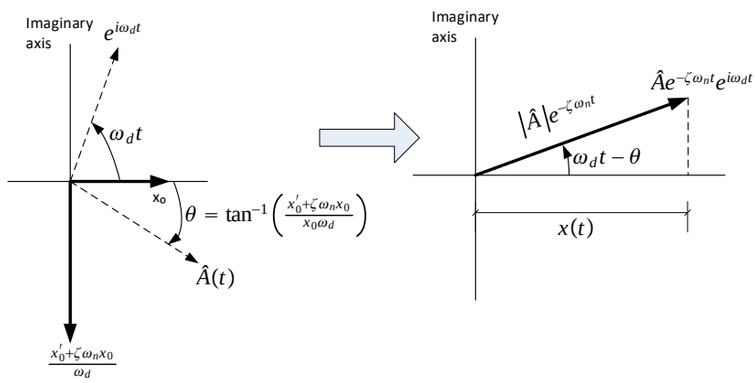
$$b = \frac{x_0' + \zeta\omega_n x_0}{\omega_d}$$

Hence 2.53 becomes

$$x(t) = \mathrm{Re}\Big((a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\Big) \tag{2.58}$$

$$= \mathrm{Re}\left(\left(x_0 + i\frac{x_0' + \zeta\omega_n x_0}{\omega_d}\right)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{2.59}$$

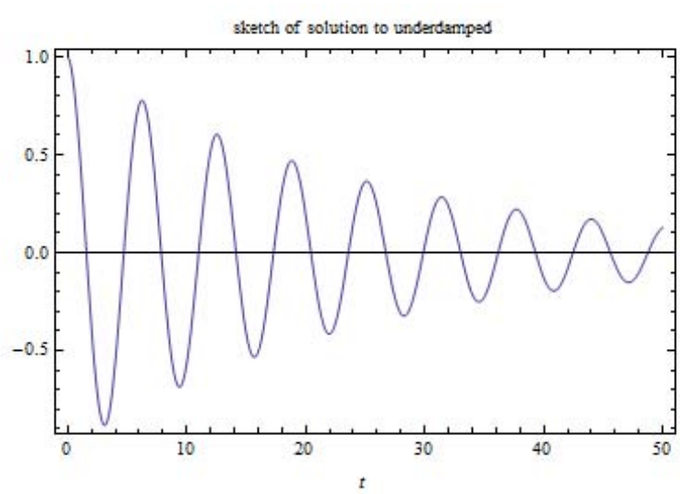
And this is the general solution. In complex plan it is



$$|\hat{A}| = \sqrt{x_0^2 + \left(\frac{x_0'+\zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$|\hat{A}(t)| = |\hat{A}|e^{-\zeta\omega_n t}$$

Hence the rotating vector will have its length become smaller with time since $|\hat{A}|$ is multiplied by $e^{-\zeta\omega_n t}$. The real part, which is the solution will eventually damp down to zero. Hence it is a damped sinusoid oscillation as follows



2.3.5.2 case 2

From

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

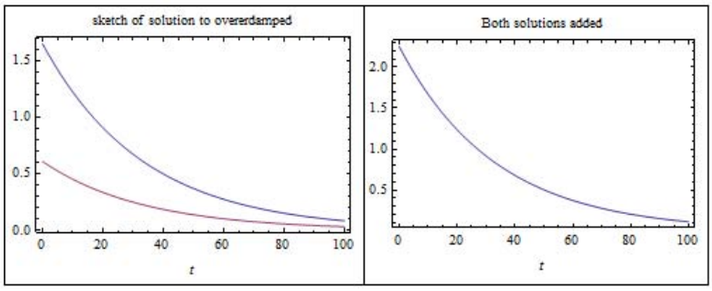
For overdamped $\zeta > 1$, hence $\sqrt{\zeta^2 - 1} > 0$ and we write the above as

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}$$

Hence the solution is

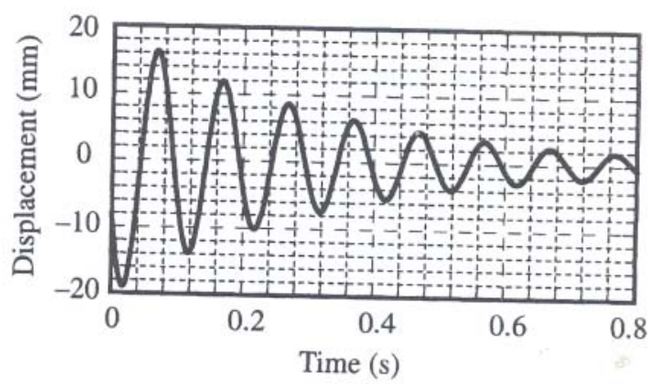
$$x(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t}$$

where $\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}$ and $\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{1-\zeta^2}$. We see that both roots are negative always, hence we have 2 exponentially damped solution being added with no oscillation. A sketch of the solution is



2.3.6 Problem 5

2.29 The measured free vibration response of a one-degree-of-freedom system is as shown in the graph.



EXERCISE 2.29

(a) Deduce from this measurement the log decrement, the natural frequency, and the critical damping ratio of the system.

(b) Estimate the value of t beyond which the displacement magnitude $|q|$ will not exceed 0.01 mm.

(c) If the damping constant C is held fixed, while the system is modified by doubling the stiffness K and halving the generalized mass M, how would that alter the answer to part (b)?

(d) The initial displacement, at $t = 0$, is $q_0 = -10$ mm. What is the initial velocity?

2.3.6.1 Part(a)

From looking at the plot above, here are the values estimated for displacement positive peaks and time they occur

t	$y(t)$
0.07	16
0.17	12
0.27	9
0.37	6

From the above we estimate the natural period $T \approx 0.1$ sec hence $f = 10$ hz hence $\omega_n = 2\pi f = 60.3$ rad/sec The log decrement is

$$\delta = \ln \frac{y_i}{y_{i+N}}$$

Select $i = 1$ and $N = 3$ gives

$$\delta = \ln \frac{16}{6} \tag{2.60}$$
$$= 0.981 \tag{2.61}$$

To find ζ we use the log decrement method

$$\delta = 2\pi N\zeta$$

Hence

$$\zeta = \frac{\delta}{2\pi N} = \frac{0.98083}{2\pi(3)} \tag{2.62}$$
$$\zeta = 0.052 \tag{2.63}$$

Hence

$$\zeta = 5.2\%$$

2.3.6.2 Part(b)

$$\ln\left(\frac{y_1}{y_{1+N}}\right) = 2\pi N\zeta$$

Where now we write $y_1 = 16$ and $y_{N+1} = 0.01$, and hence we need to find N the only unknown in the equation above

$$\ln\left(\frac{16}{0.01}\right) = 2\pi N(0.052)$$

Hence

$$N = \frac{\ln\left(\frac{16}{0.01}\right)}{2\pi(0.052)} = 22.581$$

We take $N = 23$. What this says is that after 23 periods beyond the first peak, we will satisfy the requirement. But $T = 0.1$ sec, and the first peak was at $t = 0.05$ sec, therefore

$$t = 0.07 + 23(0.1) \tag{2.64}$$
$$= 2.37\text{sec} \tag{2.65}$$

2.3.6.3 Part c

Since $\delta_N = 2\pi N\zeta$ and $\zeta = \frac{c}{c_r} = \frac{c}{2\sqrt{km}}$, then if we double k and half the mass m, then ζ would remain the same since c is held constant. Therefore the answer in part b would not change.

2.3.6.4 Part d

Since this is an underdamped system, the solution is

$$q(t) = \mathrm{Re}\Big(\hat{A}e^{(-\omega_n\zeta+i\omega_d)t}\Big)$$

Where $\hat{A}$ is the complex amplitude, say $(a + ib)$, hence

$$q(t) = \mathrm{Re}\Big((a + ib)e^{(-\omega_n\zeta+i\omega_d)t}\Big)$$

At $t = 0$ we find that

$$a = q(0) = q_0$$

Hence

$$a = -0.01$$

and the general solution is

$$q(t) = \mathrm{Re}\Big(\Big(q_0 + ib\Big)e^{(-\omega_n\zeta+i\omega_d)t}\Big)$$

Now taking derivatives of the above gives

$$q'(t) = \mathrm{Re}\Big((-\omega_n\zeta + i\omega_d)\Big(q_0 + ib\Big)e^{(-\omega_n\zeta+i\omega_d)t}\Big)$$

At $t = 0$ then, assuming q_0' is the initial velocity

$$q_0' = \mathrm{Re}\Big((-\omega_n\zeta + i\omega_d)\Big(q_0 + ib\Big)\Big) \tag{2.66}$$

$$= -\omega_n\zeta q_0 - \omega_d b \tag{2.67}$$

Hence

$$b = -\frac{q_0' + \omega_n\zeta q_0}{\omega_d}$$

Therefore the general solution is

$$q(t) = \mathrm{Re}\left(\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

and

$$q'(t) = \mathrm{Re}\left((-\omega_n\zeta + i\omega_d)\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

Now at $t_0 = 0.07$ sec the velocity is zero, since this is where the displacement is maximum (first peak). Hence now we have one equation with one unknown q_0'that we can solve for from the above

$$0 = \mathrm{Re}\left((-\omega_n\zeta + i\omega_d)\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta+i\omega_d)t_0}\right) \tag{2.68}$$

$$= e^{-\omega_n\zeta t_0}\,\mathrm{Re}\left(\left(-\omega_n\zeta q_0 + i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + iq_0\omega_d + \frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_d\right)e^{i\omega_d t_0}\right) \tag{2.69}$$

$$= e^{-\omega_n\zeta t_0}\,\mathrm{Re}\left(\left(i\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right) + q_0'\right)e^{i\omega_d t_0}\right) \tag{2.70}$$

$$= e^{-\omega_n\zeta t_0}\,\mathrm{Re}\left(i\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i\omega_d t_0} + q_0'e^{i\omega_d t_0}\right) \tag{2.71}$$

$$= e^{-\omega_n\zeta t_0}\,\mathrm{Re}\left(\frac{-1}{i}\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i(\omega_d t_0)} + q_0'e^{i\omega_d t_0}\right) \tag{2.72}$$

$$= e^{-\omega_n\zeta t_0}\left[\mathrm{Re}\left(\frac{-1}{i}\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i(\omega_d t_0)}\right) + \mathrm{Re}\Big(q_0'e^{i\omega_d t_0}\Big)\right] \tag{2.73}$$

$$= e^{-\omega_n\zeta t_0}\left[-\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)\sin(\omega_d t_0) + q_0'\cos(\omega_d t_0)\right] \tag{2.74}$$

But $q_0 = -0.01$ m/sec, $\zeta = 0.052$, $\omega_n = 60.3$ rad/sec and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 60.3\sqrt{1-0.052^2} = 60.218$, therefore $\omega_d t_0 = 60.218\times 0.07 = 4.2153$ and $q_0\omega_d = (60.218)(-0.01) = -0.60218$ hence the above equation becomes

$$0 = e^{-(60.3)(0.052)(0.07)}\left(-\left(\frac{q_0' + (60.3)(0.052)(-0.01)}{60.218}60.3\times 0.052 - 0.602\right)\sin(4.215) + q_0'\cos(4.215)\right) \tag{2.75}$$

$$= 0.80293\left(-\left(\frac{q_0' - 3.136\times 10^{-2}}{60.218}(3.1356) - 0.602\right)(-0.879) + q_0'(-0.477)\right) \tag{2.76}$$

Solving for q_0' gives

$$q_0' = -1.231 \text{ m/sec}$$

Now that we q_0',

Now that we q_0', we can find the numerical value for b and write the general solution again.

$$b = -\frac{q_0' + \omega_n\zeta q_0}{\omega_d} \tag{2.77}$$

$$= -\frac{-1.231 + 60.3(0.052)(-0.01)}{60.218} \tag{2.78}$$

$$= 2.0963\times 10^{-2} \tag{2.79}$$

Hence from

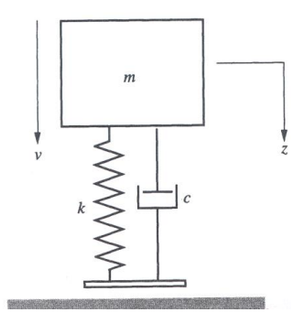
$$q(t) = \text{Re}\left(\left(q_0 + ib\right)e^{(-\omega_n\zeta + i\omega_d)t}\right) \tag{2.80}$$

$$= \text{Re}\left((-0.01 + i0.0209)e^{(-\omega_n\zeta + i\omega_d)t}\right) \tag{2.81}$$

giving $|\hat{A}| = \sqrt{0.01^2 + 0.0209^2} = 0.023$

2.3.7 Problem 6

2.32 The cushioning for a package of mass m may be represented as a spring k and dashpot c. After falling some distance, the package hits the ground with a known initial velocity v. The system is underdamped.



(a) Determine the downward displacement based on $z = 0$ being the center of mass posi at the instant when the package first contacts ground. *Hint:* Gravity cannot be ignored beca $z = 0$ is not the static equilibrium position.

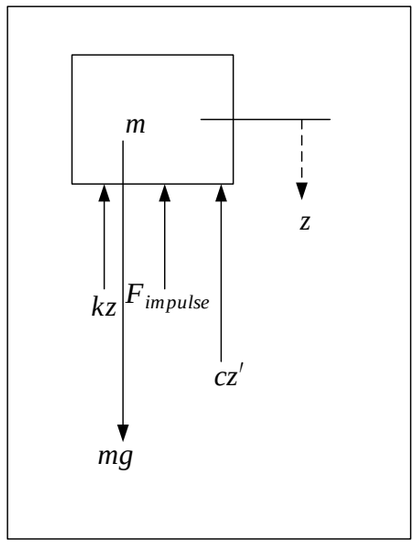
(b) Use the solution in part (a) to derive an expr sion for the force exerted by the cushioning on package mass m. How can this expression be us to determine the instant t' at which the packa will rebound from the ground?

(c) Consider the case where $m = 1$ kg, ω_{nat} Hz, and $v = 4$ m/s. Use mathematical software evaluate the maximum cushioning force in part for all t at a fixed critical damping ratio. Consid $0 < \zeta < 1$ for this evaluation. Which case lead to the most protection for the package?

2.3.7.1 Part a

Assume the system is underdamped.

When the package hits the ground, its speed becomes zero. Therefore the impulse generated on it is the change of linear momentum. Since it speed was v just before impact, then impulse= mv.



Hence the EQM is

$$mz'' = -cz' - kz - F_{impulse} + mg \tag{2.82}$$

$$mz'' + cz' + kz = mg - F_{impulse} \tag{2.83}$$

With the initial conditions now being $z = 0$ and $z' = 0$.

The response due to the force mg can be found from the response to a unit step of amplitude mg Hence the response due to the force mg is

$$u(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]\right)$$

The response due to the impulse is the response of a free system with zero initial position but with initial velocity $\frac{impulse}{m}$ in the upward (negative) direction. Hence the response due to the impulse only is

$$g(t) = \frac{mv}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{2.84}$$

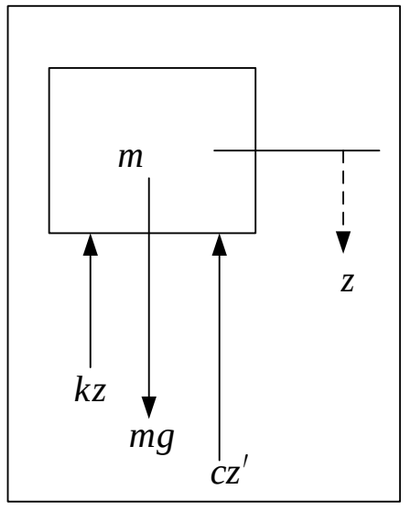
$$= \frac{v}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{2.85}$$

Hence the downward displacement is given by

$$z(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t \tag{2.86}$$

2.3.7.2 Part b

Now that the impulse have taken place and we have accounted for it in the $z(t)$ solution, then we can use this expression to find the spring force since $F_{spring} = kz(t)$ and the damping force on the mass $F_{damper} = cz'(t)$. When resultant net force F is negative then the mass will rebound from the ground.



$$F = mg - kz(t) - cz'(t)$$

But

$$z(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t \tag{2.87}$$

Hence

$$z'(t) = \frac{mg}{k}\left(\zeta\omega_n e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right] - e^{-\zeta\omega_n t}[-\omega_d\sin\omega_d t + \zeta\omega_n\cos\omega_d t]\right) - \frac{\zeta\omega_n ve^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t - ve^{-} \tag{2.88}$$

or

$$z'(t) = \frac{e^{-\zeta\omega_n t}}{k\omega_d}\left(\left(kv\omega_n\zeta + gm\left(\omega_d^2 + \omega_n^2\zeta^2\right)\right)\sin\omega_d t - kv\omega_d\cos(\omega_d t)\right) \tag{2.89}$$

Hence

$$F = mg - kz(t) - cz'(t) \tag{2.90}$$

$$= mg - k\left[\frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d}\sin\omega_d t\right] - c\frac{e^{-\zeta\omega_n t}}{k\omega_d}\left(\left(kv\omega_n\zeta + gm\left(\omega_d^2 + \omega_n^2\zeta^2\right)\right)\sin\right. \tag{2.91}$$

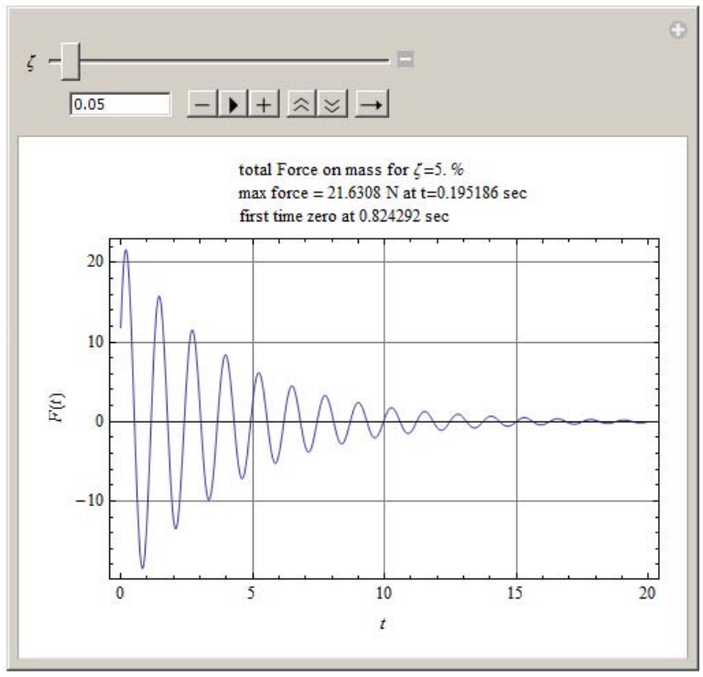
To find when this force will turn negative first time, we can take the derivative with respect to time and set it to zero and solve for first $t = t'$ that will make it zero. Since the force was positive first, then it has to become zero before turning negative.

2.3.7.3 Part(c)

Let $m = 1$ kg, $\omega_n = 5$ rad/sec[3], $v = 4$ m/s. Hence $\omega_d = 5\sqrt{1-\zeta^2}$. Since $\omega_n^2 = \frac{k}{m}$, hence $k = 25$ N/m. Also $c = \zeta c_{cr} = \zeta 2m\omega_n = 10\zeta$

Using these values, the force in part(b) is plotted for different values of ζ. For example, setting $\zeta = 5\%$ gives this plot of $F(t)$ for $t = 0$ to 20 seconds.

[3]typo in book. `hz` is assumed to mean `rad/sec`



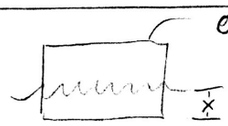
The maximum force is seen as little over 20 N.Therefore, to find which ζ gives the smallest value of maximum force, we can try different values of ζ and see how the maximum force changes as a function of ζ. Using software the following values of maximum for for different ζ are generated along with $t = t_{\max}$ when this maximum occurs and with the time $t = t'$ when the mass rebounds first time from $z = 0$

maximum force (N)	ζ%	$t_{\max}$(sec)	t'(sec)
22.1	1	0.21	0.84
21.85	3	0.206	0.834
21.45	7	0.184	0.81
21.27	10	0.16	0.779
21.4	20	0.11	0.75
25.8	40	0.01	0.68
30	50	0.001	0.64

Most protection when damping ratio is below 10%

2.3.8 Key solution for HW 2

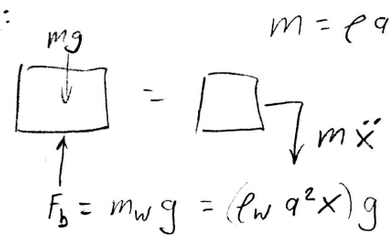
HW#2 → Floating Cube



Find EOM, nat. freq.
(submerged a distance x)

FBD:

$m = \rho a^3$



$F_b = m_w g = (\rho_w a^2 x) g$

$F_b - mg = -m\ddot{x}$

$g(\rho_w a^2 x) - mg = -m\ddot{x}$

→ static equilibrium → $\ddot{x} = 0$

$x = \frac{mg}{\rho_w a^2 g} = \frac{\rho a^3 g}{\rho_w a^2 g} = \frac{\rho}{\rho_w} a = x_s$

define new x from static eq.

$\hat{x} = x - x_s \rightarrow x = \hat{x} + x_s \qquad \ddot{\hat{x}} = \ddot{x}$

$\rho_w a^2 g(\hat{x} + x_s) - mg = -m\ddot{\hat{x}}$

$m\ddot{\hat{x}} + \rho_w a^2 g \hat{x} + \underbrace{\frac{\rho_w a^2 g \cdot \rho a}{\rho a}} - mg = 0$

$mg - mg = 0$

$$\boxed{m\ddot{\hat{x}} + \rho_w a^2 g \hat{x} = 0}$$

$\omega_n^2 = \frac{\rho_w a^2 g}{\rho a^3}$

$$\boxed{\omega_n = \sqrt{\frac{\rho_w g}{\rho a}}}$$

Units? $\sqrt{\frac{m/s^2}{m}} = 1/s = rad/s$ ✓

For a pine cube $a = 10cm$, $\rho = 400 kg/m^3$ $\rho_w = 1000 kg/m^3$

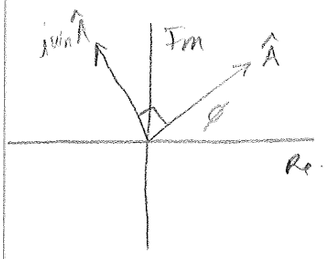
$$\boxed{\omega_n = 15.7 rad/s \rightarrow 2.5 Hz}$$

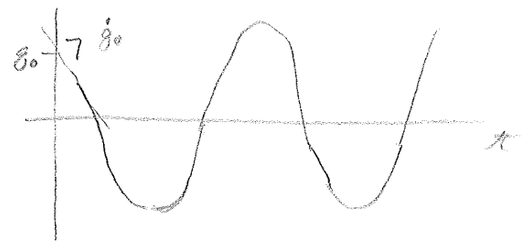
Ex. 2.17 – Solution 2/5/2013

$g(0) = 0.02\,m \qquad \dot{g}(0) = -50\,m/s$

$\omega_{nat} = 80 \cdot 2\pi\ rad/s$

From IC's, $g_0 > 0$, $\dot{g}_0 < 0$





$g(t) = Re(\hat{A} e^{i\omega_n t})$

$g(0) = Re(\hat{A}) = 0.02\,m \quad \rightarrow \hat{A} = a + ib$

$a = 0.02\,m$

$\dot{g}(0) = Re(i\omega_n \hat{A} e^{i\omega_n \cdot 0}) = Re(i\omega_n(a+ib))$

$-b\omega_n = -50\,m/s \longrightarrow b = (50\,m/s / \omega_n)$

$b = \frac{50\,m/s}{160\pi} = \frac{5}{16\pi}$

$\hat{A} = 0.02\,m + i\frac{5}{16\pi} \longrightarrow \phi = \tan^{-1}\left(\frac{5/(16\pi)}{0.02}\right) = 1.37$

from graph, max(g) when $\omega_n t = \pi - \phi$ (neg. extreme)

$\omega_n t = 2\pi - \phi$ (pos. maximum)

$t = \frac{n\pi}{\omega_n} - \frac{\phi}{\omega_n} = \frac{n\pi}{160\pi} - \frac{\phi}{160\pi} \longrightarrow t = 9.77\,ms$

from graph, max($\dot{g}$) when $(i\omega_n \hat{A} e^{i\omega_n t}) \rightarrow$ Real axis

$\longrightarrow \omega_n t = \pi - \pi/2 - \phi$ Negative

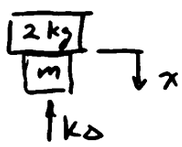
or

$2\pi - \pi/2 - \phi$ positive max

$\boxed{t = 6.64\,ms}$

✓ Plot works

Exercise 2.19



Given static $x = 0.05$ meter due to insertion of 2 kg block, ω_{nat} with 2 kg is 10π rad/s less than ω_{nat} without 2 kg. Find k & m.

Solution: Static displacement $= F/k \Rightarrow \frac{2(9.807)}{k} = 0.05$

Before 2 kg $(\omega_{nat})_0 = \left(\frac{k}{m}\right)^{1/2}$

After 2 kg $\omega_{nat} = \left(\frac{k}{m+2}\right)^{1/2} = (\omega_{nat})_0 - 10\pi$

or $\left(\frac{k}{m+2}\right)^{1/2} = \left(\frac{k}{m}\right)^{1/2} - 10\pi$

From static displacement $k = \frac{2(9.807)}{0.05} = 392.3$ N/m $\Leftarrow$

Solve $\left(\frac{392.3}{m+2}\right)^{1/2} = \left(\frac{392.3}{m}\right)^{1/2} - 10\pi \Rightarrow m = 0.1956$ kg $\Leftarrow$

HW#5 Extra Credit

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad x(t) = Ae^{\lambda t}$$

Plug into ODE

$$(\lambda^2 A + 2\zeta\omega_n\lambda A + \omega_n^2 A)e^{\lambda t} = 0$$

want sol. valid for all time, and not $A = 0$ (trivial) so

$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0$ — Two possible roots

Quadratic formula:

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(1)(\omega_n^2)}}{2(1)} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

Underdamped $\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$

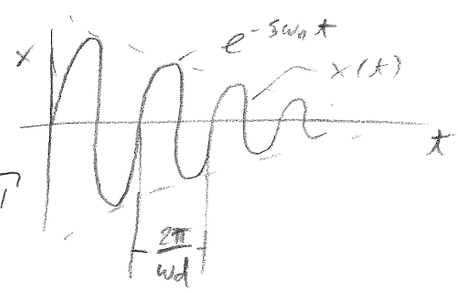
$\rightarrow \boxed{\lambda = -\zeta\omega_n \pm i\omega_d \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}}$ $\rightarrow$ if underdamped i.e. $\zeta < 1$

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

λ_1 and λ_2 are CC's, A_1 and A_2 must be conjugates as well to obtain a real $x(t)$

$$x(t) = A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_1^* e^{(-\zeta\omega_n - i\omega_d)t}$$

$$\boxed{x(t) = Re(2A_1 e^{-\zeta\omega_n t} e^{i\omega_d t})}$$



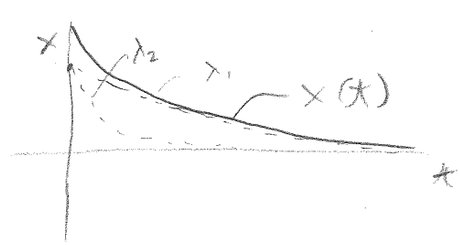
Overdamped

$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$, $\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

Both λ_1 and λ_2 will be negative real #'s for $\zeta > 1$ since $\sqrt{\zeta^2-1} < \zeta$ for $\zeta > 1$.

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

more negative, faster decay to zero.



Exercise 2.29

Given q vs t

8 peak values $\Rightarrow$ 7 cycles

$$\delta = \frac{1}{7}\ln\left(\frac{x_1}{x_8}\right) \approx \frac{1}{7}\ln\left(\frac{16\,mm}{2\,mm}\right) = 0.297 \quad \Leftarrow$$

$$\zeta = \frac{\delta}{(4\pi^2+\delta^2)^{1/2}} = 0.04723 \quad \Leftarrow$$

16 zeros $\Rightarrow$ 15 half periods $\Rightarrow 15\left(\frac{\pi}{\omega_d}\right) = (0.795 - 0.045)$ estimates

so $\omega_d = \frac{15\pi}{0.75} = 20\pi = \omega_{nat}(1-\zeta^2)^{1/2}$

$$\omega_{nat} = 62.90 \text{ rad/s} \quad \Leftarrow$$

Want max or min $|q| < 0.01$

$$\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{N+1}}\right) \Rightarrow 0.297 = \frac{1}{N}\ln\left(\frac{16}{0.01}\right)$$

$N = 24.8$ round up to nearest 0.5 $\Rightarrow N = 25$

Thus 25 damped periods beyond first peak

$$t > 0.007 + 25\left(\frac{2\pi}{\omega_d}\right) = 2.507 \text{ sec} \quad \Leftarrow$$

The number of cycles N for specified x_1 & x_{N+1} depends on $\delta = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}$, $\zeta = \frac{c}{2(KM)^{1/2}}$

If $c' = c$, $K' = 2K$ & $M' = \frac{1}{2}M \Rightarrow \zeta' = \frac{c'}{2(K'M')^{1/2}} = \zeta$

Thus $N' = N$.

However $\omega'_{nat} = \left(\frac{K'}{M'}\right)^{1/2} = 2\left(\frac{K}{M}\right)^{1/2} = 2\omega_{nat}$

Because $\zeta' = \zeta \Rightarrow \omega'_d = 2\omega_d$, so

$$t' > 0.007 + 25\left(\frac{2\pi}{\omega'_d}\right) = 1.257 \text{ sec} \quad \Leftarrow$$

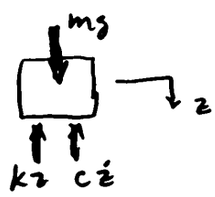
i.e. half the time from the first peak

To find $\dot{q}_0$ use eq. (2.2.24). Set $q = 0$ @ $t = 0.795$:

$$q_0\cos[\omega_d(0.795)] + \frac{\dot{q}_0 + \zeta\omega_{nat}q_0}{\omega_d}\sin[\omega_d(0.795)] = 0$$

$$\dot{q}_0 = -\zeta\omega_{nat}q_0 - \omega_d q_0\cot[\omega_d(0.795)] = -1.904 \text{ m/s} \quad \Leftarrow$$

Exercise 2.32



$$\Sigma F_z = mg - kz - c\dot{z} = m\ddot{z}$$

$$\ddot{z} + 2\zeta\omega_{nat}\dot{z} + \omega_{nat}^2 z = g$$

$$z = 0 \;\&\; \dot{z} = v \;@\; t = 0$$

$$z = z_c + z_p$$

$$z = \exp(-\zeta\omega_{nat}t)\left[c_1\cos(\omega_d t) + c_2\sin(\omega_d t)\right] + \underbrace{\frac{g}{\omega_{nat}^2}}_{=\frac{mg}{k}}$$

Satisfy initial conditions

$$c_1 + \frac{mg}{k} = 0 \quad \& \quad -\zeta\omega_{nat}c_1 + \omega_d c_2 = v$$

$$c_1 = -\frac{mg}{k} \quad \& \quad c_2 = \frac{\zeta\omega_{nat}}{\omega_d}\left(-\frac{mg}{k}\right) + \frac{v}{\omega_d}$$

$$= \frac{v}{\omega_d} - \frac{\zeta}{(1-\zeta^2)^{1/2}}\frac{mg}{k}$$

$$z = \exp(-\zeta\omega_{nat}t)\left\{-\frac{mg}{k}\cos(\omega_d t) + \left[\frac{v}{\omega_d} - \frac{\zeta}{(1-\zeta^2)^{1/2}}\frac{mg}{k}\right]\sin(\omega_d t)\right\} + \frac{mg}{k} \quad \Leftarrow$$

$$F_{susp} = kz + c\dot{z} = m(\omega_{nat}^2 z + 2\zeta\omega_{nat}\dot{z}) \quad \Leftarrow$$

where $\dot{z} = \exp(-\zeta\omega_{nat}t)\left[(\omega_d c_2 - \zeta\omega_{nat}c_1)\cos(\omega_d t) - (\omega_d c_1 + \zeta\omega_{nat}c_2)\sin(\omega_d t)\right]$

Rebound if $F_{susp} < 0$ $\Leftarrow$

If $m = 1$ kg, $\omega_{nat} = 10\pi$ rad/s $\Rightarrow k = m\omega_{nat}^2 = 100\pi^2$ N/m

For $v = 4$ m/s, $c_1 = -9.937(10^{-3})$

$$c_2(\zeta) = \frac{1}{(1-\zeta^2)^{1/2}}\left[\frac{4}{10\pi} - \zeta(9.937)(10^{-3})\right]$$

Pick a ζ in the range $0 < \zeta < 1$, evaluate $F_{susp}(t, \zeta)$ for $0 < t < T_{nat}$ (maximum force will occur in first cycle, where $z(t)$ is largest). Identify max F_{susp} for each ζ value in the range of t.

$v := 4 \qquad \omega_{nat} := 10\cdot\pi \qquad m := 1 \qquad g := 9.807 \qquad \omega_d(\zeta) := \sqrt{1-\zeta^2}\cdot\omega_{nat}$

$$c_1 := \frac{-g}{\omega_{nat}^2} \qquad c_2(\zeta) := \frac{v}{\omega_d(\zeta)} - \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\frac{g}{\omega_{nat}^2}$$

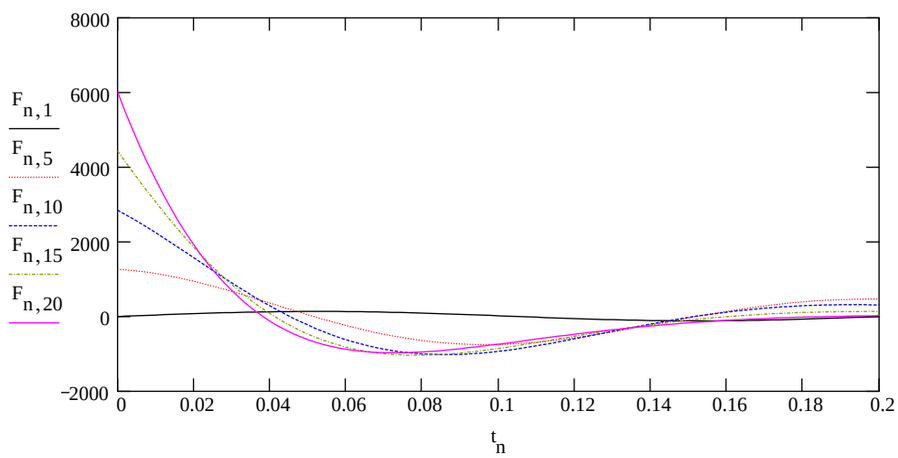
$$z(t,\zeta) := \exp\left(-\zeta\cdot\omega_{nat}\cdot t\right)\cdot\left(c_1\cdot\cos\left(\omega_d(\zeta)\cdot t\right) + c_2(\zeta)\cdot\sin\left(\omega_d(\zeta)\cdot t\right)\right) + \frac{g}{\omega_{nat}^2}$$

$$z'(t,\zeta) := \exp\left(-\zeta\cdot\omega_{nat}\cdot t\right)\cdot\left[\begin{array}{l}\left(\left(\omega_d(\zeta)\cdot c_2(\zeta) - \zeta\cdot\omega_{nat}\cdot c_1\right)\right)\cdot\cos\left(\omega_d(\zeta)\cdot t\right) \ldots \\ +\left(-\omega_d(\zeta)\cdot c_1 - \zeta\cdot\omega_{nat}\cdot c_2(\zeta)\right)\cdot\sin\left(\omega_d(\zeta)\cdot t\right)\end{array}\right]$$

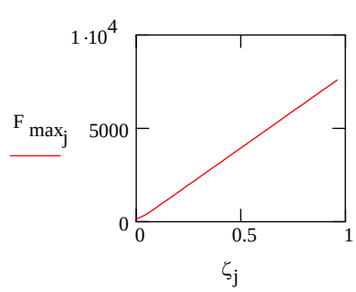
$$F_{susp}(t,\zeta) := m\cdot\left(\omega_{nat}^2\cdot z(t,\zeta) + 2\cdot\zeta\cdot\omega_{nat}^2\cdot z'(t,\zeta)\right)$$

$$j := 1..25 \qquad \zeta_j := \frac{j-1}{25} \qquad n := 1..101 \qquad t_n := \frac{n-1}{100}\cdot\frac{2\cdot\pi}{\omega_{nat}}$$

$$F_{n,j} := F_{susp}\left(t_n,\zeta_j\right)$$

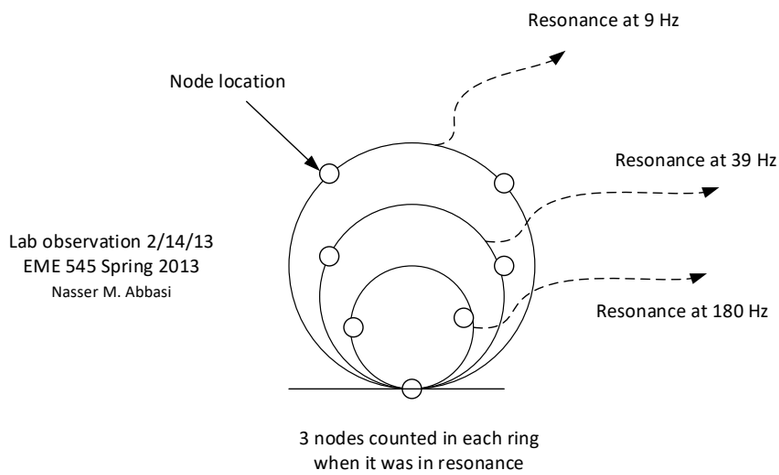


$$F_{mag_{n,j}} := \left|F_{n,j}\right| \qquad F_{max_j} := \max\left(F_{mag}^{\langle j\rangle}\right)$$



$F_{max_1} = 135.839$

2.4 HW3


Resonance at 9 Hz
Node location
Resonance at 39 Hz
Lab observation 2/14/13
EME 545 Spring 2013
Nasser M. Abbasi
Resonance at 180 Hz
3 nodes counted in each ring
when it was in resonance

2.5 HW4

Local contents

2.5.1 problem description

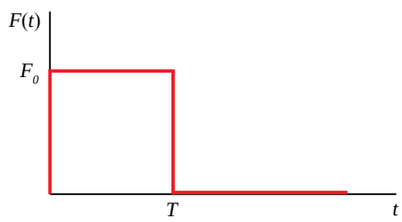
Homework #4
EMA 545, Spring 2013

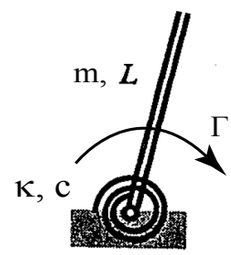
Problem 1: Exercise 2.54 from Ginsberg.

Problem 2: (**30 pts**, each part below is worth 10 pts).
The read head on a Hard Disk Drive (HDD) can be modeled as a pinned bar with a torsional spring at its base as shown below with L = 2 cm, m = 3 grams and κ = 20 N/rad. The damping ratio for the system is ζ = 0.02. The equation of motion for this system is: (later we will discuss how to find the EOM for a system like this)

$$\frac{1}{3}mL^2\ddot{\theta} + c\dot{\theta} + \kappa\theta = \Gamma(t)$$

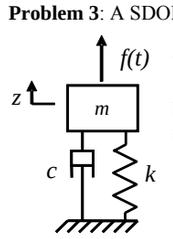
A certain read operation involves applying a step torque Γ(t) = F(t) with amplitude F_0 and duration T as shown below, where F_0 is the static torque required to displace the bar 30 degrees.





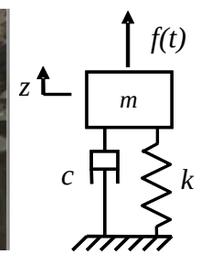
(a) Find the response of the system numerically over the time interval 0 < 1 < 10*T, with T = 2.5*T_d, where T_d is the damped period of the system. Use a numerical procedure, preferably Matlab's "ode45" function together with a suitably modified version of eom_2_12.m, which is available on the class website.
(b) Assuming an underdamped response, write down a closed-form solution for the response in terms of Heaviside-step functions, and unit step responses, $q_s(t)$. Compare this with the response that you found numerically.
(c) Plot the displacement as a function of time for the case where T = 3*T_d and T = 2.5*T_d. What do you observe? Why is the residual vibration larger in the latter case? (Hint –an undamped version of your analysis in (b) may make this easier to see.)

Problem 3: A SDOF system modeling a car bouncing on its suspension has *m*=1000kg, *k*=11 kN/m and *c*=660 N-s/m. The car is released from rest at t=0 with z(0)= –0.10m. It is possible to bring the car exactly to rest by exerting an impulsive force $f(t)=F_0\delta(t-T)$ at some instant $t=T$. (e.g. hitting it with a very large hammer at just the right instant). Find the magnitude of the impulse and the instant, *T*, at which it should be applied such the bouncing of the car stops completely after at least 2.0 seconds have elapsed but before 5.0 seconds have elapsed.



Problem 4: Suppose that the bridge over University Avenue (pictured below) can be modeled as a simply-supported beam with length L=50 m. To simplify the analysis, let's assume that the beam has rectangular cross section with height 18 inches, width 4 feet and that it is constructed from steel with ρ=7800 kg/m^3 and E=210 GPa. (Note that the stiffness for various beam configurations is given in Figure 1.1 in the text.) Model this bridge as a SDOF system with an effective mass that is one third of the total mass of the beam and a stiffness equal to the stiffness of the beam when a static force is applied at its center. The damping ratio of the system is observed to be ζ=0.01.

Suppose that a single student jumping up and down on the bridge can exert a force $f(t)$=(1000 N)cos(ωt) where ω can be between 0 and 8π rad/s depending on how quickly he jumps up and down. How many students must jump on the bridge to cause a displacement amplitude of 50 cm? What frequency should they jump at to minimize the number of students required? (Don't worry, the actual bridge is stiffer and lighter than that given in the problem statement. Extra Credit: What would be more reasonable values for its mass and natural frequency? How does this change the solution?)



Problem 5: 3.2 from Ginsberg. Note that you are approximating the radar display as a rigid mass ("mounted mass is 8 kg"), which is mounted on a spring and dashpot.

Problem 6: 3.5 from Ginsberg. Also, please sketch the force and the response of the system (by hand) over one or two cycles, taking care to properly represent the amplitude and phase difference. Do this for both cases, 0.95 kHz and 1.05 kHz.

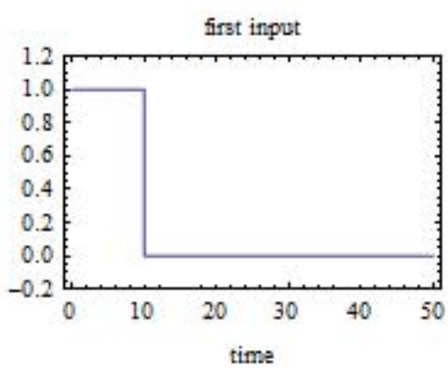
2.5.2 problem 1

2.54 An undamped system is subjected to a constant force for an interval T, after which it begins to roll off exponentially. Specifically, $Q(t) = F_0$ if $0 < t < T$, $Q(t) = F_0 \exp[-\beta(t-T)]$ if $t > T$. Derive an expression for the response.

Assuming zero initial conditions. The input to the system is made up of two inputs. We find the response to the first input, then add this response to the response due to the second input. The first input is

$$\begin{aligned} u_1(t) &= F_0 h(t) - F_0 h(t-T) \\ &= F_0(h(t) - h(t-T)) \end{aligned}$$

Which is a rectangular pulse of width T starting at $t = 0$. For example for $T = 10$ sec. and $F_0 = 1$



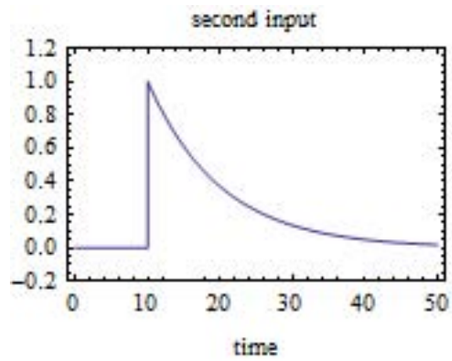
Assuming the response to unit step is $g_s(t)$ then the response to $u_1(t)$ is

$$g_1(t) = F_0\left(\overbrace{g_s(t)h(t)} - \overbrace{g_s(t-T)h(t-T)}\right)$$

From appending B, $g_s(t) = \frac{1}{m\omega_n^2}(1 - \cos(\omega_n t))$, hence the above becomes

$$g_1(t) = F_0\left(\overbrace{\frac{1}{m\omega_n^2}(1 - \cos(\omega_n t))h(t)} - \overbrace{\frac{1}{m\omega_n^2}(1 - \cos(\omega_n(t-T)))h(t-T)}\right) \tag{2.92}$$

Looking at the second input given by $u_2(t) = F_0 e^{-\beta(t-T)} h(t-T)$



From appendix B, the response to an exponential $F_0 e^{-\beta t} h(t)$ is

$$\frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta t}-\left(\cos(\omega_n t)-\frac{\beta}{\omega_n}\sin(\omega_n t)\right)\right)h(t)$$

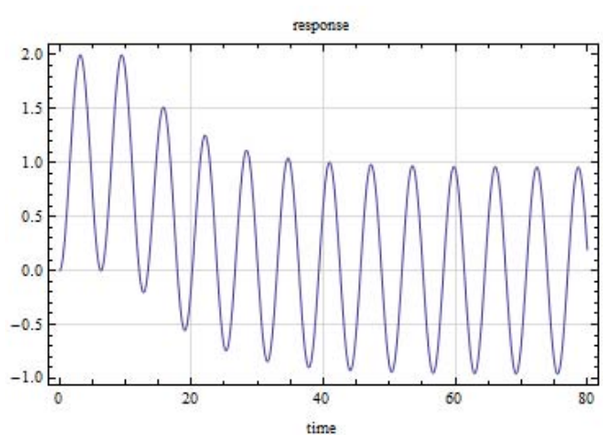
Therefore the response to $u_2(t)$ is

$$g_2(t)=\frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta(t-T)}-\left(\cos(\omega_n(t-T))-\frac{\beta}{\omega_n}\sin(\omega_n(t-T))\right)\right)h(t-T) \tag{2.93}$$

Adding Eqs 2.102 and 2.93 results in the final response

$$\begin{aligned}
g(t) &= g_1(t)+g_1(t)\\
&= F_0\left(\frac{1}{m\omega_n^2}(1-\cos(\omega_n t))h(t)-\frac{1}{m\omega_n^2}(1-\cos(\omega_n(t-T)))h(t-T)\right)+\\
&\frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta(t-T)}-\left(\cos(\omega_n(t-T))-\frac{\beta}{\omega_n}\sin(\omega_n(t-T))\right)\right)h(t-T)
\end{aligned}$$

For illustration, the following plot shows the response using some values. Using $m=1$ kg, $\omega_n=1$ rad/sec, $T=10$ sec, $\beta=1, F_0=1$ Volt.

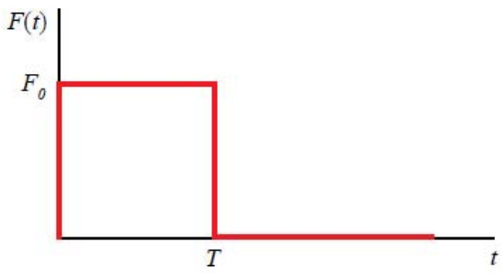


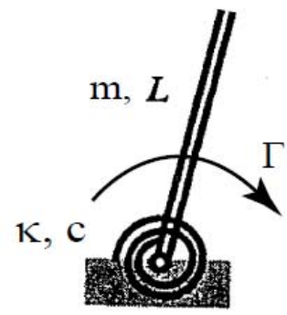
2.5.3 Problem 2

Problem 2: (**30 pts**, each part below is worth 10 pts).
The read head on a Hard Disk Drive (HDD) can be modeled as a pinned bar with a torsional spring at its base as shown below with L = 2 cm, m = 3 grams and κ = 20 N/rad. The damping ratio for the system is $\zeta = 0.02$. The equation of motion for this system is: (later we will discuss how to find the EOM for a system like this)

$$\frac{1}{3}mL^2\ddot{\theta} + c\dot{\theta} + \kappa\theta = \Gamma(t)$$

A certain read operation involves applying a step torque Γ(t) = F(t) with amplitude F_0 and duration T as shown below, where F_0 is the static torque required to displace the bar 30 degrees.





(a) Find the response of the system numerically over the time interval 0 < 1 < 10*T, with T = 2.5*T_d, where T_d is the damped period of the system. Use a numerical procedure, preferably Matlab's "ode45" function together with a suitably modified version of eom_2_12.m, which is available on the class website.
(b) Assuming an underdamped response, write down a closed-form solution for the response in terms of Heaviside-step functions, and unit step responses, $q_s(t)$. Compare this with the response that you found numerically.
(c) Plot the displacement as a function of time for the case where T = 3*T_d and T = 2.5*T_d. What do you observe? Why is the residual vibration larger in the latter case? (Hint –an undamped version of your analysis in (b) may make this easier to see.)

2.5.3.1 part(a)

The differential equation is

$$\frac{1}{3}ML^2\theta''(t) + c\theta'(t) + k\theta(t) = F_0(h(t) - h(t-T)) \tag{2.94}$$

The initial conditions are not given, and assumed to be zero, therefore $\theta(0) = 0^\circ$ and $\theta'(0) = 0$ rad/sec. The system is underdamped, hence

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Let T_d, be the damped period of oscillation given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

To obtain an expression for ω_n, Eq 2.110 is changed to a standard form $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{F_0(h(t)-h(t-T))}{\frac{1}{3}ML^2}$

$$\theta''(t) + \frac{\overbrace{3c}^{2\zeta\omega_n}}{ML^2}\theta'(t) + \frac{\overbrace{3k}^{\omega_n^2}}{ML^2}\theta(t) = \frac{F_0(h(t)-h(t-T))}{\frac{1}{3}ML^2} \tag{2.95}$$

Therefore

$$\omega_n^2 = \frac{3k}{ML^2}$$

Using $k = 20$ N/rad, $L = 0.02$ meter, $M = 0.003$ kg

$$\omega_n^2 = \frac{3(20)}{(0.003)(0.02)^2} = 5.0 \times 10^7$$

or

$$\omega_n = \sqrt{5.0 \times 10^7} = \boxed{7071 \text{ rad/sec}}$$

and

$$T_d = \frac{2\pi}{7071.1\sqrt{1-0.02^2}} = \boxed{0.8888 \text{ ms}}$$

Therefore

$$T = 2.5T_d = 2.5 \times 0.88857 = \boxed{2.221 \text{ ms}}$$

To find F_0 it is assumed the head was initially at rest. Therefore

$$\begin{aligned} F_0 &= k\theta_0 \\ &= 20\left(\frac{\pi}{6}\right) = \boxed{10.472 \text{ N-meter}} \end{aligned}$$

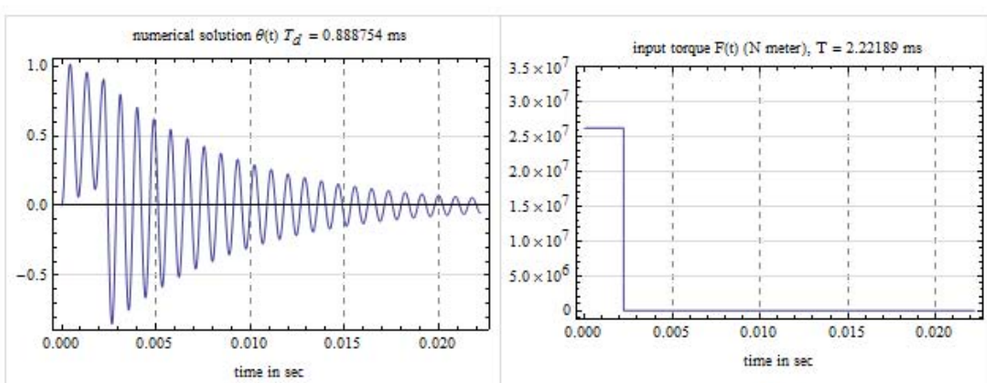
Eq 2.111 becomes

$$\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{F_0(h(t) - h(t-2.5T_d))}{\frac{1}{3}ML^2}$$

$$\theta''(t) + 2(0.02)(7071)\theta'(t) + \left(5 \times 10^7\right)\theta(t) = \frac{3 \times 20\left(\frac{\pi}{6}\right)(h(t) - h(t-2.5T_d))}{(0.003)(0.02)^2}$$

$$\theta''(t) + 283\theta'(t) + 5 \times 10^7\theta(t) = 2.618 \times 10^7(h(t) - h(t-0.0022219))$$

This is solved numerically for $0 < t < 10T$ with the initial conditions $\theta(0) = 0^\circ$ and $\theta'(0) = 0$ rad/sec. Here is a plot of the solution and the input on a second plot.



A computational software was used to numerically solve the above differential equation for the solution $\theta(t)$.

```
params = {m → 0.003, L -> 0.02, ξ -> 0.02, k → 20};
ωn = Sqrt[3 k / (m L^2)] ; ωd = ωn Sqrt[1 - ξ^2] ; f0 = k π/6 ; Td = 2 π/ωd ; T = 2.5 Td;
input = f0/(1/3 m L^2) (UnitStep[t] - UnitStep[t - 2.5 Td]);
eq = θ''[t] + 2 ξ ωn θ'[t] + (ωn)^2 θ[t] == input;
sol = θ[t] /. First@NDSolve[{eq /. params, θ'[0] == 0, θ[0] == 0}, θ[t], {t, 0, 10 * T /. params}];

: Grid[{
   {Plot[sol, {t, 0, 10 * T /. params}, Frame → True, GridLines → Automatic,
     GridLinesStyle → {Dashed, LightGray},
     FrameLabel →
      {{None, None},
       {"time in sec", Row[{"numerical solution θ(t) Td = ", (Td /. params) * 1000, " ms"}]}},
     ImageSize → 300],
    Plot[input /. params, {t, 0, 10 * T /. params}, Frame → True, GridLines → Automatic,
     GridLinesStyle → {Dashed, LightGray}, PlotRange → {All, 3.5 * 10^7},
     FrameLabel →
      {{None, None}, {Row[{"time in sec"}],
        Row[{"input torque F(t) (N meter), T = ", (T /. params) * 1000, " ms"}]}}, ImageSize → 300]
   }
  }, Spacings → {1, 1}, Frame → All, FrameStyle → LightGray]
```

2.5.3.2 Part(b)

From appendix B the response to underdamped second order system to a unit step $u(t)$ is

$$q_s(t) = \frac{1}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right)h(t)$$

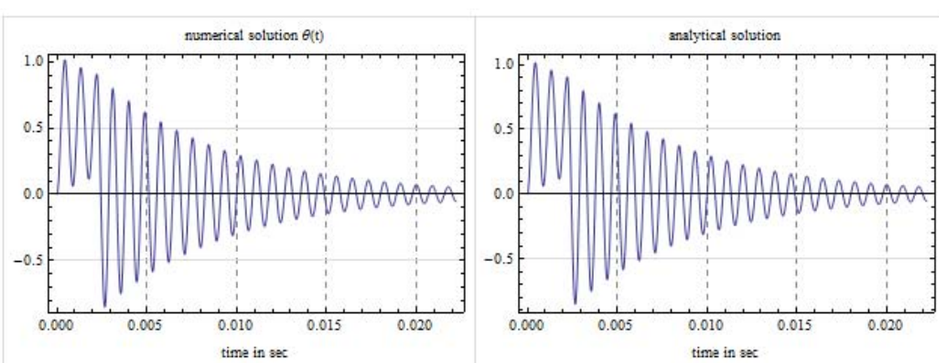
Hence the response $U(t)$ due to $\frac{F_0}{\left(\frac{1}{3}mL^2\right)}(h(t) - h(t-T))$ is given by

$$U(t) = \frac{F_0}{\left(\frac{1}{3}L^2\right)}\left(\overbrace{q_s(t)h(t)} - \overbrace{q_s(t-T)h(t-T)}\right)$$

Notice the factor $\frac{F_0}{\frac{1}{3}L^2}$. This was used since appendix B solution on based on equation of motion $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{1}{m}$ while in this case, the equation of motion is $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{F_0}{\frac{1}{3}mL^2}$, hence a factor of $\frac{F_0}{\frac{1}{3}L^2}$ is needed to scale the solution. Therefore the analytical solution is

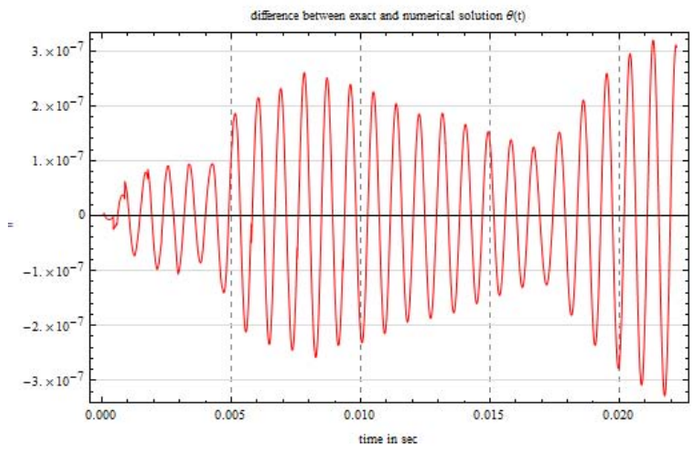
$$\begin{aligned} U(t) = &\frac{3F_0/L^2}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right)h(t) \\ &- \frac{3F_0/L^2}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n (t-2T)}\left(\cos(\omega_d (t-T)) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d (t-T))\right)\right)h(t-T) \end{aligned}$$

To compare this solution with the numerical solution found in part(a), the two solutions are plotted side-by-side for the case $T = 2.5T_d$



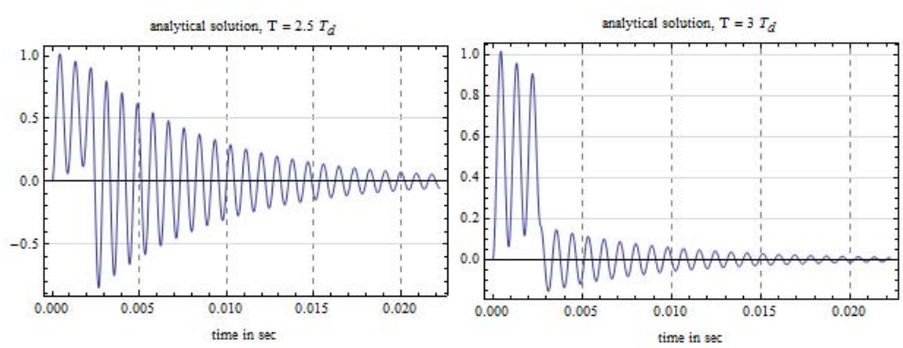
We see that solutions are in good approximate. Here is a plot of the difference. The error is in the order of 10^{-7}

```
Plot[sol - analyticalSolution /. params, {t, 0, 10 * T /. params}, Frame → True, GridLines → Automatic,
 GridLinesStyle → {Dashed, LightGray},
 FrameLabel → {{None, None}, {"time in sec", "difference between exact and numerical solution θ(t)"}},
 ImageSize → 500, PlotStyle → Red]
```



2.5.3.3 Part(c)

The analytical solutions for $T = 2.5T_d$ and $T = 3.0T_d$ are



We see when the step load duration is $T = 2.5T_d$, the disk head will vibrate with larger amplitudes than when the step duration was $T = 3T_d$.

To understand the reason for this, analysis was done on the undamped version of the solution for part b

From appendix B the response to undamped second order system to a unit step $u(t)$ is

$$q_s(t) = \frac{1}{M\omega_n^2}(1 - \cos(\omega_n t))h(t)$$

Therefore the solution for $0 < t < T$ is $\frac{3F_0/L^2}{M\omega_n^2}(1 - \cos(\omega_n t))$. This means at $t = T$ which is when the step load is removed, $\theta(T) = \frac{3F_0/L^2}{M\omega_n^2}(1 - \cos(\omega_n T))$ and $\theta'(T) = -\frac{3F_0/L^2}{M\omega_n^2}(\omega_n \sin(\omega_n T))$.

For $t > T$, the load is not present any more and we have free vibration response but with the above initial conditions obtained at the end of the T. The solution to free vibration

of an undamped system for $\tilde{t} = t - T \geq 0$ is given by

$$\begin{aligned}
\theta(\tilde{t}) &= \frac{\theta'(T)}{\omega_n}\sin\omega_n\tilde{t} + \theta(T)\cos\omega_n\tilde{t} \\
&= \frac{-\frac{3F_0/L^2}{M\omega_n^2}(\omega_n\sin(\omega_n T))}{\omega_n}\sin\omega_n\tilde{t} + \frac{3F_0/L^2}{M\omega_n^2}(1-\cos(\omega_n T))\cos\omega_n\tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2}\sin(\omega_n T)\sin\omega_n\tilde{t} + \left(\frac{3F_0/L^2}{M\omega_n^2} - \frac{3F_0/L^2}{M\omega_n^2}\cos(\omega_n T)\right)\cos\omega_n\tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2}\sin(\omega_n T)\sin\omega_n\tilde{t} - \frac{3F_0/L^2}{M\omega_n^2}\cos(\omega_n T)\cos\omega_n\tilde{t} + \frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2}\left(\sin(\omega_n T)\sin\omega_n\tilde{t} + \cos(\omega_n T)\cos\omega_n\tilde{t}\right) + \frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \qquad (2.96)
\end{aligned}$$

We have obtained a solution for the time after the step load was removed. We now investigate the result observed. We see that when T is close to an integer multiple of the period of the system, where we call the period of the system $\tilde{T}$ to differentiate it from T, then

$$\sin\left(\omega_n n\tilde{T}\right) = \sin\left(\frac{2\pi}{\tilde{T}}n\tilde{T}\right) = \sin(n2\pi) = 0$$

Also

$$\cos\left(\omega_n n\tilde{T}\right) = \cos\left(\frac{2\pi}{\tilde{T}}n\tilde{T}\right) = \cos(n2\pi) = 1$$

Hence the response given by equation 2.96 becomes

$$\begin{aligned}
\theta(\tilde{t}) &= -\frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} + \frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \\
&= 0 \qquad (2.97)
\end{aligned}$$

But if T occurs at multiple of halves of the period of the system (for example, $T = 0.5\tilde{T}, 1.5\tilde{T}, 2.5\tilde{T}$, etc...) then now

$$\sin\left(\omega_n\left(n\frac{\tilde{T}}{2}\right)\right) \to \sin\left(\frac{2\pi}{\tilde{T}}\left(n\frac{\tilde{T}}{2}\right)\right) \to \sin(n\pi) \to 0$$

However

$$\cos\left(\omega_n n\frac{\tilde{T}}{2}\right) \to \cos\left(\frac{2\pi}{\tilde{T}}n\frac{\tilde{T}}{2}\right) \to \cos(n\pi) \to -1$$

We notice that the sign is now negative. This means equation 2.96 becomes

$$\begin{aligned}
\theta(\tilde{t}) &= -\frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} - \frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \\
&= -\frac{6F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \qquad (2.98)
\end{aligned}$$

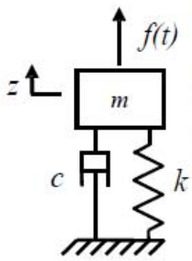
Comparing Eqs 2.97 and 2.98 we see that $\boxed{\text{when } T \text{ is an integer multiple of the period of the system}}$, then the response after T is minimal (zero for the case on undamped)

While $\boxed{\text{when } T \text{ occurs at multiple of halves of the period of the system}}$,the response is large beyond the time T.

The above analysis was done for undamped system, but the same idea carries to the underdamped case. This explains why the response dies out quickly when $T = 3T_d$ while it was large when $T = 2.5T_d$

2.5.4 Problem 3

Problem 3: A SDOF system modeling a car bouncing on its suspension has m=1000kg, k=11 kN/m and c=660 N-s/m. The car is released from rest at t=0 with z(0)= −0.10m. It is possible to bring the car exactly to rest by exerting an impulsive force $f(t)$=$F_0\delta(t\text{-}T)$ at some instant t=T. (e.g. hitting it with a very large hammer at just the right instant). Find the magnitude of the impulse and the instant, T, at which it should be applied such the bouncing of the car stops completely after at least 2.0 seconds have elapsed but before 5.0 seconds have elapsed.



First lets look at the free vibration response (zero input response, called u_{zi}). The damping ratio $\zeta = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{660}{2\sqrt{11000\times1000}} = 9.9499\times10^{-2} \approx \boxed{0.1}$ and $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{1000}}$ hence $\omega_n = 3.162$ rad/sec, and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.1623\sqrt{1-0.1^2}$. Hence $\boxed{\omega_d = 3.146 \text{ rad/sec}}$. The damped period of the system is $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{3.146} = \boxed{1.997}$ seconds and the natural period is $T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.162} = \boxed{1.987}$ seconds.

Hence the system is underdamped and the solution is

$$u_{zi} = \text{Re}\left(\hat{A}e^{(i\omega_d-\zeta\omega_n)t}\right)$$

Where $\hat{A} = a + ib$ is the complex amplitude. At $t = 0$ we have

$$a = u_{zi}(0) = -0.1$$

and $u'_{zi}(0) \equiv u'_0 = \text{Re}((i\omega_d - \zeta\omega_n)(a+ib)) = -b\omega_d - a\zeta\omega_n$ therefore $b = \frac{-u'_0 - a\zeta\omega_n}{\omega_d}$. Since car was dropped from rest, then we take $u'_0 = 0$ which leads to $b = -\frac{(-0.1)(0.1)3.162}{3.146} = \boxed{0.1}$

Hence, since $a = u'_0(0) \equiv u_0$ and

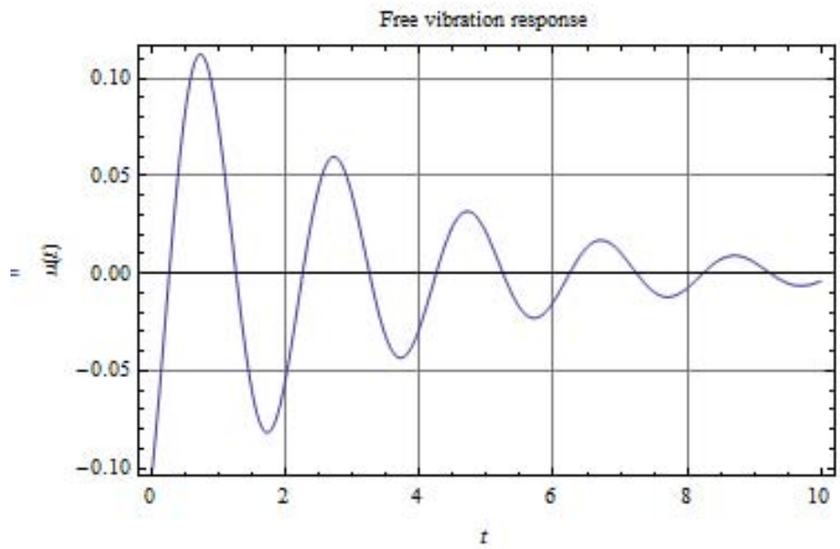
$$b = \frac{-u'_0 - a\zeta\omega_n}{\omega_d} = 0.1$$

then

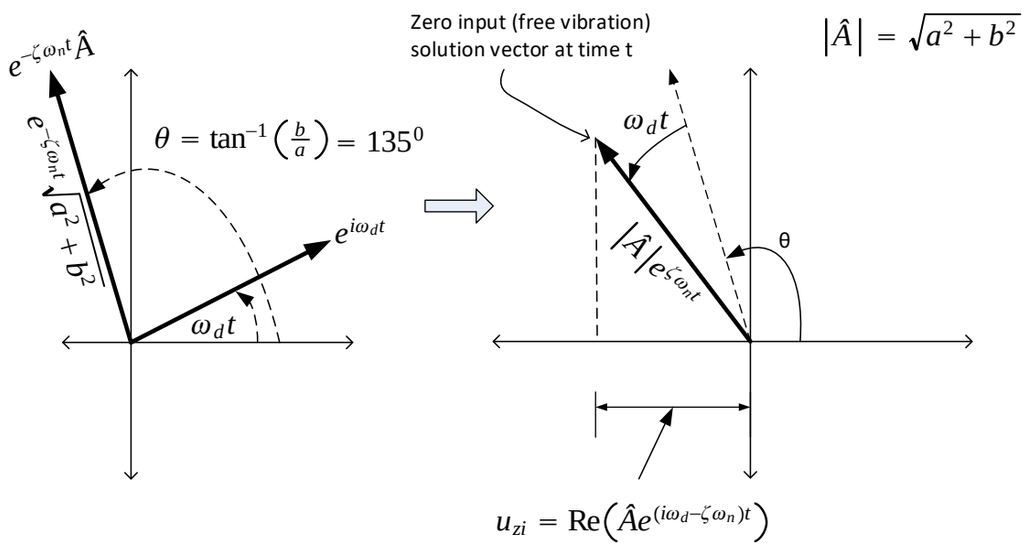
$$\begin{aligned} u_{zi}(t) &= \text{Re}\left((a-ib)e^{(i\omega_d-\zeta\omega_n)t}\right) \\ &= \text{Re}\left(e^{-\zeta\omega_n t}\left(u_0 - i\left(\frac{u'_0 + a\zeta\omega_n}{\omega_d}\right)\right)e^{i\omega_d t}\right) \\ &= e^{-\zeta\omega_n t}\left(u_0(0)\cos\omega_d t + \left(\frac{u'_0 + a\zeta\omega_n}{\omega_d}\right)\sin\omega_d t\right) \end{aligned} \tag{2.99}$$

For the numerical values gives, we now can plot this solution

$$u_{zi}(t) = e^{-0.1(3.162)t}(-0.1\cos 3.146t + 0.1\sin 3.146t)$$



The phase is given by $\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{0.1}{-0.1}\right) = 2.356\ rad = 135^0$, In complex plane, $u_h(t)$ is



Now we add the zero initial conditions response, also called zero state response u_{zs} for an input which is an impulse using appendix B.

$$u_{zs}(t) = e^{-\zeta\omega_n t}\left(\frac{F_0}{m\omega_d}\sin\omega_d t\right)h(t)$$

Hence u_{zs} for an impulse that occurs at time T is

$$u_{zs}(t) = e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T) \tag{2.100}$$

Hence the solution is found by combining Eq. 2.117 and Eq 2.113

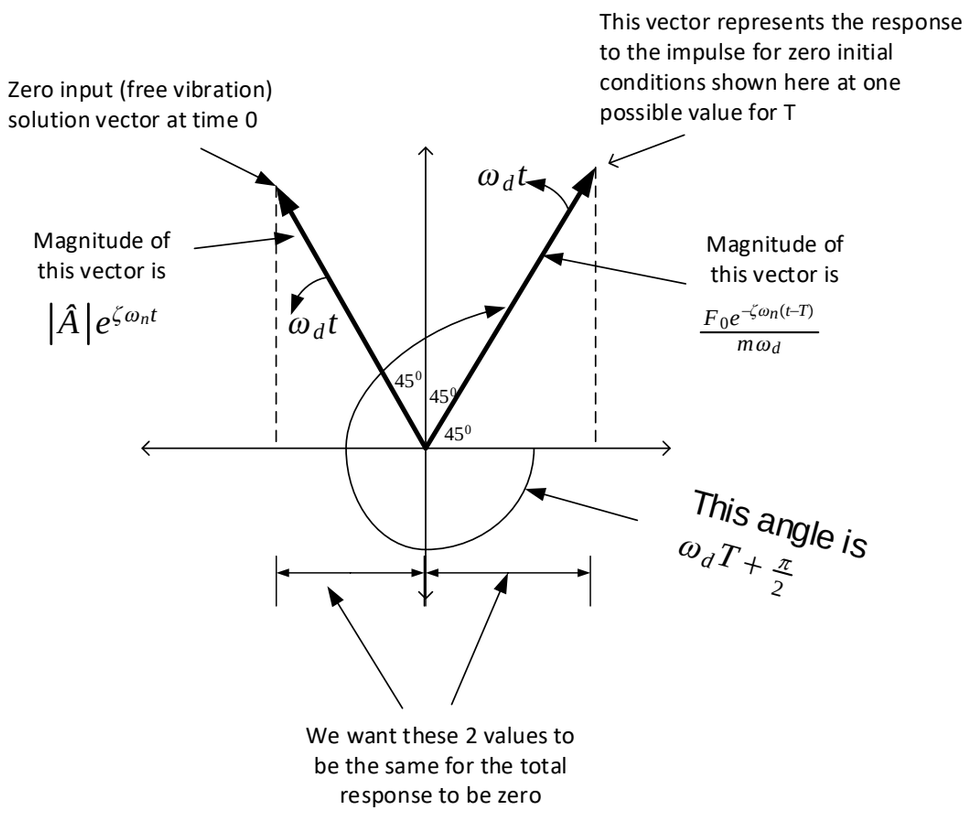
$$\begin{aligned} u(t) &= u_{zi} + u_{zs} \\ &= e^{-\zeta\omega_n t}\left(u_0(0)\cos\omega_d t + \left(\frac{u_0' + a\zeta\omega_n}{\omega_d}\right)\sin\omega_d t\right)h(t) + e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T) \end{aligned}$$

We need now to solve for T and F_0 in order to meet the requirements that $u(t)$ should become zero between for $2 < t < 5$. To do this in the complex plane, we draw the zero

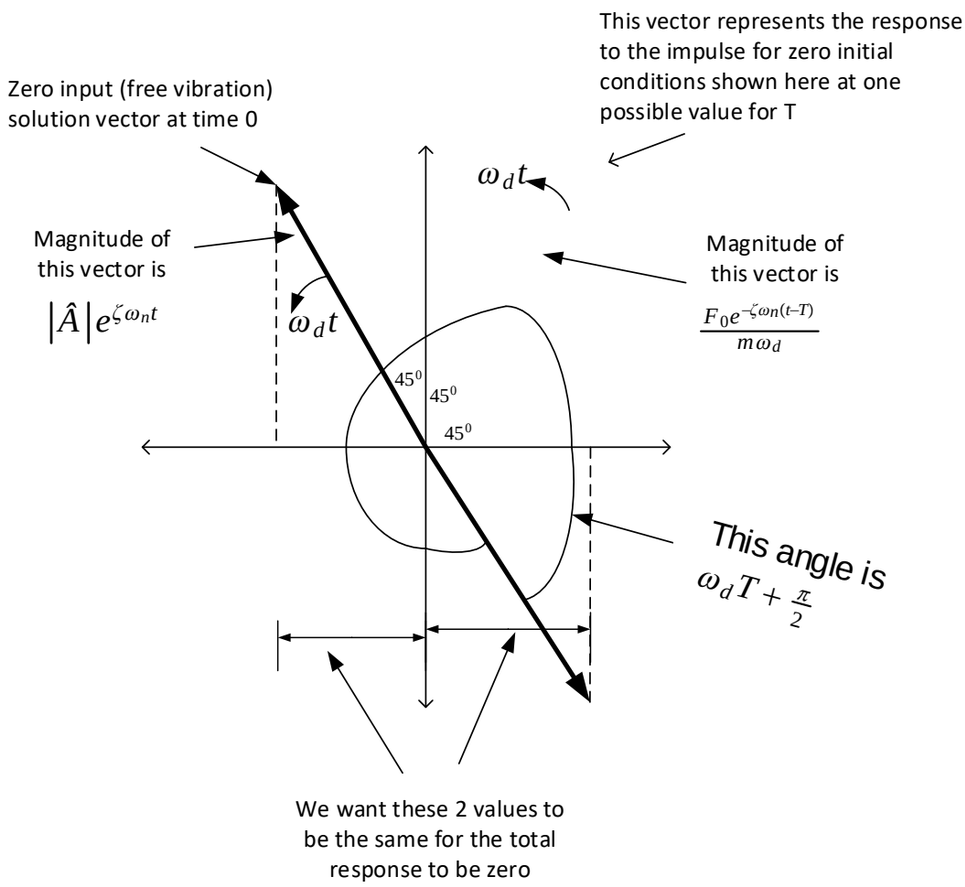
state response as a vector

$$
\begin{aligned}
u_{zs} &= e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T)\\
&= \mathrm{Re}\left(\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}\frac{1}{i}e^{i\omega_d(t-T)}\right)h(t-T)\\
&= \mathrm{Re}\left(\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}e^{i\left(\omega_d(t-T)-\frac{\pi}{2}\right)}\right)h(t-T)
\end{aligned}
$$

Hence u_{zs} vector has $\boxed{\text{magnitude } \frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}}$ and phase $\boxed{\omega_d t - \omega_d T - \frac{\pi}{2}}$ Now to solve the problem of finding T and F_0: To make the response become *zero* we need the magnitude of u_{zs} to be equal but opposite in sign to the magnitude of u_{zi} so that the projection on the x-axis cancel out (the projection on the x-axis of the vector is the real part which is the solution). Therefore, for the projection of u_{zs} to be the same as the projection of u_{zi} but of different sign, the following diagram shows all the possible T values that allows this. We will pick the first T value which is larger than 2 seconds to use.



From the above diagram, we need $\omega_d T + \frac{\pi}{2} = 2\pi - \frac{\pi}{4}$, hence $T = \frac{2\pi - \frac{\pi}{4} - \frac{\pi}{4}}{\omega_d} = \frac{\frac{3}{2}\pi}{\pi} = 1.5$ seconds. Hence this value of T is not acceptable. We now look for the next possible T.



From the above diagram we see it will be $\omega_d T + \frac{\pi}{2} = 2\pi + \frac{\pi}{4}$ hence $T = \frac{2\pi + \frac{\pi}{4} - \frac{\pi}{2}}{\pi} = 1.75$ seconds. Hence this is still too early to apply the impulse. We look at the next possible case. We see that now we must rotate the vector all the way it was in the first diagram above to get the projection on the x-axis canceling the projection of the free vibration vector. Hence now the relation to solve for is

$$\omega_d T + \frac{\pi}{2} = 4\pi - \frac{\pi}{4}$$

Where in the above we added full 2π to the first case we considered above. This gives

$$T = \frac{4\pi - \frac{\pi}{4} - \frac{\pi}{2}}{\pi} = 3.25 \text{ sec}$$

.We have found T which brings the system to halt after at least 2 seconds has elapsed. Now we find F_0 This is done by equating the amplitudes of the vectors as follows

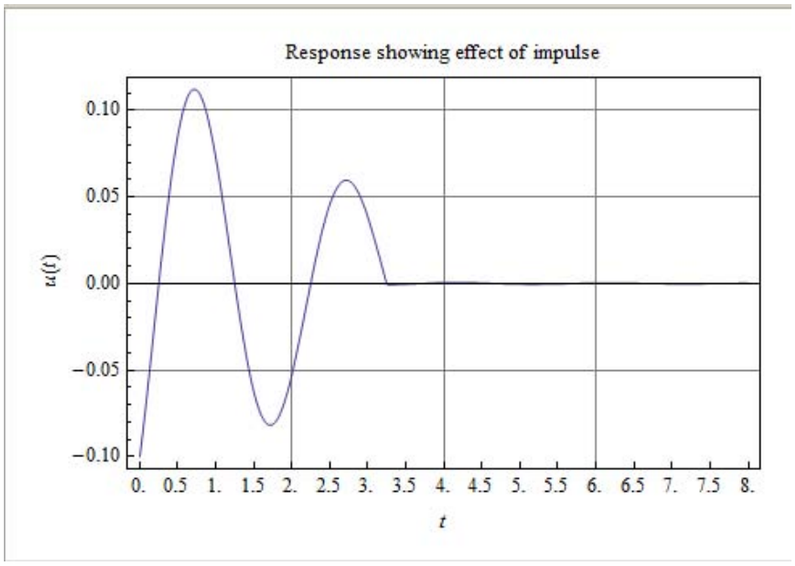
$$\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d} = e^{-\zeta\omega_n t}|\hat{A}|$$

Now for$t = T = 3.25$ second, plug-in numerical values

$$\begin{aligned}
\frac{F_0}{1000(3.146)} &= e^{-(0.1)(3.162)(3.25)}\sqrt{0.1^2 + 0.1^2} \\
\frac{F_0}{3146.0} &= 5.0607 \times 10^{-2} \\
F_0 &= 159.21
\end{aligned}$$

To verify, here is a plot of the response when the impulse hit with

$$\boxed{F_0 = 159.21 \text{ N at } t = 3.25 \text{ seconds}}$$



2.5.5 Problem 4

Problem 4: Suppose that the bridge over University Avenue (pictured below) can be modeled as a simply-supported beam with length L=50 m. To simplify the analysis, let's assume that the beam has rectangular cross section with height 18 inches, width 4 feet and that it is constructed from steel with ρ=7800 kg/m^3 and E=210 GPa. (Note that the stiffness for various beam configurations is given in Figure 1.1 in the text.) Model this bridge as a SDOF system with an effective mass that is one third of the total mass of the beam and a stiffness equal to the stiffness of the beam when a static force is applied at its center. The damping ratio of the system is observed to be ζ=0.01.

Suppose that a single student jumping up and down on the bridge can exert a force $f(t)$=(1000 N)cos(ωt) where ω can be between 0 and 8π rad/s depending on how quickly he jumps up and down. How many students must jump on the bridge to cause a displacement amplitude of 50 cm? What frequency should they jump at to minimize the number of students required? (Don't worry, the actual bridge is stiffer and lighter than that given in the problem statement. Extra Credit: What would be more reasonable values for its mass and natural frequency? How does this change the solution?)

2.5.5.1 First part

Let A be the area of the cross section and ρ the mass density and L the length, then actual mass is

$$\begin{aligned} m_{actual} &= LA\rho \\ &= 50(18 \times 0.0254)(4 \times 0.3048)(7800) \\ &= 217393 \text{ kg} \end{aligned}$$

Hence we will use

$$m = \frac{217393}{3} = 72464 \text{ kg}$$

The actual stiffness for a simply supported mean with loading at the center is $\frac{48EI}{L^3}$ where

I is the area moment of inertia. Hence

$$I = \frac{wh^3}{12} = \frac{(4 \times 0.3048)(18 \times 0.0254)^3}{12} = 0.00971 \text{ m}^4$$

Therefore the stiffness of the beam is

$$\begin{aligned} k &= \frac{48EI}{L^3} \\ &= \frac{48\left(210 \times 10^9\right)(0.00971)}{50^3} = 783014 \text{ N/m} \end{aligned}$$

The natural frequency is

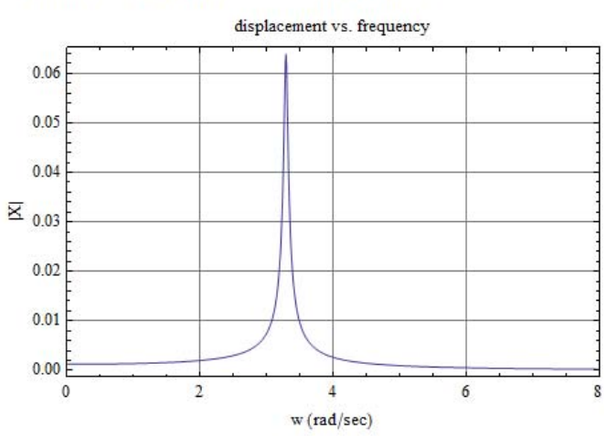
$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{783014}{72464}} = 3.287 \text{rad/sec} \\ f_n &= 0.523 \text{ Hz} \end{aligned}$$

Therefore, assuming the loading is given by $F_0 \cos(\bar{\omega} t)$ where $\bar{\omega}$ is the forcing frequency. The dynamic response at any time is given by

$$\left|\hat{X}\right| = \frac{F_0/k}{\sqrt{\left(1 - r^2\right)^2 + (2\zeta r)^2}}$$

Where $r = \frac{\bar{\omega}}{\omega_n}$. We start by drawing $\left|\hat{X}\right|$ vs. $\bar{\omega}$ for the load of 1000 N by changing $\bar{\omega}$ from 0 to 8π, Hence for a single student the displacement vs. forcing frequency is

```
parms = {f → 1000, k → 783 014, wn → 3.287, z → 0.01};
y[w_] := f / k / Sqrt[(1 - (w/wn)^2)^2 + (2 z w/wn)^2];
Plot[y[w] /. parms, {w, 0, 8 Pi}, PlotRange → {{0, 8}, All},
 GridLines → Automatic, Frame → True,
 FrameLabel → {{"|X|", None}, {"w (rad/sec)", "displacement vs. frequency"}}]
```



Hence we see that for one student, the maximum displacement is around 6 cm when the student is jumping at resonance frequency.

To answer the question of how many students are needed to cause $|X|$ to be 50 cm then that will depend on what forcing frequency is used. Now we will find the minimum number of students needed.

The minimum number will be when they all jump at the resonance frequency which is

found from solving for $\bar{\omega}_{resonance}$ in

$$\begin{aligned}\frac{\bar{\omega}_{resonance}}{\omega_n} &= \sqrt{1-2\zeta^2}\\ \bar{\omega}_{resonance} &= \omega_n\sqrt{1-2\zeta^2}\\ &= 3.287\sqrt{1-2(0.01)^2}\\ &= 3.28667 \text{ rad/sec}\end{aligned}$$

Therefore, at this forcing frequency, we now solve for F_0 to determine the number of students

$$\begin{aligned}|\hat{X}| &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2}}\\ F_0 &= k|\hat{X}|\sqrt{\left(1-\left(\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2}\\ &= (783014)(0.5)\sqrt{\left(1-\left(\frac{3.28667}{3.287}\right)^2\right)^2+\left(2(0.01)\frac{3.28667}{3.287}\right)^2}\\ &= 7829.75 \text{ N}\end{aligned}$$

Therefore we need at least $\boxed{\text{8 students}}$ all jumping at 3.287 rad/sec to cause a displacement of at least 50 cm.

2.5.6 Extra part

To make the structure avoid resonance, we need to make sure the ratio $\frac{\omega}{\omega_n}$ stays away from one. This is the ratio of the forcing frequency to the natural frequency. One way is to make ω_n much larger than any expected ω that can occur is typical use of this structure.

But to make $\omega_n = \sqrt{\frac{k}{m}}$ large, means either making m small or making k large. It is hard to reduce the mass of the structure. Therefore, making the structure more stiff will be a better solution.

The bridge can be made more stiff in many ways, such as by adding additional truss structure to it (assuming this will add minimal weight). For this example, suppose we double the stiffness. Hence $\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2(783014)}{72464}} = 4.649$rad/sec.

Therefore now $\bar{\omega}_{resonance} = \omega_n\sqrt{1-2\zeta^2} = 4.649\sqrt{1-2(0.01)^2} = 4.65$ rad/sec. Now the same number of students (8) as before, jumping at same frequency of 3.28667 will cause displacement of

$$\begin{aligned}|\hat{X}| &= \frac{8F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}}\\ &= \frac{8000/783014}{\sqrt{\left(1-\left(\frac{3.28667}{4.649}\right)^2\right)^2+\left(2(0.01)\frac{3.28667}{4.649}\right)^2}}\\ &= 0.02 \text{ meter}\end{aligned}$$

Therefore by making the bridge twice as stiff, now the same 8 students at $\bar{\omega} = 3.287$ will cause only 2 cm displacement instead of 50 cm.

2.5.7 Problem 5

A radar display is to be tested by mounting it on spring-dashpot suspension and subjecting it to harmonic force $Q = F\cos(\bar{\omega}t)$. The mounted mass is 8 kg and $\zeta = 0.25$. A free vibration shows that damped natural frequency $f_d = 5$hz.It is observed that when the force is applied at very low frequency the displacement amplitude is 2 mm. The test is to be performed at 5.2 Hz. What will be the steady state response?

We are given are the following

$$\begin{aligned} m &= 8 \text{ kg} \\ \zeta &= 0.25 \\ \omega_d &= \omega_n\sqrt{1-\zeta^2} = 2\pi(5) \text{ rad/sec} \\ F_0/k &= 0.002 \text{ meter} \\ \bar{\omega} &= 2\pi(5.2) \text{ rad/sec} \end{aligned}$$

Hence $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi(5)}{\sqrt{1-0.25^2}} = 32.446$ rad/sec. The steady state response is given by

$$u_{ss} = \text{Re}\left(\hat{X}e^{i\bar{\omega}t}\right)$$

where $\hat{X} = \left|\hat{X}\right|e^{i\theta}$. Hence

$$\begin{aligned} \left|\hat{X}\right| &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}} \\ &= \frac{0.002}{\sqrt{\left(1-\left(\frac{2\pi(5.2)}{32.446}\right)^2\right)^2 + \left(2(0.25)\frac{2\pi(5.2)}{32.446}\right)^2}} \\ &= 0.00397 \end{aligned}$$

and

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \\ &= \tan^{-1}\left(\frac{2(0.25)}{0}\right) \\ &\tan^{-1}(\infty) \end{aligned}$$

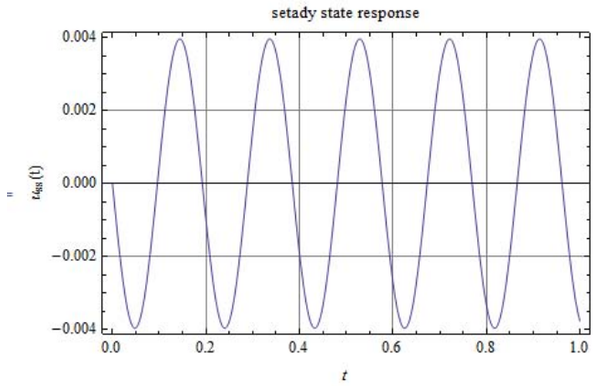
Since $0 \le \theta \le \pi$ then the phase is

$$\theta = \frac{\pi}{2}$$

Hence

$$\begin{aligned} u &= \text{Re}\left(\hat{X}e^{i\bar{\omega}t}\right) \\ &= \text{Re}\left(0.00397e^{i\frac{\pi}{2}}e^{i\bar{\omega}t}\right) \\ &= 0.00397\cos\left(\bar{\omega}t + \frac{\pi}{2}\right) \\ &= -0.00397\sin(\bar{\omega}t) \end{aligned}$$

```
Plot[-0.00397 Sin[2 * Pi * 5.2 t], {t, 0, 1}, Frame → True, GridLines → Automatic,
  FrameLabel → {{"uss(t)", None}, {t, "setady state response"}}]
```



2.5.8 Problem 6

A one degree of freedom system whose mass is 10 kg and whose natural frequency is 1 khz is subjected to a harmonic excitation $1.2 \sin \bar{\omega} t$ kN. The steady state amplitude when $\bar{\omega} = 1$ khz is observed to be 2.4 mm. Determine the steady state response at $\bar{\omega} = 0.95$ khz and $\bar{\omega} = 1.05$ khz.

We are given

$$\begin{aligned} m &= 10 \text{ kg} \\ \omega_n &= 2\pi(1000) \text{ rad/sec} \\ F_0 &= 1200 \text{ N} \\ |X| &= 2.4 \times 10^{-3} \text{ meter when } \bar{\omega} = \omega_n \end{aligned}$$

Since $\omega_n^2 = \frac{k}{m}$, hence $k = \omega_n^2 m = (2\pi(1000))^2(10)$, therefore

$$k = 3.949 \times 10^8 \text{ N/m}$$

Now when $\bar{\omega} = \omega_n$ we have

$$\begin{aligned} |\hat{X}| &= \frac{F_0/k}{\sqrt{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\bar{\omega}}{\omega_n}\right)^2}} \\ 2.4 \times 10^{-3} &= \frac{1200/\left(3.949 \times 10^8\right)}{\sqrt{(2\zeta)^2}} \\ &= \frac{3.039 \times 10^{-6}}{2\zeta} \end{aligned}$$

Hence

$$\begin{aligned} \zeta &= \left(\frac{3.039 \times 10^{-6}}{2 \times 2.4 \times 10^{-3}}\right) \\ &= 0.000633 \end{aligned}$$

2.5.8.1 Part (1)

when $\boxed{\bar{\omega} = 2\pi(950)}$ now $r = \frac{\bar{\omega}}{\omega_n} < 1$ hence dynamic magnification factor is positive. Therefore loading and displacement will be in phase with each others. (i.e. displacement is in same direction as force). Since the force is sin then the response will be sin with same frequency but different phase and amplitude. Hence let

$$u_{ss} = X \sin(\bar{\omega} t - \theta)$$

Where

$$
\begin{aligned}
X &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}} \\
&= \frac{1200/\left(3.949\times 10^8\right)}{\sqrt{\left(1-\left(\frac{2\pi(950)}{2\pi(1000)}\right)^2\right)^2+\left(2(0.000633)\frac{2\pi(950)}{2\pi(1000)}\right)^2}} \\
&= 3.116\times 10^{-5} \text{ meter}
\end{aligned}
$$

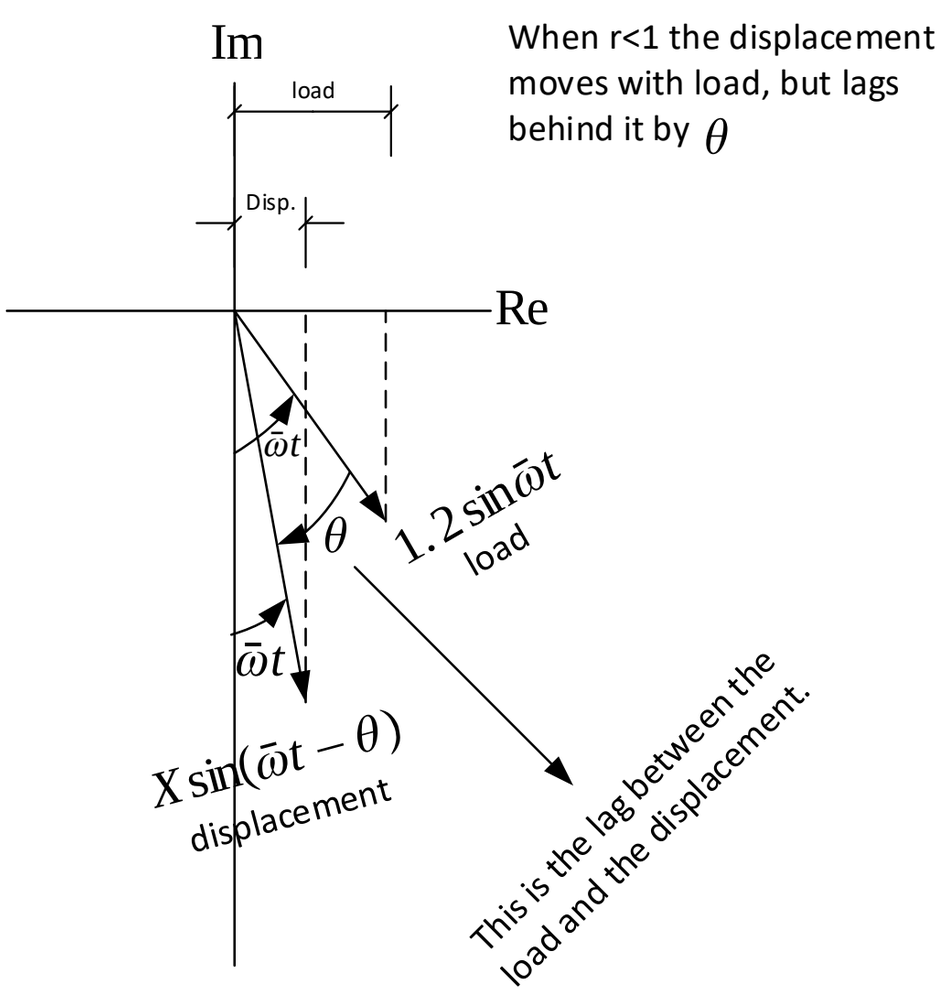
and

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \\
&= \tan^{-1}\left(\frac{2(0.000633)\frac{2\pi(950)}{2\pi(1000)}}{1-\left(\frac{2\pi(950)}{2\pi(1000)}\right)^2}\right) \\
&= \tan^{-1}\left(1.234\times 10^{-2}\right) \\
&= 0.01235 \text{ radians} \\
&= 0.71^\circ
\end{aligned}
$$

Hence steady state response is

$$
u_{ss} = 3.116\times 10^{-5}\sin(\bar{\omega}t - 0.71^\circ)
$$

Hence we see that the displacement is lagging the load by 0.71°. On complex plane it looks as follows



2.5.8.2 Part (2)

When $\boxed{\bar{\omega} = 2\pi(1050)}$ now $r = \frac{\bar{\omega}}{\omega_n} > 1$ hence dynamic magnification factor is negative. Therefore loading and displacement will be out of phase with loading. (i.e .displacement is in opposite direction to force). Doing the same calculations are done as above

$$u_{ss} = X \sin(\bar{\omega} t - \theta)$$

where X

$$\begin{aligned}
X &= \frac{F_0/k}{\sqrt{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\zeta \frac{\bar{\omega}}{\omega_n}\right)^2}} \\
&= \frac{1200/\left(3.949 \times 10^8\right)}{\sqrt{\left(1 - \left(\frac{2\pi(1050)}{2\pi(1000)}\right)^2\right)^2 + \left(2(0.000633)\frac{2\pi(1050)}{2\pi(1000)}\right)^2}} \\
&= 2.964 \times 10^{-5} \text{ meter}
\end{aligned}$$

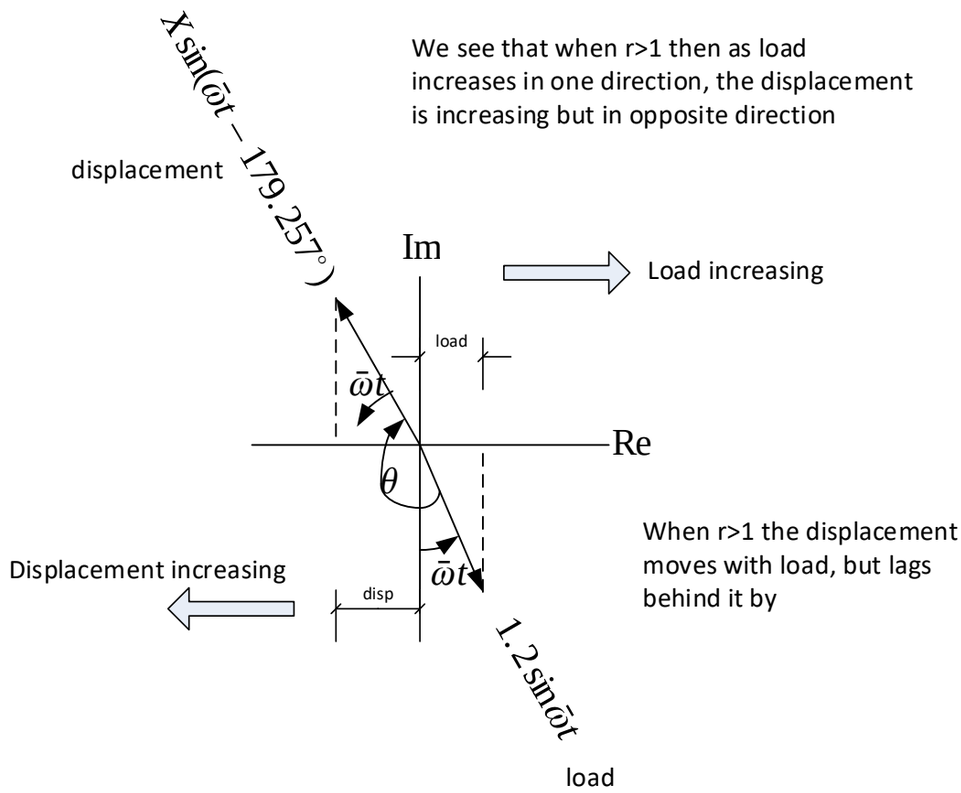
and

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \\
&= \tan^{-1}\left(\frac{2(0.000633)\frac{2\pi(1050)}{2\pi(1000)}}{1 - \left(\frac{2\pi(1050)}{2\pi(1000)}\right)^2}\right) \\
&= \tan^{-1}\left(\frac{0.0013293}{-0.1025}\right) \\
&= 3.12862 \text{ radians} \\
&= 179.257^\circ
\end{aligned}$$

Hence steady state response is

$$u_{ss} = 2.964 \times 10^{-5} \sin(\bar{\omega} t - 179.257^\circ)$$

On complex plane it looks as follows



Here is a plot by hand for the above 2 cases. First, the period that the loading is using $T = \frac{2\pi}{\omega} = \frac{1}{950} = 1.0526 \times 10^{-3}$sec

$$T = 1.053 \text{ ms}$$

2.5.9 Key solution for HW 4

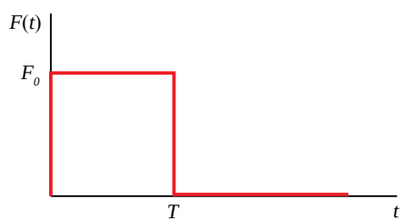
Homework #4
EMA 545, Spring 2013

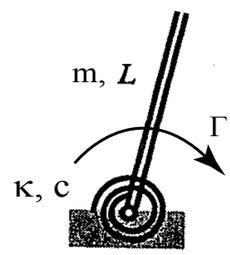
Problem 1: Exercise 2.54 from Ginsberg.

Problem 2: (**30 pts**, each part below is worth 10 pts).
The read head on a Hard Disk Drive (HDD) can be modeled as a pinned bar with a torsional spring at its base as shown below with L = 2 cm, m = 3 grams and κ = 20 N/rad. The damping ratio for the system is ζ = 0.02. The equation of motion for this system is: (later we will discuss how to find the EOM for a system like this)

$$\frac{1}{3}mL^2\ddot{\theta} + c\dot{\theta} + \kappa\theta = \Gamma(t)$$

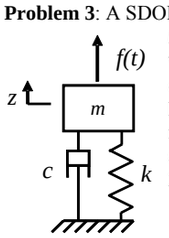
A certain read operation involves applying a step torque Γ(t) = F(t) with amplitude F_0 and duration T as shown below, where F_0 is the static torque required to displace the bar 30 degrees.





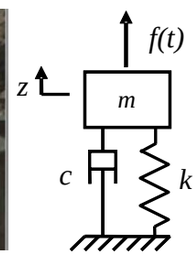
(a) Find the response of the system numerically over the time interval 0 < 1 < 10*T, with T = 2.5*T_d, where T_d is the damped period of the system. Use a numerical procedure, preferably Matlab's "ode45" function together with a suitably modified version of eom_2_12.m, which is available on the class website.
(b) Assuming an underdamped response, write down a closed-form solution for the response in terms of Heaviside-step functions, and unit step responses, $q_s(t)$. Compare this with the response that you found numerically.
(c) Plot the displacement as a function of time for the case where T = 3*T_d and T = 2.5*T_d. What do you observe? Why is the residual vibration larger in the latter case? (Hint –an undamped version of your analysis in (b) may make this easier to see.)

Problem 3: A SDOF system modeling a car bouncing on its suspension has *m*=1000kg, *k*=11 kN/m and *c*=660 N-s/m. The car is released from rest at t=0 with z(0)= –0.10m. It is possible to bring the car exactly to rest by exerting an impulsive force $f(t)=F_0\delta(t-T)$ at some instant *t*=*T*. (e.g. hitting it with a very large hammer at just the right instant). Find the magnitude of the impulse and the instant, *T*, at which it should be applied such the bouncing of the car stops completely after at least 2.0 seconds have elapsed but before 5.0 seconds have elapsed.



Problem 4: Suppose that the bridge over University Avenue (pictured below) can be modeled as a simply-supported beam with length L=50 m. To simplify the analysis, let's assume that the beam has rectangular cross section with height 18 inches, width 4 feet and that it is constructed from steel with ρ=7800 kg/m^3 and E=210 GPa. (Note that the stiffness for various beam configurations is given in Figure 1.1 in the text.) Model this bridge as a SDOF system with an effective mass that is one third of the total mass of the beam and a stiffness equal to the stiffness of the beam when a static force is applied at its center. The damping ratio of the system is observed to be ζ=0.01.

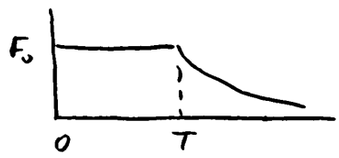
Suppose that a single student jumping up and down on the bridge can exert a force $f(t)$=(1000 N)cos(ωt) where ω can be between 0 and 8π rad/s depending on how quickly he jumps up and down. How many students must jump on the bridge to cause a displacement amplitude of 50 cm? What frequency should they jump at to minimize the number of students required? (Don't worry, the actual bridge is stiffer and lighter than that given in the problem statement. Extra Credit: What would be more reasonable values for its mass and natural frequency? How does this change the solution?)



Problem 5: 3.2 from Ginsberg. Note that you are approximating the radar display as a rigid mass ("mounted mass is 8 kg"), which is mounted on a spring and dashpot.

Problem 6: 3.5 from Ginsberg. Also, please sketch the force and the response of the system (by hand) over one or two cycles, taking care to properly represent the amplitude and phase difference. Do this for both cases, 0.95 kHz and 1.05 kHz.

Exercise 2.54



$$Q = F_0[h(t) - h(t-T)] + F_0 \exp[-\beta(t-T)]\,h(t-T)$$

$m\ddot{q} + kq = Q$, Assume $q(0) = \dot{q}(0) = 0$

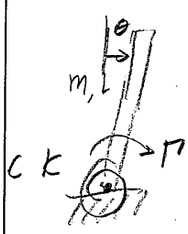
Let u be the unit step response, and x be the unit exponential response. Then

$$q = F_0\, u(t) - F_0\, u(t-T) + F_0\, x(t-T)$$

where $u(t) = \frac{1}{M\omega_{nat}^2}[1 - \cos(\omega_{nat} t)]\,h(t)$

$$x(t) = \frac{1}{M(\omega_{nat}^2 - 2\beta\omega_{nat} + \beta^2)}\Big\{\exp(-\beta t) - \Big[\cos(\omega_{nat} t) - \frac{\beta}{\omega_{nat}}\sin(\omega_{nat} t)\Big]\Big\}\,h(t)$$

HW 2e-01 HDP



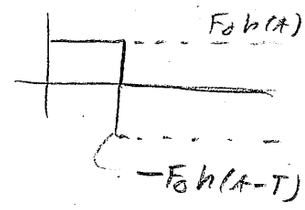
→ EOM $\frac{1}{3}mL^2\ddot{\theta} + c\dot{\theta} + k\theta = \Gamma(t)$

$M = \frac{1}{3}mL^2 \longrightarrow$

$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{\Gamma(t)}{M}$

$\zeta = 0.02, \quad \omega_n = \sqrt{\frac{k}{\frac{1}{3}mL^2}}$

$F(t) = F_0 h(t) - F_0 h(t-T)$



$g(t) = F_0 g_s(t) - F_0 g_s(t-T)$

$g_s(t)$ given in Appendix B
has $1/\mu$ term included

→ Do the rest in malab → a) b)

- time vector – sample 10 times per period T_d

c) for an undamped system,

$g_s(t) = \text{const.}(1 - \cos(\omega_n t))\, h(t)$

The response is a sum of 2 step responses

$g(t) = c_1(1 - \cos(\omega_n t)) - c_1(1 - \cos(\omega_n(t-T)))h(t-T)$

$= -c_1\cos(\omega_n t) + c_1\cos(\omega_n t - \omega_n T)$

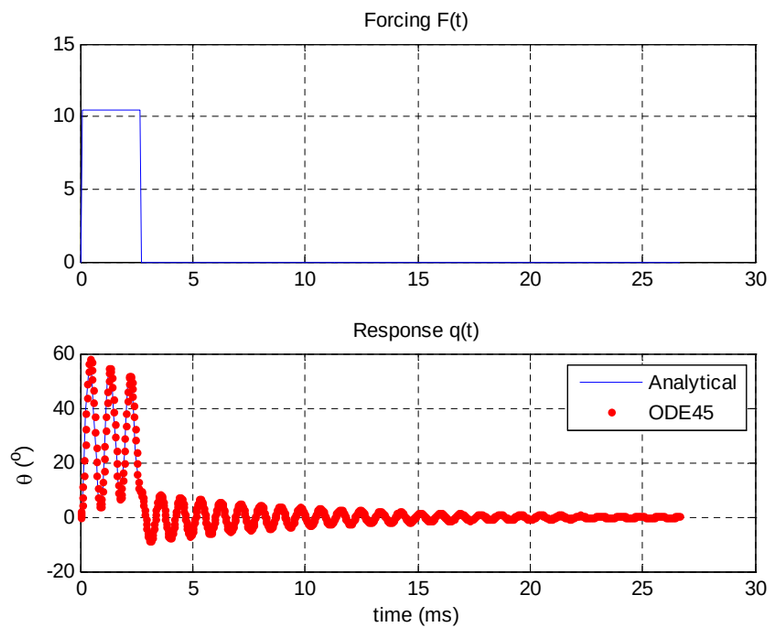
if $T = nT_d$, $\omega_n T = n \cdot 2\pi$

- and the two cosine terms are in phase and cancel completely
- if $T = (n + \frac{1}{2})T_d$ the terms reinforce and the residual vibration is largest.

* In the damped case, the constants multiplying the two terms, are slightly different, so there is some residual vibration, even when $T = nT_d$

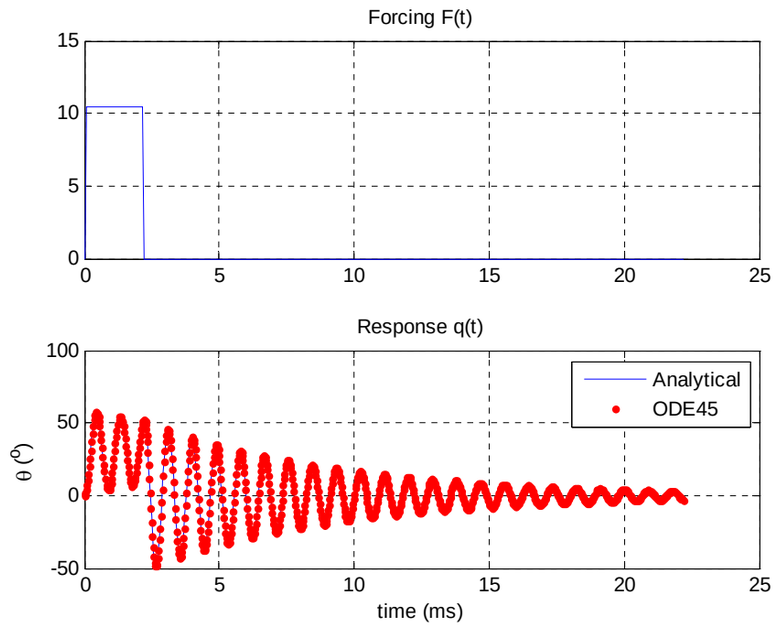
HW#6, Problem 2e-01, Hard Disk Drive
SOLUTION
MSA – Mar. 2009

Response with T = 3*Td



The transient response due to the step up is in phase with that of the transient response due to the step down, so the two almost cancel. They do not quite cancel because the response has damped somewhat, so the second step is larger than what would be needed to cancel the residual vibration from the first step.

Response with T = 2.5*Td



Matlab code:

```
% Solution to HW 2e-01 Hard Disk Drive Head

clear all; close all

% Parameter values
L = 0.02; % m
k = 20; % N/rad (torsional)
m = 0.003; %
M = (1/3)*m*L^2; K = k; % SDOF parameters
wn = sqrt(K/M)
zt = 0.02

F0 = 30*pi/180*k;
q_0 = 0; q_dot_0 = 0;
Td = 2*pi/wn;
Tc = 1/(zt*wn); % time constant of the system

T = 3*Td
% T = 2.5*Td;

global S
vns = whos; % put into a global variable
for k = 1:length(vns);
    eval(['S.',vns(k).name, ' = ',vns(k).name,';']);
end
```

```
% Time Vector
ts = [0:Td/10:10*T]; % 4*Tctime vector, sample 10x per period and over
4 time constants.

% Forcing - sum of step and ramp
% note h(t) written as (t>0) in Matlab
F = F0*(ts>0) - F0.*(ts-T>0);

% Analytical Solution

% Unit step and ramp responses from Ginsberg - includes particular and
% complimentary solutions
qs = inline(['(1/(M*wn^2))*(1-exp(-zt*wn*t)*(cos(wn*sqrt(1-
zt^2)*t)+',...
    '(zt/sqrt(1-zt^2))*sin(wn*sqrt(1-
zt^2)*t)))*(t>0)'],'t','M','wn','zt');

% Response is a sum of step and ramp responses
q = zeros(size(ts));
for k = 1:length(ts)
    q(k) = F0*qs(ts(k),M,wn,zt) - F0*qs(ts(k)-T,M,wn,zt);
end

figure(1)
subplot(2,1,1)
plot(ts*1e3,F); grid on;
title('Forcing F(t)');
subplot(2,1,2);
plot(ts*1e3,q*180/pi); grid on;
title('Response q(t)');
xlabel('time (ms)'); ylabel('\theta (^o)');

% Solution using ODE45

% Define equations of motion in eom_2_12.m
% Note - ode45 requires only the time span, not the whole time vector
tic
[tout,yout] = ode45('eom_2e_1',[ts(1),ts(end)],[q_0; q_dot_0]);
t_ode = toc

q_ode = yout(:,1); % the first of the y variables is q(t), the second
is q_dot(t)

% Add red dots to plot above
hold on; plot(tout*1e3,q_ode*180/pi,'r.'); hold off;
legend('Analytical','ODE45');
```

%%% Equations of Motion:

```
function [xdot] = eom_2e_1(t,x)

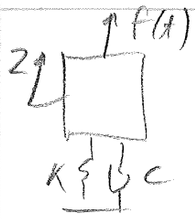
global S % bring in parameters

% Forcing - sum of step and ramp
```

```
F = S.F0*(t>0) - S.F0.*(t-S.T>0);

% Equations of Motion
xdot(1,1) = x(2);
xdot(2,1) = -S.wn^2*x(1)-2*S.zt*S.wn*x(2) + F/S.M;
```

Solution Pe2-03 Car vib. ①



m, k, c known

$\longrightarrow \omega_n = \sqrt{\frac{k}{m}}$ $\quad 2\zeta\omega_n = c/m$

$\zeta = \frac{c}{2m\omega_n}$ $\longrightarrow \boxed{\omega_n = 3.32\ \text{rad/s}}$ $\boxed{\zeta = 0.099}$

$$m\ddot{z} + c\dot{z} + kz = f(t)$$

$z(0) = -0.1\,m, \quad \dot{z}(0) = 0$

apply $f(t) = F_0\,\delta(t-T)$ to get to rest

Free response:

$z = Re(A e^{\lambda t})$

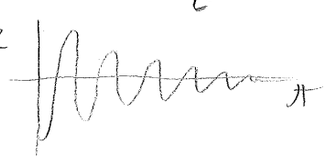
$z(0) = Re(A) = -0.01 \longrightarrow \boxed{A = -0.1 + ib}$

$\dot{z}(0) = Re(\lambda A) = 0$

$Re\left((-0.1+ib)(-\zeta\omega_n + i\omega_d)\right) = (+0.1\zeta\omega_n - b\omega_d) = 0$

$\boxed{b = \frac{0.1\,\zeta\omega_n}{\omega_d}}$

$z(t) = Re\left[\left(-0.1 + i\,\frac{0.1\,\zeta\omega_n}{\omega_d}\right) e^{\lambda t}\right]$



(Note, could solve this with impulse - momentum from EMA 202.)

Here we'll use superposition

Notice that the impulse response from Appendix B

$z_F = \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)\,h(t) \qquad \lambda = -\zeta\omega_n + i\omega_d$

has the exact same form! (Obvious in complex expon. form:)

$z_F = Re(B e^{\lambda t})\,h(t) \quad , \quad B = \frac{F_0}{m\omega_d}(-i)$

So it should be possible to cancel the response for $t > T$

$z_{tot} = z + z_F(t-T) = Re\left(A e^{\lambda t} + B e^{\lambda(t-T)}\,h(t-T)\right)$

for $t \geq T$, $\quad z_{tot} = Re\left[\left(A + B e^{-\lambda T}\right) e^{\lambda t}\right]$

(2)

Response zero for all $t > T$ iff

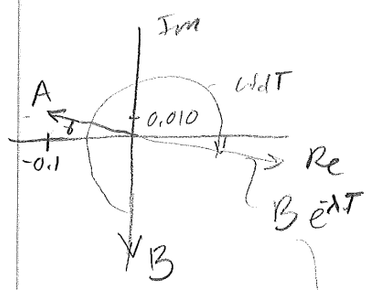
$$A + \underline{Be^{-\lambda T}} = 0 \qquad \lambda = -\xi\omega_n + i\omega_d$$

✓Known Don't Know F_0 or T → first find T

$$A = -0.1 + i0.010 \quad \text{(Matlab)}$$

$$\gamma = \tan^{-1}\left(\frac{0.010}{0.1}\right) = \pi - \text{angle}(A)$$

$$\gamma = 0.0997 \text{ rad}$$



$$B = \frac{-iF_0}{m\omega_d} e^{+\xi\omega_n T} e^{-i\omega_d T}$$

Choose T so these cancel

$$\omega_d T = \frac{3\pi}{2} + \gamma + n(2\pi)$$

could have gone through any number of cycles

$$T = \frac{\frac{3\pi}{2} + \gamma + n(2\pi)}{\omega_d} = 1.46, 3.36, 5.3, \ldots$$

$$T = 3.36 \text{ sec}$$

Match magnitude:

$$|A| = |Be^{-\lambda T}| = \frac{F_0}{m\omega_d} e^{\xi\omega_n T}$$

$$F_0 = m\omega_d |A| e^{-\xi\omega_n T} = 109.4 \text{ N-s}$$

See Matlab for plot

```
% Solution to HW e2-03, Car impulse

m=1000;
wn=sqrt(11000/1000)
zt=660/1000/2/wn
wd=wn*sqrt(1-zt^2);
lam=-zt*wn+1i*wd;

A=-0.1+1i*0.1*zt*wn/(wn*sqrt(1-zt^2))

gam=pi-angle(A)

T=(3*pi/2+gam+1*2*pi)/wd

F0=m*wd*abs(A)*exp(-zt*wn*T)
% Notice, the value of F0 seems rather small.  Remember that it is an
% impulse, so to get realistic units we need to integrate over the
impulse.
% For example, if the impulse were a constant force, Fc, that is
0.001sec long,
% then the integral of Fc*0.001 would equal F0, or in other words,

Fc=F0/0.001 % N
% Maybe that still seems a little small but it seems to be correct.

B=(-1i*F0/(m*wd))

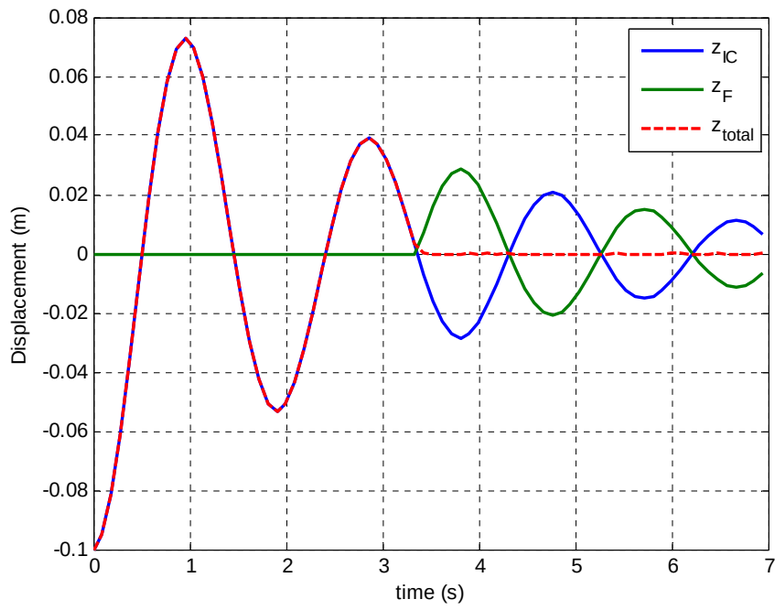
% Check to see if this works:
dt=(2*pi/wd)/20; % 20 samples per period
ts=[0:dt:7];
z_IC=real(A*exp(lam*ts));
z_F=real(B*exp(lam*(ts-T))).*(ts>T);

figure(1);
plot(ts,z_IC, ts,z_F, ts, z_IC+z_F,'--'); grid on;
legend('z_{IC}','z_F','z_{total}');
set(get(gca,'Children'),'LineWidth',2);
xlabel('time (s)'); ylabel('Displacement (m)');
```

Command Window Output:

```
wn =
       3.3166
zt =
     0.099499
A =
         -0.1 +  0.0099995i
gam =
     0.099664
T =
         3.362
F0 =
       109.36
Fc =
```

```
   1.0936e+05
B =
             0 -   0.033138i
```



Solution Pe2-04 Bridge

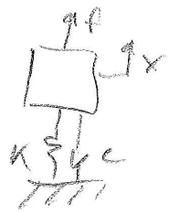
(1)

$\rho = 7800$

$E = 210\ GPa$

$\zeta = 0.01$

$K = \frac{48EI}{L^3}$ (P.5 in book)



$m\ddot{x} + c\dot{x} + Kx = f(t)$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m}$

$\omega_n^2 = \sqrt{\frac{K}{m}} = ?$

$I = \frac{bh^3}{12} = \frac{(18')(12'')^3}{12} \rightarrow 9.7\times10^{-3}\ m^4$

$m = \left(\frac{1}{3}\right)\rho \cdot bhL = 72.4\times10^3\ kg$

given

$\longrightarrow \boxed{\omega_n = 3.28\ rad/s \Rightarrow 0.52\ Hz}$

Steady state response $x(t) = Re(X e^{i\omega t})$

$X = \frac{(1000\ N/m)}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$

Largest response when $\omega = \omega_n\sqrt{1-2\zeta}$ eq. 3.1.11

or $\omega \approx \omega_n$ for $\zeta \ll 1$

$X = \frac{(1000\ N/m)}{i\omega_n^2 2\zeta} \longrightarrow \boxed{|X| = 0.064\ m \text{ for 1 student}}$

$\longrightarrow$ **8** students required to get 50 cm amplitude

For reference, to get the same displacement by standing on the bridge,

$X_{static} = \frac{F}{K} = 1.2\ mm \longrightarrow$ 392 students!

Exercise 3.2

$$m\ddot{q} + c\dot{q} + kq = F\cos(\omega t)$$

$m = 8$ kg, $\zeta = 0.25$, $\omega_d = 10\pi$ rad/s, $F/k = 0.002$ meter

$$\omega_{nat} = \frac{\omega_d}{(1-\zeta^2)^{1/2}} = 32.446 \text{ rad/s}$$

$$r = \frac{5.2(2\pi)}{\omega_{nat}} = 1.006976$$

$$q = \frac{F}{k}|D(r,\zeta)|\cos(\omega t - \phi)$$

$$|D(r,\zeta)| = \frac{1}{[(1-r^2)^2 + 4\zeta^2 r^2]^{1/2}} = 1.9854$$

$$\phi = \tan^{-1}\frac{2\zeta r}{(1-r^2)} = -1.5430 + \pi = 1.5986 \text{ rad} = 91.593^\circ$$

Thus $q = 0.003971 \cos(10.4\pi t - 1.5986)$ meter

3.5 $m = 10\,kg$, $f_n = 1\,kHz$ $\quad F(t) = 1.2\sin(\omega t)\ kN$

$\omega = 1\,kHz$, $|\underline{X}| = 2.4\,mm$, find $|\underline{X}|$ when $\omega = 0.95\,kHz$, $\omega = 1.05\,kHz$

$m\ddot{x} + c\dot{x} + kx = f(t) \rightarrow f(t) = Re(F e^{i\omega t}) \rightarrow x(t) = Re(\underline{X} e^{i\omega t})$

$F = -i \cdot 1200\ N$ (sine)

$$\underline{X} = \frac{F/m}{\omega_n^2 - \omega^2 + i\omega \cdot 2\zeta\omega_n}$$

$$\omega = \omega_n,\ |\underline{X}| = \left|\frac{F/m}{i2\zeta\omega_n^2}\right| = \frac{|F|}{2m\omega_n^2 \cdot \zeta}$$

$$\zeta = \frac{|F|}{|\underline{X}| \cdot 2 \cdot m \cdot \omega_n^2} = \frac{1200\ \cancel{kg\text{-}m/s^2}}{0.0024\,\cancel{m} \cdot 2 \cdot 10\,kg\,(1000 \cdot 2\pi)^2\ \cancel{1/s^2}}$$

$$\boxed{\zeta = 6.33 \times 10^{-4}}$$

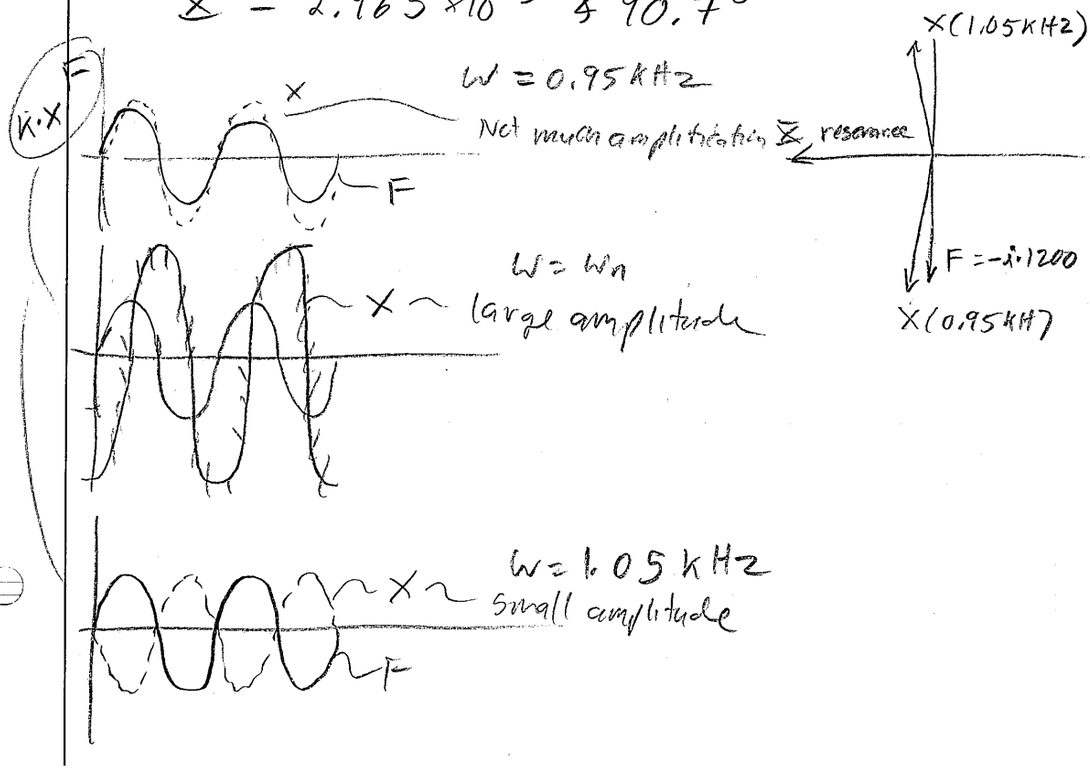
$\omega = 0.95\ kHz$

$$\underline{X} = \frac{-i \cdot 1200/10}{(2\pi)^2(1000^2 - 950^2 + i \cdot 2 \cdot \zeta \cdot 1000 \cdot 950} = 3.117 \times 10^{-5}$$

$\measuredangle -90.7°$

$\omega = 1.05\ kHz$

$\underline{X} = 2.965 \times 10^{-5}\ \measuredangle 90.7°$



2.6 HW5

Local contents

2.6.1 problem description

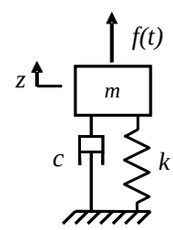
Homework #5, EMA 545, Spring 2013
Due Thursday 2/28

Comment: As I mentioned in class, I strongly encourage you to avoid hunting for formulas on forced response. All of these problems can be solved simply by knowing the differential equation and that the force and steady state response have the form:

$$f(t) = \mathrm{Re}\left(Fe^{i\omega t}\right) \qquad \rightarrow \qquad x(t) = \mathrm{Re}\left(Xe^{i\omega t}\right)$$

Problem 1: 3.9 from Ginsberg (Hint: assume that the motion of every component of the system is harmonic. Derive the equation(s) of motion and show the full derivation used to obtain the complex amplitude(s) from the equation(s) of motion.)

Problem 2: (3e1) (**20 points**)
A 450 kg generator, modeled as a rigid mass, must be installed on the same floor as some sensitive laboratory equipment. The operation of the generator results in a vertical force, $f(t)$, being applied to the generator (rigid mass) whose amplitude is 20kN and whose frequency is 1800 rpm. Use a damping ratio of $\zeta = 0.03$ for both (a) and (b) below.

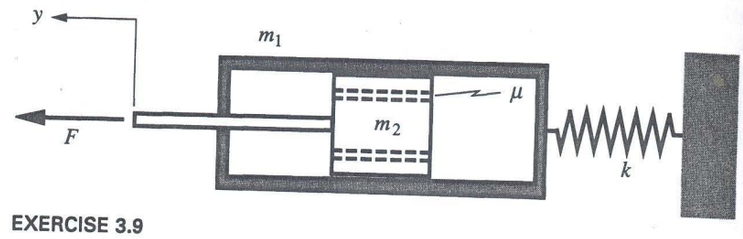


a.) Find the stiffness of the support, k, such that the force transmitted to the ground is no more than 2kN.
b.) Take your result from part (a) and compute the amplitude of the generator as the machine starts up. (As the machine starts up, assume that the force amplitude is constant at 20kN, but the frequency increases very slowly from zero to 1800 rpm. Do a worst-case analysis – just assure that the steady-state amplitude of the machine is less than 10mm for any forcing frequency between 0 and 1800 rpm.)
c.) Using your results from (a) and (b), suppose that the startup amplitude must not exceed 10mm. The startup amplitude can be decreased by adding mass to the generator while also increasing the stiffness of the support to keep the natural frequency of the system constant. How much mass must be added to keep the amplitude below 10mm?

Problem 3: 3.19 from Ginsberg

Problem 4: 3.23 from Ginsberg

2.6.2 problem 1



EXERCISE 3.9

3.9 The shock absorber consists of an outer tube of mass m_1 that is restrained by spring k and an inner piston of mass m_2. Orifices in the piston permit passage of a viscous fluid that fills the tube; the coefficient of damping between the piston and the tube is μ. An unknown force $F(t)$ is applied to the piston, with the result that the absolute displacement of the piston is $y = A\sin(\omega t)$. The parameters of the system are: $m_1 = 0.5$ kg, $m_2 = 1.0$ kg, $k = 3.2$ kN/m, $\mu = 40$ N-s/m, and $A = 20$ mm. Determine the amplitude and phase of the force $F(t)$ relative to the displacement $y(t)$ when $\omega = 75$ rad/s and $\omega = 85$ rad/s.

Assuming the 2 masses move together (else we will have 2 systems and 2 equations of motions. Hence I assumed that they move together as one body).

$$(m_1 + m_2)y'' + y'\mu + ky = f(t)$$

Since $y(t) = A\sin(\omega t)$ hence

$$y(t) = \text{Re}\left(\frac{A}{i}e^{i\omega t}\right)$$

Let

$$f(t) = \text{Re}\left(\frac{\hat{F}}{i}e^{i(\omega t)}\right)$$

Where $\hat{F}$ is the complex amplitude of the force. Now we substitute all these in the differential equation above.

$$\begin{aligned} y' &= \text{Re}\left(\omega A e^{i\omega t}\right) \\ y'' &= \text{Re}\left(i\omega^2 A e^{i\omega t}\right) \end{aligned}$$

$$\begin{aligned} (m_1 + m_2)y'' + y'\mu + ky &= \text{Re}\left(\frac{\hat{F}}{i}e^{i(\omega t)}\right) \\ \text{Re}\left(i\omega^2 A e^{i\omega t}\right)(m_1 + m_2) + \text{Re}\left(\omega A e^{i\omega t}\right)\mu + k\,\text{Re}\left(\frac{A}{i}e^{i\omega t}\right) &= \text{Re}\left(\frac{\hat{F}}{i}e^{i(\omega t)}\right) \\ \text{Re}\left[\left(i\omega^2(m_1 + m_2) + \omega\mu + \frac{1}{i}k\right)Ae^{i\omega t}\right] &= \text{Re}\left(\frac{\hat{F}}{i}e^{i(\omega t)}\right) \\ \left(i\omega^2(m_1 + m_2) + \omega\mu + \frac{1}{i}k\right)A &= \frac{\hat{F}}{i} \end{aligned}$$

Hence

$$\hat{F} = \left(-\omega^2(m_1 + m_2) + i\omega\mu + k\right)A$$

$k = 3.2\times10^3$ Nm, $\mu = 40$ Ns/m, $A = 0.02$ meter. When $\omega = 75$rad/sec the above becomes

$$\begin{aligned} \hat{F} &= \left(-75^2(1.5) + i75\times 40 + 3.2\times10^3\right)0.02 \\ &= -104.75 + 60.0i \end{aligned}$$

Hence $\text{Re}\left(\hat{F}\right) = -104.75\text{N}$ and the phase is $\tan^{-1}\left(-\frac{60.0}{104.75}\right) = 2.62$ rad/sec.

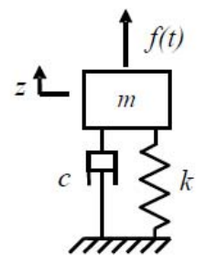
When $\omega = 85$

$$\begin{aligned}\hat{F} &= 1.5\left(-85^2 + i85\frac{40}{1.5} + 2133.3\right)0.02\\ &= -152.75 + 68.0i\end{aligned}$$

$\text{Re}\left(\hat{F}\right) = -152.75$ N and the phase is $\tan^{-1}\left(-\frac{68}{152.75}\right) = 2.722$ rad/sec.

2.6.3 problem 2

Problem 2: (3e1) (**20 points**)
A 450 kg generator, modeled as a rigid mass, must be installed on the same floor as some sensitive laboratory equipment. The operation of the generator results in a vertical force, $f(t)$, being applied to the generator (rigid mass) whose amplitude is 20kN and whose frequency is 1800 rpm. Use a damping ratio of $\zeta = 0.03$ for both (a) and (b) below.



a.) Find the stiffness of the support, k, such that the force transmitted to the ground is no more than 2kN.
b.) Take your result from part (a) and compute the amplitude of the generator as the machine starts up. (As the machine starts up, assume that the force amplitude is constant at 20kN, but the frequency increases very slowly from zero to 1800 rpm. Do a worst-case analysis – just assure that the steady-state amplitude of the machine is less than 10mm for any forcing frequency between 0 and 1800 rpm.)
c.) Using your results from (a) and (b), suppose that the startup amplitude must not exceed 10mm. The startup amplitude can be decreased by adding mass to the generator while also increasing the stiffness of the support to keep the natural frequency of the system constant. How much mass must be added to keep the amplitude below 10mm?

2.6.3.1 Part(a)

Force transmitted to floor is given by

$$F_{tr} = cz' + kz$$

Let $f(t) = F\cos(\omega t) = \text{Re}\left(Fe^{i\omega t}\right) = \text{Re}\left(Fe^{i\omega t}\right)$ where we are given that $F = 20\times 10^3$ N. $\omega = 2\pi\left(\frac{1800}{60}\right) = 60\pi = 188.50$ rad/sec or 30 Hz.

Let $z_{ss} = \text{Re}\left(\frac{F}{k}|D|e^{i(\omega t-\phi)}\right)$ where $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $|D| = \frac{1}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$. and $r = \frac{\omega}{\omega_n}$

Hence $z' = \text{Re}\left(i\omega\frac{F}{k}|D|e^{i(\omega t-\phi)}\right) = \text{Re}\left(\omega\frac{F}{k}|D|e^{i\left(\omega t-\phi+\frac{\pi}{2}\right)}\right)$. Therefore

$$F_{tr} = c\,\text{Re}\left(\omega\frac{F}{k}|D|e^{i\left(\omega t-\phi+\frac{\pi}{2}\right)}\right) + k\,\text{Re}\left(\frac{F}{k}|D|e^{i(\omega t-\phi)}\right)$$

Where $c = 2\zeta\omega_n m$ and When $F_{tr} = 2\times 10^3$N . We now solve for k from

$$2\times 10^3 \geq 2\zeta\omega_n m\,\text{Re}\left(\omega\frac{F}{k}|D|e^{i\left(\omega t-\phi+\frac{\pi}{2}\right)}\right) + k\,\text{Re}\left(\frac{F}{k}|D|e^{i(\omega t-\phi)}\right)$$

Taking the maximum case for RHS where exponential are unity magnitude, hence

$$\begin{aligned}2\times 10^3 &= 2\zeta\omega_n m\omega\frac{F}{k}|D| + F|D|\\ &= \left(2\zeta\omega_n m\omega\left(\frac{F}{k}\right) + F\right)|D|\\ &= \frac{F\left(1 + 2\zeta\omega_n\frac{m}{k}\omega\right)}{\sqrt{\left(1-r^2\right)^2 + (2\zeta r)^2}}\end{aligned}$$

Where $r = \frac{\omega}{\omega_n} = \frac{\omega}{\sqrt{\frac{k}{m}}}$. Hence the above becomes

$$2\times 10^3 = \frac{F\left(1 + 2\zeta\omega_n \frac{m}{k}\omega\right)}{\sqrt{\left(1 - \frac{\omega^2}{\frac{k}{m}}\right)^2 + \left(2\zeta\frac{\omega}{\sqrt{\frac{k}{m}}}\right)^2}}$$

In the above everything is known except for k which we solve for. Plugging the numerical values given. $\omega = 2\pi\left(\frac{1800}{60}\right)$,$m = 450, F = 20\times 10^3, \zeta = 0.03$ hence

$$2\times 10^3 = \frac{20\times 10^3\left(1 + 2(0.03)\sqrt{\frac{k}{450}}\,\frac{450}{k}(60\pi)\right)}{\sqrt{\left(1 - \frac{450(60\pi)^2}{k}\right)^2 + \left(2(0.03)\frac{60\pi}{\sqrt{\frac{k}{450}}}\right)^2}}$$

Hence $k = 1.2135\times 10^6$ N/m. Hence $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.2135\times 10^6}{450}} = 51.929$ rad/sec or 8.265 Hz.

2.6.3.2 part(b)

The total displacement is given by

$$\begin{aligned} z(t) &= z_{transient}(t) + z_{ss}(t) \\ &= e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + \mathrm{Re}\left(\frac{F}{k}|D|e^{i(\omega t-\phi)}\right) \end{aligned}$$

Where

$$z_{transient}(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

Assuming at $t = 0$ the system is relaxed hence $z(0) = 0$ and $z'(0) = 0$ we can determine A, B from Eq **??**.

At $t = 0$,

$$\begin{aligned} z(0) &= 0 \\ &= A + \mathrm{Re}\left(\frac{F}{k}|D|e^{-i\phi}\right) \end{aligned}$$

Hence

$$A = -\mathrm{Re}\left(\frac{F}{k}|D|e^{-i\phi}\right)$$

and

$$\begin{aligned} z'(t) = &-\zeta\omega_n e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\zeta\omega_n t}(-\omega_d A\sin\omega_d t + \omega_d B\cos\omega_d t) \\ &+ \mathrm{Re}\left(\omega\frac{F}{k}|D|e^{i\left(-\phi+\frac{\pi}{2}\right)}\right) \end{aligned}$$

Hence at $t = 0$

$$\begin{aligned} z'(0) &= 0 \\ &= -\zeta\omega_n A + \omega_d B + \mathrm{Re}\left(\omega\frac{F}{k}|D|e^{i\left(-\phi+\frac{\pi}{2}\right)}\right) \end{aligned}$$

Hence

$$
\begin{aligned}
B &= \frac{\zeta\omega_n}{\omega_d}A - \frac{1}{\omega_d}\operatorname{Re}\left(\omega\frac{F}{k}|D|e^{i\left(-\phi+\frac{\pi}{2}\right)}\right)\\
&= -\frac{\zeta\omega_n}{\omega_d}\operatorname{Re}\left(\frac{F}{k}|D|e^{-i\phi}\right) - \frac{1}{\omega_d}\operatorname{Re}\left(\omega\frac{F}{k}|D|e^{i\left(-\phi+\frac{\pi}{2}\right)}\right)
\end{aligned}
$$

Therefore the displacement is

$$
\begin{aligned}
z(t) = e^{-\zeta\omega_n t}&\left[-\operatorname{Re}\left(\frac{F}{k}|D|e^{-i\phi}\right)\cos\omega_d t + \left\{-\frac{\zeta\omega_n}{\omega_d}\operatorname{Re}\left(\frac{F}{k}|D|e^{-i\phi}\right) - \frac{1}{\omega_d}\operatorname{Re}\left(\omega\frac{F}{k}|D|e^{i\left(-\phi+\frac{\pi}{2}\right)}\right)\right\}\sin\omega_d t\right]\\
&+ \operatorname{Re}\left(\frac{F}{k}|D|e^{i\left(\omega t-\frac{\pi}{2}-\phi\right)}\right)
\end{aligned}
$$

Hence expressed in sin and cos

$$
\begin{aligned}
z(t) &= -\left(\frac{F}{k}|D|\cos\phi\right)e^{-\zeta\omega_n t}\cos\omega_d t + e^{-\zeta\omega_n t}\left\{-\frac{\zeta\omega_n}{\omega_d}\left(\frac{F}{k}|D|\cos\phi\right) - \frac{1}{\omega_d}\left(\omega\frac{F}{k}|D|\sin\phi\right)\right\}\sin\omega_d t\\
&\quad + \frac{F}{k}|D|\cos\left(\omega t-\phi\right)\\
&= -\left(\frac{F}{k}|D|\cos\phi\right)e^{-\zeta\omega_n t}\cos\omega_d t + e^{-\zeta\omega_n t}\frac{F|D|}{\omega_d k}\left(-\zeta\omega_n\cos\phi - \omega\sin\phi\right)\sin\omega_d t + \frac{F}{k}|D|\cos\left(\omega t-\phi\right)\\
&= \frac{F}{k}|D|e^{-\zeta\omega_n t}\left[-\cos\phi\cos\omega_d t + \frac{1}{\omega_d}\left(-\zeta\omega_n\cos\phi - \omega\sin\phi\right)\sin\omega_d t\right] + \frac{F}{k}|D|\cos\left(\omega t-\phi\right)
\end{aligned}
$$

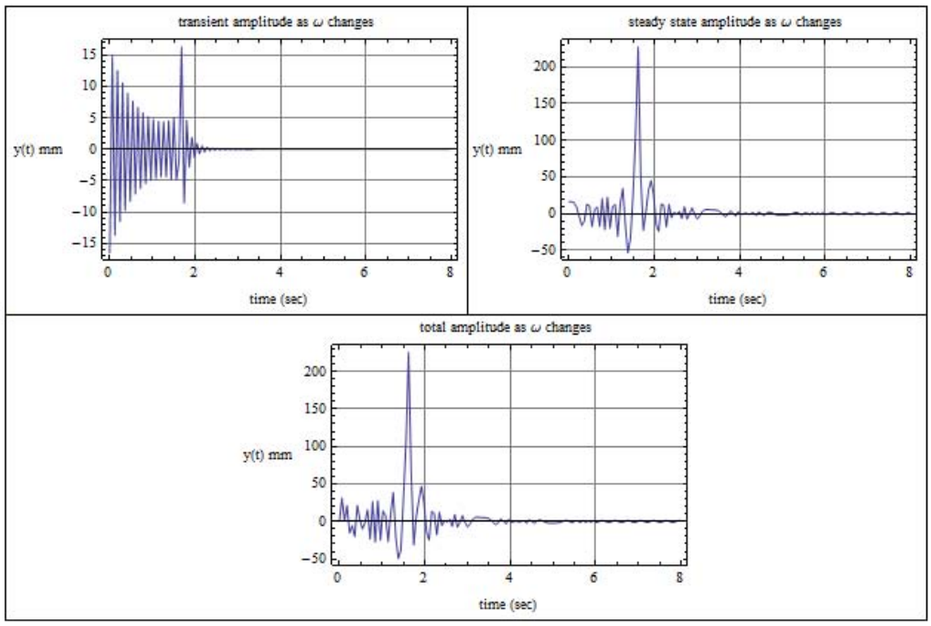
Since $\zeta = 0.03$ then $\omega_d = \omega_n\sqrt{1-\zeta^2} = \omega_n\sqrt{1-0.03^2} = 0.99955(\omega_n)$. Therefore in the above we can just replace ω_d by ω_n with very good approximation, hence we now obtain

$$
\begin{aligned}
z(t) &= \frac{F}{k}|D|e^{-\zeta\omega_n t}\left[-\cos\phi\cos\omega_n t + \frac{1}{\omega_n}\left(-\zeta\omega_n\cos\phi - \omega\sin\phi\right)\sin\omega_n t\right] + \frac{F}{k}|D|\cos\left(\omega t-\phi\right)\\
&= \frac{F}{k}|D|e^{-\zeta\omega_n t}\left[-\cos\phi\cos\omega_n t - \left(\zeta\cos\phi + \frac{\omega}{\omega_n}\sin\phi\right)\sin\omega_n t\right] + \frac{F}{k}|D|\cos\left(\omega t-\phi\right)
\end{aligned}
$$

This is the amplitude. In the above $|D| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$, and $\phi = \tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}$. The transient solution usually goes away after 5 or 6 cycles. Hence let us assume that the start up time takes $6\times\frac{2\pi}{\omega_n} = 6\times\frac{2\pi}{\sqrt{\frac{k}{m}}} = 6\times\frac{2\pi}{\sqrt{\frac{1.2135\times10^6}{450}}} = 0.72597$ seconds. Or 1 second at worst.

Therefore we can now plot the amplitude for $t = 0$ to $t = 1$ second in increments of 0.1 second, and each time advance, we can increment ω from 0 to 60π in linear fashion, hence each 0.1 second we update ω by an amount 6π. After 1 second has passed, the system is assumed to be in steady state, and then we keep ω fixed at 60π rad/sec. This is a plot showing $z(t)$ for $t = 0$ to 2 seconds given the above method of changing ω

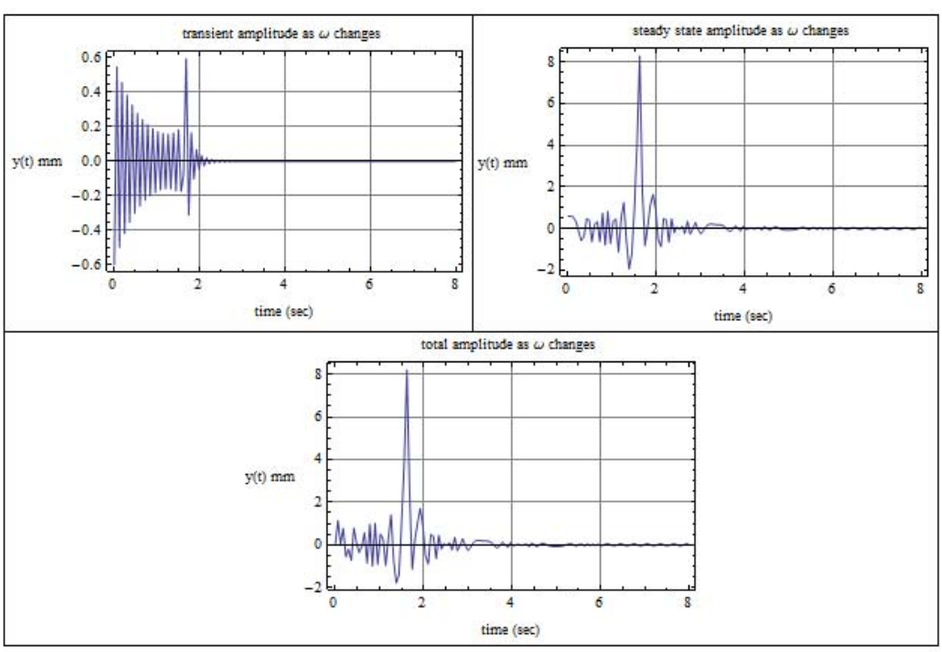
To avoid going over $10mm$, this means we have to avoid the case of $r = 1$ or $\omega = \omega_n$. When I first just incremented ω_n such that $r = 1$ was not avoided, resonance caused the amplitude to go over $10mm$ as given in this plot. The transient solution itself stayed just below 10mm but the steady state solution went over 10mm due to resonance



2.6.3.3 Part(c)

To insure that the amplitude does not go over 10mm, we need to add mass to the generator. Maximum amplitude is given by $\frac{F}{k}\frac{1}{2\zeta} = \frac{20\times10^3}{1.2135\times10^6}\frac{1}{2(0.03)} = 0.27469$ meter or 274mm

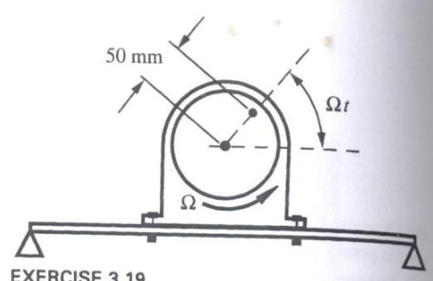
So to insure maximum does not exceed 10mm , solve for new k from $0.01 = \frac{20\times10^3}{k_n}\frac{1}{2(0.03)}$, hence $k_n = 3.3333\times10^7$. Since $\omega_n = 51.929 = \sqrt{\frac{k_n}{m_n}}$ then new mass is $m_n = \frac{3.3333\times10^7}{51.929^2} = \boxed{12361 \text{ kg}}$ using these values, the above plot now are redone. This is the result



We see that now the maximum displacement remained below 10 mm.

2.6.4 Problem 3

3.19 The total mass of a motor is 80 kg. It is observed that gently placing the motor on a beam produces a static downward displacement of 40 mm. When the motor rotates at an angular speed of 145 rev/min, the steady-state amplitude of the beam is 10 mm. It also is observed that at a rotation rate of 145 rev/min, the radial line to the center of mass of each rotor is 75° above horizontal when the beam is at its static reference position (where $q = 0$).



EXERCISE 3.19

(a) Determine the damping ratio ζ for the system.
(b) Determine the imbalance εm.
(c) Determine the smallest possible amplitude of vibration of the beam if the motor turns at a rate that is much larger than the natural frequency of the system.

Let $\varepsilon = 50mm = 0.05m$ be the distance of the unbalance mass m. Let $M = 80kg$ be the mass of the motor. The equation of motion is given by

$$(M+m)y'' + cy' + ky = m\varepsilon\Omega^2 \sin(\Omega t)$$
$$y'' + 2\zeta\omega_n y' + \omega_n^2 y = \frac{m}{m+M}\varepsilon\Omega^2 \operatorname{Re}\left(\frac{1}{i}e^{i\Omega t}\right)$$

Where $\omega_n = \sqrt{\frac{k}{m+M}}$ and $\zeta = \frac{c}{2(M+m)\omega_n}$. Let $y = \operatorname{Re}\left(\frac{Y}{i}e^{i\Omega t}\right)$. This leads to

$$Y = \frac{m}{m+M}\frac{\varepsilon\Omega^2}{\omega_n^2 - \Omega^2 + 2i\zeta\omega_n\Omega}$$

Since static deflection is $40mm$, then

$$\frac{(M+m)g}{k} = 0.04$$
$$k = \frac{(M+m)g}{0.04}$$

But $\omega_n^2 = \frac{k}{m+M} = \frac{(M+m)g}{0.04(M+m)} = \frac{g}{0.04}$, hence $\omega_n = \sqrt{\frac{9.81}{0.04}} = 15.66$ rad/sec or 2.492 Hz.

2.6.4.1 part(a)

Since at steady state the displacement is 10 mm, then $\Omega = 2\pi\frac{145}{60} = 15.184$ or $\boxed{2.4167 \text{ Hz}}$ hence

$$y = \operatorname{Re}\left(\frac{Y}{i}e^{i\Omega t}\right) = \operatorname{Re}\left(\frac{\varepsilon m}{m+M}\frac{r^2}{\left(1 - r^2 + 2i\zeta r\right)}e^{i\left(\Omega t - \frac{\pi}{2}\right)}\right)$$
$$= \frac{\varepsilon m r^2}{m+M}\frac{1}{\sqrt{\left(\left(1-r^2\right)^2 + (2\zeta r)^2\right)}}\operatorname{Re}\left(e^{-i\phi}e^{i\left(\Omega t - \frac{\pi}{2}\right)}\right)$$

Where $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$. $r = \frac{\Omega}{\omega_n} = \frac{15.184}{15.66} = \boxed{0.9696}$ hence the above becomes, at steady state

$$\begin{aligned} 0.01 &= \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2\zeta 0.9696)^2}}\operatorname{Re}\left(e^{i\left(15.184\ t-\frac{\pi}{2}-\phi\right)}\right) \\ &= \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2\zeta 0.9696)^2}}\sin\left(15.184\ t-\phi\right) \end{aligned} \tag{2.101}$$

We are now told that at $\Omega = 15.184$ and when $\Omega t = 75^0$ then the displacement is zero, hence

$$0 = \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2\zeta 0.9696)^2}}\sin\left(75^0-\phi\right)$$

or

$$\begin{aligned} \sin\left(75^0-\phi\right) &= 0 \\ 75^0-\phi &= 0 \\ \phi &= 75^0 \end{aligned}$$

Since $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ then

$$75\left(\frac{\pi}{180}\right) = \tan^{-1}\left(\frac{2\zeta 0.9696}{1-0.9696^2}\right)$$

Hence

$$\begin{aligned} \tan^{-1}\left(\frac{2\zeta 0.9696}{1-0.9696^2}\right) &= 1.3090 \\ \frac{2\zeta 0.9696}{1-0.9696^2} &= \tan(1.3090) \\ \frac{2\zeta 0.9696}{1-0.9696^2} &= 3.7321 \end{aligned}$$

Hence $\boxed{\zeta = 0.11523}$

2.6.4.2 Part(b)

From Eq 2.101

$$0.01 = \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2\zeta 0.9696)^2}}\sin\left(15.184\ t-\phi\right)$$

The maximum amplitude is when

$$0.01 = \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2\zeta 0.9696)^2}}$$

But $\zeta = 0.11523$, hence we now solve for m

$$0.01 = \frac{(0.05)m}{m+80}\frac{0.9696^2}{\sqrt{\left(1-0.9696^2\right)^2+(2(0.11523)0.9696)^2}}$$

Hence

$$\boxed{m = 4.1\ kg}$$

Hence $\varepsilon m = (0.05)(4.1) = \boxed{0.20 \text{ kg meter}}$

2.6.4.3 Part(c)

since

$$y = \frac{\varepsilon m r^2}{m+M} \frac{1}{\sqrt{\left(\left(1-r^2\right)^2 + (2\zeta r)^2\right)}} \operatorname{Re}\left(e^{i\left(\Omega t - \frac{\pi}{2} - \phi\right)}\right)$$

As Ω becomes much larger than ω_n then $\left(1-r^2\right)^2 \to r^4$. Now dividing numerator and denominator by r^2 gives

$$\begin{aligned} y &= \frac{\varepsilon m}{m+M} \frac{1}{\sqrt{\frac{\left(r^4+(2\zeta r)^2\right)}{r^4}}} \sin\left(\Omega t - \phi\right) \\ &= \frac{\varepsilon m}{m+M} \frac{1}{\sqrt{\left(1+\frac{4\zeta^2}{r^2}\right)}} \sin\left(\Omega t - \phi\right) \end{aligned}$$

as r becomes large then $\frac{4\zeta^2}{r^2} \to 0$ hence

$$y \simeq \frac{\varepsilon m}{m+M} \sin\left(\Omega t - \phi\right)$$

The smallest possible amplitude is

$$\begin{aligned} |y| &= \frac{0.20}{4.1+80} \\ &= 2.3781 \times 10^{-3} \text{ meter} \end{aligned}$$

or

$$|y| = 2.38 \text{ mm}$$

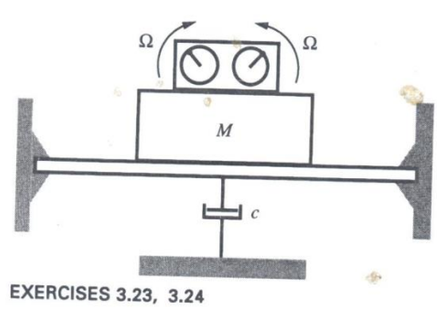
2.6.5 problem 4

3.23 A counter-rotating eccentric mass exciter is attached to a block, which is supported by a light-weight beam. Stroboscopic measurement at an angular speed $\omega = 900$ rev/min indicates that the block passes its static equilibrium position with an upward velocity at the instant when the eccentric masses are at their highest position. The amplitude of vertical displacement at this speed is 8.5 mm. The total mass of the system is $m = 200$ kg and the rotating imbalance of each rotor is 0.5 kg-m. Determine

(a) The natural frequency of the system.

(b) The damping constant c.

(c) For the case where $\Omega = 1000$ rev/min, the amplitude of the vertical displacement of the mass and the angular position of the eccentric masses when the block passes its equilibrium position.



EXERCISES 3.23, 3.24

2.6.5.1 Part(a)

(note: total mass of system includes the small unbalanced masses) Since static deflection is $8.5mm$, then

$$\begin{aligned} \frac{Mg}{k} &= 0.0085 \\ k &= \frac{Mg}{0.0085} \end{aligned}$$

But $\omega_n^2 = \frac{k}{M} = \frac{Mg}{0.0085M} = \frac{g}{0.0085}$, hence $\omega_n = \sqrt{\frac{9.81}{0.0085}} = 33.972$ rad/sec or 5.4068 Hz

2.6.5.2 Part(b)

The equation of motion is (angle Ω is now measured from horizontal, anti-clock wise positive)

$$My'' + cy' = 2m\varepsilon\Omega^2 \sin(\Omega t) = \text{Re}\left(\frac{1}{i}2m\varepsilon\Omega^2 e^{i(\Omega t)}\right)$$

Let $y(t) = \text{Re}\left(\frac{1}{i}Ye^{i\Omega t}\right)$ hence $y'(t) = \text{Re}\left(Y\Omega e^{i\Omega t}\right)$,$y''(t) = \text{Re}\left(iY\Omega^2 e^{i\Omega t}\right)$, hence the above becomes

$$\begin{aligned}
\text{Re}\left(iY\Omega^2 e^{i\Omega t}\right) + \frac{c}{M}\text{Re}\left(Y\Omega e^{i\Omega t}\right) &= \text{Re}\left(\frac{1}{i}\frac{2m\varepsilon\Omega^2}{M}e^{i\Omega t}\right)\\
\text{Re}\left(\left(i\Omega^2 + \frac{c\Omega}{M}\right)Ye^{i\Omega t}\right) &= \text{Re}\left(\frac{1}{i}\frac{2m\varepsilon\Omega^2}{M}e^{i\Omega t}\right)\\
\left[i\Omega^2 + \frac{c\Omega}{M}\right]Y &= \frac{1}{i}\frac{2m\varepsilon\Omega^2}{M}\\
Y &= \frac{1}{i}\frac{\frac{2m\varepsilon\Omega^2}{M}}{i\Omega^2 + \frac{c\Omega}{M}}\\
&= \frac{2m\varepsilon\Omega^2}{ic\Omega - M\Omega^2}
\end{aligned}$$

Hence

$$\begin{aligned}
y_{ss}(t) &= \text{Re}\left(\frac{1}{i}Ye^{i(\Omega t)}\right)\\
&= \text{Re}\left(\frac{1}{i}\frac{2m\varepsilon\Omega^2}{ic\Omega - M\Omega^2}e^{i(\Omega t)}\right)
\end{aligned}$$

Now we are told when $\Omega t = \frac{\pi}{2}$ (upright position) then $y = 0$(since it passes static equilibrium). At this moment $\Omega = 2\pi\frac{900}{60} = \boxed{94.248 \text{ rad/sec}}$, At this moment the centripetal forces equal the damping force downwards (since the mass was moving upwards). Hence

$$\boxed{m\varepsilon\Omega^2 = cy'(t)}$$

But from above we found that

$$\begin{aligned}
y'(t) &= \text{Re}\left(\frac{2m\varepsilon\Omega^2}{ic\Omega - M\Omega^2}\Omega e^{i\Omega t}\right)\\
&= \text{Re}\left(\frac{(0.5)94.248^2}{ic(94.248) - 200(94.248)^2}94.248e^{i\frac{\pi}{2}}\right)\\
&= \text{Re}\left(\frac{8.3718\times 10^5}{94.248ic - 1.7765\times 10^6}e^{i\frac{\pi}{2}}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
m\varepsilon\Omega^2 &= c\left|y'(t)\right|\\
(0.5)94.248^2 &= c\frac{8.3718\times 10^5}{\sqrt{(94.248c)^2 + \left(1.7765\times 10^6\right)^2}}\\
4441.3 &= 8.3718\times 10^5\frac{c}{\sqrt{8882.7c^2 + 3.1560\times 10^{12}}}
\end{aligned}$$

Solving numerically for c gives

$$c = 1.0882\times 10^4 \text{ N second per meter}$$

2.6.5.3 Part(c)

When $\Omega = \left(2\pi\frac{1000}{60}\right) = 104.72$ rad/sec or 16.667 Hz. From

$$
\begin{aligned}
y &= \mathrm{Re}\left(\frac{1}{i}\frac{2m\varepsilon\Omega^2}{ic\Omega - M\Omega^2}e^{i(\Omega t)}\right) \\
&= \mathrm{Re}\left(\frac{2(0.5)(104.72)^2}{i\left(1.0882\times10^4\right)104.72 - 200(104.72)^2}e^{i\left(104.72t-\frac{\pi}{2}\right)}\right) \\
&= \mathrm{Re}\left(\frac{10966.}{i1.1396\times10^6 - 2.1933\times10^6}e^{i\left(104.72t-\frac{\pi}{2}\right)}\right) \\
|y| &= 4.4 \text{ mm}
\end{aligned}
$$

2.6.6 Key solution for HW 5

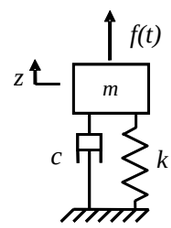
Homework #5, EMA 545, Spring 2013
Due Thursday 2/28

Comment: As I mentioned in class, I strongly encourage you to avoid hunting for formulas on forced response. All of these problems can be solved simply by knowing the differential equation and that the force and steady state response have the form:

$$f(t) = \mathrm{Re}\left(Fe^{i\omega t}\right) \qquad \rightarrow \qquad x(t) = \mathrm{Re}\left(Xe^{i\omega t}\right)$$

Problem 1: 3.9 from Ginsberg (Hint: assume that the motion of every component of the system is harmonic. Derive the equation(s) of motion and show the full derivation used to obtain the complex amplitude(s) from the equation(s) of motion.)

Problem 2: (3e1) (**20 points**)
A 450 kg generator, modeled as a rigid mass, must be installed on the same floor as some sensitive laboratory equipment. The operation of the generator results in a vertical force, $f(t)$, being applied to the generator (rigid mass) whose amplitude is 20kN and whose frequency is 1800 rpm. Use a damping ratio of $\zeta = 0.03$ for both (a) and (b) below.

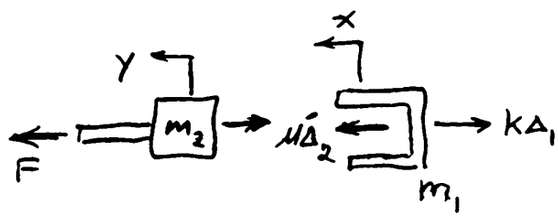


a.) Find the stiffness of the support, k, such that the force transmitted to the ground is no more than 2kN.
b.) Take your result from part (a) and compute the amplitude of the generator as the machine starts up. (As the machine starts up, assume that the force amplitude is constant at 20kN, but the frequency increases very slowly from zero to 1800 rpm. Do a worst-case analysis – just assure that the steady-state amplitude of the machine is less than 10mm for any forcing frequency between 0 and 1800 rpm.)
c.) Using your results from (a) and (b), suppose that the startup amplitude must not exceed 10mm. The startup amplitude can be decreased by adding mass to the generator while also increasing the stiffness of the support to keep the natural frequency of the system constant. How much mass must be added to keep the amplitude below 10mm?

Problem 3: 3.19 from Ginsberg

Problem 4: 3.23 from Ginsberg

Exercise 3.9



Given $y = A\sin\omega t$,
$m_1 = 0.5$ kg,
$m_2 = 1.0$ kg,
$k = 3.2(10^3)$ N/m,
$\mu = 40$ N-s/m,
$A = 0.02$ meter

Find amplitude & phase of F relative to y for $\omega = 150\pi$ & 170π.

Solution: Eqs of motion: $\Delta_1 = x$, $\Delta_2 = y - x$

Piston: $F - \mu(\dot{y} - \dot{x}) = m_2 \ddot{y}$

Tube: $\mu(\dot{y} - \dot{x}) - kx = m_1 \ddot{x}$

It is given that $y = A\sin(\omega t) = Re\left[\frac{A}{i} e^{i\omega t}\right]$

Harmonic variation $\Rightarrow$ Assume $x = Re\left[\frac{X}{i} e^{i\omega t}\right]$, $F = Re\left[\frac{\hat{F}}{i} e^{i\omega t}\right]$

Note: Using $\frac{1}{i}$ as a factor for x & F means that the polar angles of $\hat{F}$ and X are phase angles relative to y.

Substitute into eqs of motion & cancel $e^{i\omega t}$:

$$\hat{F} - \mu(i\omega)(A - X) = -m_2\omega^2 A \qquad (1)$$

$$\mu(i\omega)(A - X) - kX = -m_1\omega^2 X \qquad (2)$$

These are two simultaneous equations for F and X at any ω because $A = 0.02$:

From (2): $X = \dfrac{i\omega\mu}{k + i\omega\mu - m_1\omega^2} A$

From (1):

$$\hat{F} = \left[(i\omega\mu - m_2\omega^2) + \frac{\mu^2\omega^2}{k + i\omega\mu - m_1\omega^2}\right] A$$

For $\omega = 75$ rad/s:

$F = -104.88 + 0.99i = 104.88 e^{i\,3.132}$

or $|F| = 104.88$ N @ $\phi = 3.132 \frac{180}{\pi} = 179.46°$ ahead of y $\Leftarrow$

For $\omega = 85$ rad/s:

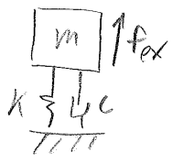
$F = -152.63 + 0.99i = 152.63 e^{3.135i}$

or $|F| = 152.63$ N @ $\phi = 3.135 \frac{180}{\pi} = 179.63°$ ahead of y $\Leftarrow$

HW 7 Solution V2 - new damping

P3e-01) Generator

$m = 450\,kg$ $|f| = 20\,kN$ @ 1800 rpm



$\zeta = 0.03$

EOM:

$$m\ddot{x} + c\dot{x} + kx = f_{ex} \qquad (1)$$

$$f_{tr} = c\dot{x} + kx \qquad (2)$$

from (1)

$$\boxed{X = \frac{F_{ex}/m}{\omega_n^2 - \omega^2 + i\omega \cdot 2\zeta\omega_n}}$$

(since $f_{ex} = Re(F_{ex} e^{i\omega t})$)

$$F_{tr} = (i\omega c + k)\, X$$

$$F_{tr} = \frac{(i\omega c + k)/m \cdot F_{ex}}{\cdots}$$

$$F_{tr} = \frac{(i2\zeta\omega_n\omega + \omega_n^2)}{(\omega_n^2 - \omega^2 + i2\zeta\omega\omega_n)} F_{ex}$$

define $r = \omega/\omega_n$

$$\boxed{F_{tr} = \frac{(i2\zeta r + 1)}{(1 - r^2 + i2\zeta r)} F_{ex}}$$

a) $|F_{tr}| = 0.1|F_{ex}|$ $\quad 0.1 = \frac{\sqrt{(1 + 4\zeta^2 r^2)}}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}$

$$0.01\left[(1 - 2r^2 + r^4) + 4\zeta^2 r^2\right] = 1 + 4\zeta^2 r^2$$

$$0.01 r^4 + (0.01 \cdot 4\zeta^2 - 0.02 - 4\zeta^2) r^2 - 0.99 = 0$$

polynomial solver, or plot $\rightarrow r^2 = 11.198$

$\frac{\omega}{\omega_n}$ $\quad r \geq 3.3463$

$\omega = 1800 \frac{rev}{min} \cdot \frac{1\,min}{60\,sec} \cdot \frac{2\pi\,rad}{rev} = 60\pi\ rad/s \approx 188.5\ rad/s$

$\omega_n \leq \frac{60\pi}{3.35} \rightarrow \omega_n \leq 56.33\ rad/s$ $\quad \omega_n = \sqrt{\frac{K}{m}}$

$K \leq \omega_n^2 m \rightarrow \boxed{K \leq 1.428 \times 10^6\ N/m}$

b) $X = \frac{20{,}000\ N / 450\ kg}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$ $\Rightarrow$ Peak @ $\omega = \omega_n\sqrt{1 - 2\zeta^2}$

let $\omega = 56.33\sqrt{1 - 2 \cdot 0.03^2} \rightarrow X = 0.007 - 0.233i$

P3e-01 sol.

b) continued

$abs(X) = 0.234\ m$

exceeds 10mm → add mass

* assume Damping ratio unchanged,

$$X^{new} = \frac{F/m'}{W_n^2 - w^2 + jw \cdot 2\zeta w_n} = X \cdot \frac{m}{m'}$$

$$\frac{m}{m'} = \frac{10mm}{234mm} \rightarrow m' = \frac{234}{10}\ m$$

$$\boxed{m' = 10{,}510\ kg \sim \text{or add } 10{,}060\ kg}$$

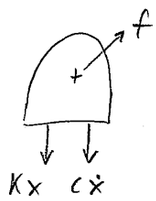
New stiffness:

$$\boxed{K^{new} = W_n^2 \cdot m' = 3.33 \times 10^7\ N/m}$$

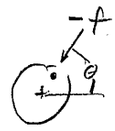
$$1/s^2 \cdot kg = kg/s^2 = kg\text{-}m/s^2 \cdot 1/m = N/m \checkmark$$

3.19 — 1 —

$m = 80\,Kg$ ~ Motor + rotor $\qquad m_1 = $ motor $\quad m_2 = $ rotor



$= m_1 \ddot{x}$



$= m_2 \vec{a}_g$

$\vec{a}_g = \ddot{x}\hat{j} - \varepsilon\Omega^2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$-f = m_2\ddot{x}\hat{j} - m_2\varepsilon\Omega^2(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$m_1\ddot{x} + c\dot{x} + Kx = f\cdot\hat{j}$

$= -m_2\ddot{x} + m_2\varepsilon\Omega^2\sin\theta \qquad \theta = \Omega t$

$$\boxed{(m_1+m_2)\ddot{x} + c\dot{x} + Kx = m_2\varepsilon\Omega^2\sin(\Omega t)} \qquad EOM$$

Note: pos of rotor & let $F = -i m_2\varepsilon\Omega^2 \rightarrow f(t) = Re(F e^{i\omega t})$ — $f(t) = 0$ when $t = 0$

- Steady state: $x(t) = Re(X e^{i\Omega t})$

$$X = \frac{-i m_2 \varepsilon \Omega^2}{-(m_1+m_2)\Omega^2 + c(i\Omega) + K}$$

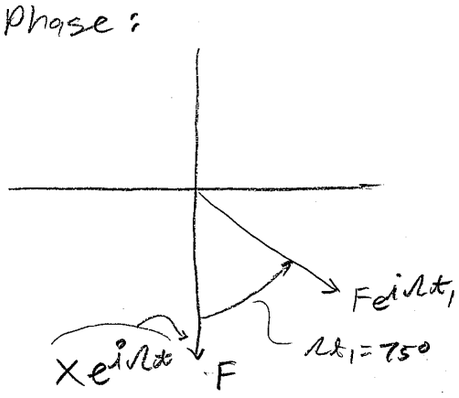
given $\rightarrow$ static deflection due to $F = 80\,Kg \cdot 9.81\,m/s^2$ is $40\,mm$

$f = Kx \longrightarrow K = \dfrac{80\,Kg \cdot 9.81\,m/s^2}{0.040\,m} = 19{,}620\ N/m$

$m_1 + m_2 = 80\,Kg$

unknowns: ξ and m_2

Phase:



Given @ t_1, $\theta = 75°$

Note: $\theta = 0$, $f(t) = 0$

$\theta = 90°$ $f(t) = +m_2\varepsilon\Omega^2$

So, The angle below is $75°$

Similarly, $x(t)$ is zero and increasing at t_1,

$x(t) = Re(X e^{i\Omega t_1})$

so it must also be purely imaginary

In other words, it is lagging the force by $75°$

3.19 cont. — 2 —

so,

$$\text{angle}(Xe^{i\Omega t_1}) = \text{angle}\left(\frac{-i m_2 \varepsilon \Omega^2 e^{i\Omega t_1}}{-(m_1+m_2)\Omega^2 + i\Omega c + K}\right)$$

Ωc $\angle$den

$K-(m_1+m_2)\Omega^2$

$-i = e^{-\pi/2}$ $\angle$num − $\angle$den

$$-\cancel{90^\circ} = -\cancel{90} + 75^\circ - \text{atan}\left(\frac{\Omega c}{K-(m_1+m_2)\Omega^2}\right)$$

$$K-(m_1+m_2)\Omega^2 \tan(75^\circ) = \Omega c$$

$$(\omega_n^2 - \Omega^2)\tan 75 = \Omega \cdot 2\zeta\omega_n$$

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{19620}{80}}$$

$$\omega_n = 15.66 \text{ rad/s}$$

$$\zeta = \frac{\omega_n^2 - \Omega^2}{\Omega \cdot 2 \cdot \omega_n} \cdot \tan(75^\circ)$$

a) $\boxed{\zeta = 0.1152}$

$$|X| = 0.010 \quad @ \quad \Omega = \frac{145 \cdot 2\pi}{60}$$

$$|X| = m_2 \varepsilon \left| \frac{-i\Omega^2}{-(m_1+m_2)\Omega^2 + ic\Omega + K} \right| \qquad c = (m_1+m_2)2\zeta\omega_n$$

$$0.010 = m_2 \varepsilon \cdot 0.0508$$

b) $\boxed{m_2 \varepsilon \approx \frac{0.010}{0.0508} = 0.1969}$

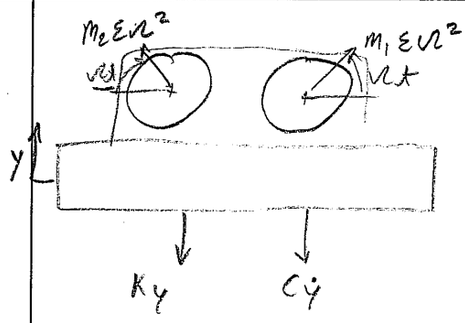
c) as $\Omega \to \infty$ X on prev page becomes

$$X\Big|_{\Omega\to\infty} = \frac{-i m_2 \varepsilon \cancel{\Omega^2}}{-(m_1+m_2)\cancel{\Omega^2}}$$

$$\boxed{|X|_{\Omega\to\infty} = \frac{m_2\varepsilon}{(m_1+m_2)} \approx \frac{0.1969}{80} \approx 0.00246 \text{ m}}$$

(Min of $|X|$ occurs at $\Omega = \infty$ if $\Omega > \omega_n$)

3.23 — 1 —



Given: $\Omega = 900\,rpm = \frac{900 \cdot 2\pi}{60}\,rad/s$

- Block passes static eq. with $\dot{y} > 0$

$|Y| = 8.5\,mm$, $m_{tot} = 200\,kg$

$m_1 \varepsilon = 0.5\,kg\text{-}m$

a) find ω_n b) find c

c) find $|Y|$ and angle of rotors at $\Omega = 1000\,rpm$.

Power Balance: (Beam mass negligible → spring)

$T = \frac{1}{2} m_{tot} \dot{y}^2 \quad V = \frac{1}{2} k y^2 \quad P_{diss} = c\dot{y}^2$

$P_{in} = m_1 \varepsilon \Omega^2 (\cos \Omega t\, \hat{i} + \sin \Omega t\, \hat{j}) \cdot \dot{y}\hat{j} +$

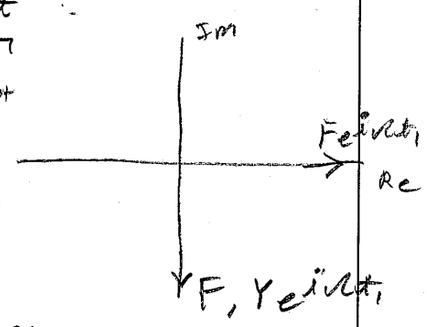
$+ m_1 \varepsilon \Omega^2 (-\cos \Omega t\, \hat{i} + \sin \Omega t\, \hat{j}) \cdot \dot{y}\hat{j}$

$= \underbrace{2 m_1 \varepsilon \Omega^2 \sin \Omega t}_{Q}\, \dot{y}$

$m_{tot} \ddot{y} + c\dot{y} + ky = 2 m_2 \varepsilon \Omega^2 \sin \Omega t$

$y(t) = Re(Y e^{i\Omega t}) \quad , \omega_n = \sqrt{\frac{k}{m_{tot}}}$

$$Y = \frac{2 m_2 \varepsilon \Omega^2}{k + i\Omega c - \Omega^2 m_{tot}}$$



let $2 m_2 \varepsilon \Omega^2 \sin \Omega t = Re(F e^{i\Omega t})$

$F = -i 2 m_2 \varepsilon \Omega^2$

from description: @t_1 masses at highest pos,

so $Fe^{i\Omega t_1}$ is real, $\Omega t_1 = 90° + n360°$, n = any integer

$y(t) = 0$ and increasing, so $Ye^{i\Omega t_1}$ = purely imaginary

from above — only possible if $\underline{\Omega^2 = \omega_n}$

$$|Y|_{\Omega = 30\pi} = \left| \frac{2 m_2 \varepsilon \Omega^2}{i \Omega c} \right|$$

$$c = \frac{2 m_2 \varepsilon \Omega}{|Y|} = \frac{2 \cdot 0.5\,kg\text{-}m \cdot 30\pi\,rad/s}{0.0085\,m}$$

b) $\boxed{c = 11{,}088\,kg/s}$ a) $\boxed{\omega_n = \Omega = 30\pi\,rad/s}$

$\zeta = 0.2941 \rightarrow$ heavily damped

3.23 cont — 2 —

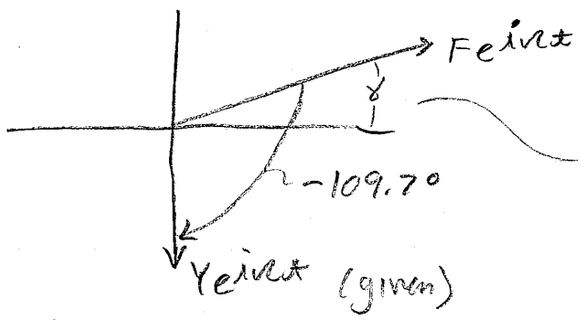
$\Omega = 1000\,rpm = \frac{1000}{60} \cdot 2\pi$ rad/s

$$Y = \frac{2 m_2 \varepsilon \Omega^2}{m + e_t}\left(\frac{1}{\omega_n^2 + i\Omega \cdot 2\zeta\omega_n - \Omega^2}\right)$$

matlab

$$\boxed{Y = -0.00300 - 0.00837 = 0.00889 \measuredangle -109.7^\circ}$$

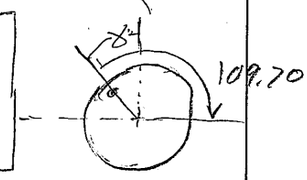
Lags force by 109.7°,



γ = angle of masses past vertical

$\gamma = 29.7^\circ$

$$\boxed{|Y| = 8.89\,mm, \quad \Omega t = 109.7^\circ \text{ or } \gamma = 29.7^\circ}$$



2.7 HW6

Local contents

2.7.1 problem description

Homework #6
EMA 545, Spring 2013

1.) **Problem 3.41** in Ginsberg. Check your answer for λ=1.0 using FFT techniques with the fft_easy.m Matlab function from the course website.

2.) **Problem 3.50** in Ginsberg. DO PART (a) ONLY.

3.) (**20 points**) Find the steady-state response of the system in **Problems 3.45** and **3.46** from Ginsberg using FFT techniques. Perform your analysis with $\tau = \pi/(3\omega_n)$ as stated in the problem <u>and also repeat the analysis for</u> $\tau = 3\pi/\omega_n$. Which harmonic is dominant in the response in each case? Why? Create a plot of the steady-state displacement for each case.

2.7.2 problem 1

3.41 in text: A periodic disturbance consists of a sequence of exponentially pulse repeated at intervals T, such that $Q(t) = Fe^{\frac{-\lambda t}{T}}$ for $0 < t < T$, and $Q(t \pm T) = Q(t)$. The parameter λ is nondimensional. Determine the complex Fourier series representing the force. Evaluate the first 5 coefficients when $\lambda = 0.1, 1, 10$. What does this reveal regarding the influence of λ on the frequency spectrum?

Let $\tilde{Q}(t)$ be the Fourier series approximation to $Q(t)$ given by

$$\tilde{Q}(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} F_n e^{in\frac{2\pi}{T}t} \tag{2.102}$$

Where

$$\begin{aligned}
F_n &= \frac{2}{T}\int_0^T Q(t)e^{-in\frac{2\pi}{T}t}dt \\
&= \frac{2}{T}\int_0^T Fe^{\frac{-\lambda t}{T}}e^{-in\frac{2\pi}{T}t}dt = \frac{2F}{T}\int_0^T e^{-t\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}dt = \frac{2F}{T}\left(\frac{e^{-t\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}}{in\frac{2\pi}{T}-\frac{\lambda}{T}}\right)_0^T \\
&= \frac{2F}{in2\pi-\lambda}\left(e^{-T\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}-1\right) \\
&= \frac{2F}{in2\pi-\lambda}\left(e^{-in2\pi}e^{-\lambda}-1\right)
\end{aligned}$$

But $e^{-in2\pi} = 1$, hence

$$F_n = \frac{2F}{in2\pi-\lambda}\left(e^{-\lambda}-1\right)$$

Hence Eq 2.102 becomes

$$\begin{aligned}
\tilde{Q}(t) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{2F}{in2\pi-\lambda}\left(e^{-\lambda}-1\right)e^{in\frac{2\pi}{T}t} \\
&= F\sum_{n=-\infty}^{\infty}\frac{\left(e^{-\lambda}-1\right)}{in2\pi-\lambda}e^{in\frac{2\pi}{T}t} \\
&= F\sum_{n=-\infty}^{\infty}\frac{1-e^{-\lambda}}{\lambda+in2\pi}e^{in\frac{2\pi}{T}t}
\end{aligned}$$

For $n = -2, -1, 0, 1, 2$ we obtain

$$\begin{aligned}
\tilde{Q}(t) &= F\sum_{n=-2}^{2}\frac{1-e^{-\lambda}}{\lambda+in2\pi}e^{in\frac{2\pi}{T}t} \\
&= F\left(\frac{1-e^{-\lambda}}{\lambda-i4\pi}e^{-i\frac{4\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda-i2\pi}e^{-i\frac{2\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda}+\frac{1-e^{-\lambda}}{\lambda+i2\pi}e^{i\frac{2\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda+i4\pi}e^{i\frac{4\pi}{T}t}\right)
\end{aligned}$$

For $\lambda = 0.1$

$$\begin{aligned}
\tilde{Q}(t) &= F\left(\frac{1-e^{-0.1}}{0.1-i4\pi}e^{-i\frac{4\pi}{T}t}+\frac{1-e^{-0.1}}{0.1-i2\pi}e^{-i\frac{2\pi}{T}t}+\frac{1-e^{-0.1}}{0.1}+\frac{1-e^{-0.1}}{0.1+i2\pi}e^{i\frac{2\pi}{T}t}+\frac{1-e^{-0.1}}{0.1+i4\pi}e^{i\frac{4\pi}{T}t}\right) \\
&= F\{\left(6.026\times10^{-5}+7.572\times10^{-3}i\right)e^{-i\frac{4\pi}{T}t} \\
&+ \left(2.41\times10^{-4}+1.514\times10^{-2}i\right)e^{-i\frac{2\pi}{T}t} \\
&+ 0.952 \\
&+ \left(2.4099\times10^{-4}-1.5142\times10^{-2}i\right)e^{i\frac{2\pi}{T}t} \\
&+ \left(6.026\times10^{-5}-7.572\times10^{-3}i\right)e^{i\frac{4\pi}{T}t}\}
\end{aligned}$$

For $\lambda = 1$

$$\begin{aligned}\tilde{Q}(t) &= F\left(\frac{1-e^{-1}}{1-i4\pi}e^{-i\frac{4\pi}{T}t} + \frac{1-e^{-1}}{1-i2\pi}e^{-i\frac{2\pi}{T}t} + \frac{1-e^{-1}}{1} + \frac{1-e^{-1}}{1+i2\pi}e^{i\frac{2\pi}{T}t} + \frac{1-e^{-1}}{1+i4\pi}e^{i\frac{4\pi}{T}t}\right)\\ &= F\{(0.00398+0.05i)e^{-i\frac{4\pi}{T}t} + (0.016+0.098i)e^{-i\frac{2\pi}{T}t} + 0.632 + (0.016+0.098i)e^{i\frac{2\pi}{T}t} + (0.00398+0.05i)e^{i\frac{4\pi}{T}}\end{aligned}$$

For $\lambda = 10$

$$\begin{aligned}\tilde{Q}(t) &= F\left(\frac{1-e^{-10}}{10-i4\pi}e^{-i\frac{4\pi}{T}t} + \frac{1-e^{-10}}{10-i2\pi}e^{-i\frac{2\pi}{T}t} + \frac{1-e^{-10}}{10} + \frac{1-e^{-10}}{10+i2\pi}e^{i\frac{2\pi}{T}t} + \frac{1-e^{-10}}{10+i4\pi}e^{i\frac{4\pi}{T}t}\right)\\ &= F\{\left(3.877\times10^{-2}+4.872\times10^{-2}i\right)e^{-i\frac{4\pi}{T}t}\\ &+ \left(7.169\times10^{-2}+4.505\times10^{-2}i\right)e^{-i\frac{2\pi}{T}t}\\ &+ 0.1\\ &+ \left(7.169\times10^{-2}-4.505\times10^{-2}i\right)e^{i\frac{2\pi}{T}t}\\ &+ \left(3.877\times10^{-2}-4.872\times10^{-2}i\right)e^{i\frac{4\pi}{T}t}\}\end{aligned}$$

We notice that as λ became larger, the DC term became smaller. Since the DC term represents average value of the whole signal, then we can say that as λ gets larger, then the average becomes smaller. This means the energy of the signal becomes smaller as λ becomes larger.

2.7.2.1 Verification using Matlab ffteasy.m

From above, we found for $\lambda = 1$

$$\begin{aligned}F_n &= \frac{2F}{in2\pi-\lambda}\left(e^{-\lambda}-1\right)\\ &= \frac{2F}{in2\pi-1}\left(e^{-1}-1\right)\end{aligned}$$

and the first 5 found to be

n	F_n
-2	$0.00398+0.05i$
-1	$0.016+0.098i$
0	0.632
1	$0.016-0.098i$
2	$0.00398-0.05i$

To verify the result with ffteasy.m using $\lambda = 1$, Using $F = 1$, and using $T = 1$. This below shows the result for F_0, F_1, F_2 and we see that the DC term F_0 agrees, and that complex component of F_1, F_2 also agrees. The real parts are little larger than what I obtained using the above. This might be a scaling issue, and I was not able to determine the reason for it at this time.

```
EDU>> T=1; del=0.01; t=0:del:T; lambda=1; xt=exp(-lambda*t/T);
EDU>> (1/length(t))*fft_easy(xt,t)

ans =

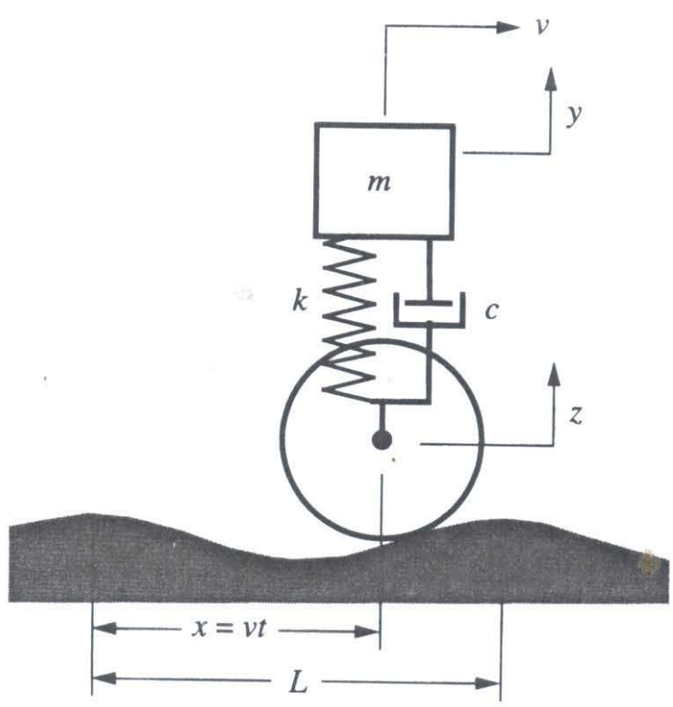
   0.6326 + 0.0000i
   0.0190 - 0.0986i
   0.0072 - 0.0502i
```

2.7.3 problem 2

3.50 The sketch depicts a one-degree-of-freedom model of an automobile traveling to the right at constant speed v when the road is not smooth. The mass is 1200 kg, the natural frequency of the system is 5 Hz, and the critical damping ratio is 0.4. The elevation of a certain road is a sequence of periodic 50 mm high bumps spaced at a distance of 4 m, specifically, $z = (x - 5x^2)$ if $0 < x < 0.2$ m, $z = 0$ if $0.2 < x < 4$ m, $z(x + 4) = z(x)$.

(a) What speeds v would cause the vertical displacement y to be resonant if the dashpot were not present?

(b) Determine the steady-state displacement y when $v = 5$ m/s.



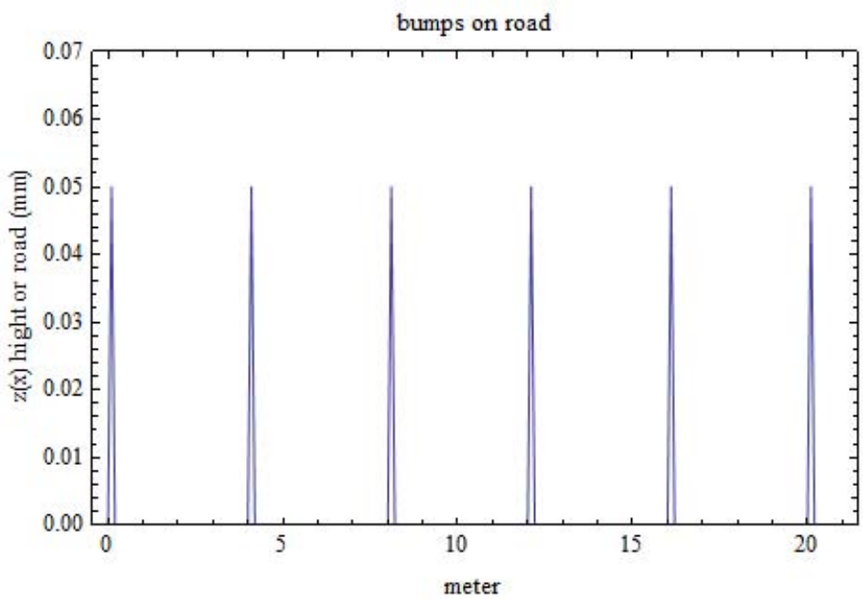
We are given that $m = 1200$ kg, $f = 5$ Hz, $\zeta = 0.4$ and

$$z(x) = \begin{cases} x - 5x^2 & 0 < x < 0.2 \\ 0 & 0.2 < x < 4 \end{cases}$$

A plot of $z(x)$ for first 20 meters is

```
z[x_] := Piecewise[{{x - 5 x^2, 0 <= x < 0.2}, {0, 0.2 <= x <= 4}}]
z[x_] /; x > 4 := z[Mod[x, 4]];
Table[{x, z[x]}, {x, 0, 21, .1}];
```

```
ListLinePlot[%, PlotRange -> {All, {0, .07}}, Frame -> True,
 FrameLabel -> {{"z(x) hight or road (mm)", None}, {"meter",
    "bumps on road"}}]
```



We need to be able to express $z(t)$ as $\mathrm{Re}\left\{Ze^{i\frac{2\pi}{T}t}\right\}$ where T is the period of the function $z(t)$. Hence we need to represent $z(x)$ as Fourier series approximation then replace $x = vt$ and use the result.

The period $T = 4$ meter. Let $\tilde{z}(x)$ be the Fourier series approximation to $z(x)$, hence

$$\tilde{z}(x) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)$$

Where

$$F_n = \frac{2}{T}\int_0^T z(x)e^{-in\frac{2\pi}{T}x}dx = \frac{1}{2}\int_0^{2/10}\left(x - 5x^2\right)e^{-in\frac{\pi}{2}x}dx = \frac{1}{2}\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx - \frac{5}{2}\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx$$

Using integration by parts $\int udv = uv - \int vdu$, letting $u = x$ and $dv = e^{-in\frac{\pi}{2}x}$ then

$v = \int e^{-in\frac{\pi}{2}x}dx = \frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}$hence

$$\begin{aligned}
\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx &= x\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}\Bigg|_0^{\frac{2}{10}} - \int_0^{2/10}\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}dx \\
&= \frac{2}{10}\frac{ie^{-in\frac{\pi}{2}\frac{2}{10}}}{n\frac{\pi}{2}} - \frac{2}{n\pi}\int_0^{2/10} ie^{-in\frac{\pi}{2}x}dx \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} - \frac{i2}{n\pi}\left(\frac{e^{-in\frac{\pi}{2}x}}{-in\frac{\pi}{2}}\right)_0^{\frac{2}{10}} \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} + \frac{4}{n^2\pi^2}\left(e^{-in\frac{\pi}{2}x}\right)_0^{\frac{2}{10}} \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} + \frac{4}{n^2\pi^2}\left(e^{-in\frac{\pi}{2}\frac{2}{10}} - 1\right) \\
&= \frac{4i}{10n\pi}e^{-in\frac{\pi}{10}} + \frac{4}{n^2\pi^2}e^{-in\frac{\pi}{10}} - \frac{4}{n^2\pi^2} \\
&= e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2} + \frac{2i}{5n\pi}\right) - \frac{4}{n^2\pi^2}
\end{aligned}$$

Now we do the second integral $\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx$.

Integration by parts, $\int udv = uv - \int vdu$, letting $u = x^2$ and $dv = e^{-in\frac{\pi}{2}x}$ then $v = \frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}$ hence

$$\begin{aligned}
\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx &= \left[x^2\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}\right]_0^{\frac{2}{10}} - \int_0^{2/10} 2x\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}dx \\
&= \frac{8}{100}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} - \frac{4i}{n\pi}\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx
\end{aligned}$$

But $\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx$ was solved before and its results is Eq 2.1, hence

$$\begin{aligned}
\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx &= \frac{4}{100}\frac{ie^{-in\frac{\pi}{10}}}{n\frac{\pi}{2}} - \frac{4i}{n\pi}\left(e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2} + \frac{2i}{5n\pi}\right) - \frac{4}{n^2\pi^2}\right) \\
&= \frac{8i}{100n\pi}e^{-in\frac{\pi}{10}} - e^{-in\frac{\pi}{10}}\left(\frac{16i}{n^3\pi^3} - \frac{8}{5n^2\pi^2}\right) + \frac{16i}{n^3\pi^3} \\
&= e^{-in\frac{\pi}{10}}\left(\frac{8i}{100n\pi} - \frac{16i}{n^3\pi^3} + \frac{8}{5n^2\pi^2}\right) + \frac{16i}{n^3\pi^3}
\end{aligned}$$

Putting all the above together, we obtain F_n as

$$
\begin{aligned}
F_n &= \frac{1}{2}\int_0^{2/10} x e^{-in\frac{\pi}{2}x}dx - \frac{5}{2}\int_0^{2/10} x^2 e^{-in\frac{\pi}{2}x}dx \\
&= \frac{1}{2}\left[e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2}+\frac{2i}{5n\pi}\right)-\frac{4}{n^2\pi^2}\right]-\frac{5}{2}\left[e^{-in\frac{\pi}{10}}\left(\frac{8i}{100n\pi}-\frac{16i}{n^3\pi^3}+\frac{8}{5n^2\pi^2}\right)+\frac{16i}{n^3\pi^3}\right] \\
&= e^{-in\frac{\pi}{10}}\left(\frac{2}{n^2\pi^2}+\frac{i}{5n\pi}\right)-\frac{2}{n^2\pi^2}-e^{-in\frac{\pi}{10}}\left(\frac{20i}{100n\pi}-\frac{40i}{n^3\pi^3}+\frac{20}{5n^2\pi^2}\right)-\frac{40i}{n^3\pi^3} \\
&= e^{-in\frac{\pi}{10}}\left[\frac{2}{n^2\pi^2}+\frac{i}{5n\pi}-\frac{20i}{100n\pi}+\frac{40i}{n^3\pi^3}-\frac{4}{n^2\pi^2}\right]-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3} \\
&= e^{-in\frac{\pi}{10}}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3}
\end{aligned}
$$

Now

$$
F_0 = \frac{2}{T}\int_0^T z(x)dx = \frac{1}{2}\int_0^{2/10}\left(x-5x^2\right)dx = \frac{1}{300}
$$

Hence

$$
\begin{aligned}
\tilde{z}(x) &= \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right) \\
&= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{N}\left(e^{-in\frac{\pi}{10}}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3}\right)e^{in\frac{\pi}{2}x}\right) \\
&= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{N} e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-e^{in\frac{\pi}{2}x}\left(\frac{2}{n^2\pi^2}+\frac{40i}{n^3\pi^3}\right)\right) \\
&= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{N}\frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{in\frac{\pi}{2}x}+\frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi}{2}x}\right)
\end{aligned}
$$

But $x = vt$, hence

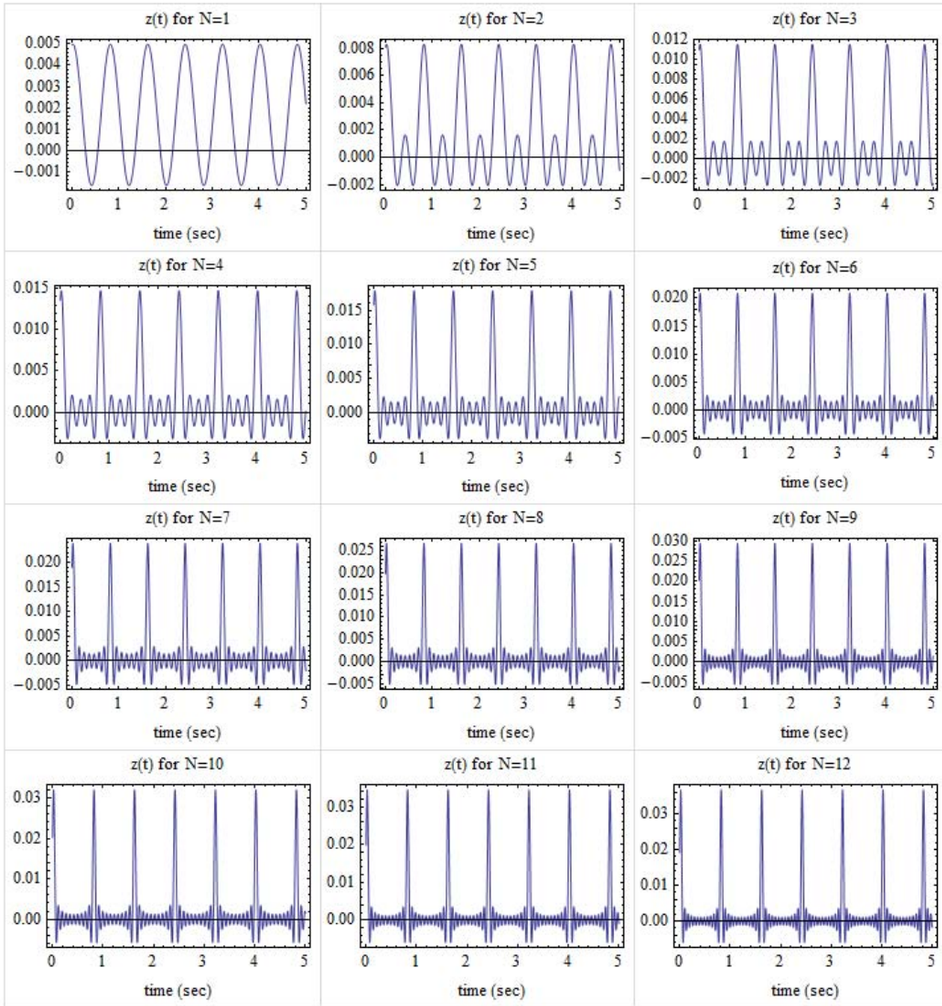
$$
\tilde{z}(t) = \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{N}\frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi v}{2}t-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi v}{2}t-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{in\frac{\pi v}{2}t}+\frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi v}{2}t}\right)
$$

Therefore the forcing frequency is $n\varpi_1 = n\frac{\pi v}{2}$ or from $2\pi f_1 = \frac{\pi v}{2}$, hence $f_1 = \frac{v}{4}$Hz.The above can be written as

$$
\begin{aligned}
\tilde{z}(t) &= \frac{1}{600} + \sum_{n=1}^{N}\operatorname{Re}\left(\frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi v}{2}t-\frac{n\pi}{10}\right)}\right)-\sum_{n=1}^{N}\operatorname{Re}\left(\frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi v}{2}t-\frac{n\pi}{10}\right)}\right) \\
&\quad -\sum_{n=1}^{N}\operatorname{Re}\left(\frac{2}{n^2\pi^2}e^{in\frac{\pi v}{2}t}\right)+\sum_{n=1}^{N}\operatorname{Re}\left(\frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi v}{2}t}\right) \\
&= \frac{1}{600} + \sum_{n=1}^{N}\frac{-40}{n^3\pi^3}\sin\left(n\varpi_1 t-\frac{n\pi}{10}\right)-\sum_{n=1}^{N}\frac{2}{n^2\pi^2}\cos\left(n\varpi_1 t-\frac{n\pi}{10}\right) \\
&\quad -\sum_{n=1}^{N}\frac{2}{n^2\pi^2}\cos(n\varpi_1 t)+\sum_{n=1}^{N}\frac{40}{n^3\pi^3}\sin(n\varpi_1 t) \\
&= \frac{1}{600} - \frac{40}{\pi^3}\sum_{n=1}^{N}\frac{1}{n^3}\sin\left(n\varpi_1 t-\frac{n\pi}{10}\right)-\frac{2}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2}\cos\left(n\varpi_1 t-\frac{n\pi}{10}\right) \\
&\quad -\frac{2}{\pi^2}\sum_{n=1}^{N}\frac{1}{n^2}\cos(n\varpi_1 t)+\frac{40}{\pi^3}\sum_{n=1}^{N}\frac{1}{n^3}\sin(n\varpi_1 t)
\end{aligned}
$$

Where $\varpi_1 = \frac{\pi v}{2}$

To verify the above, here is a plot for different number of fourier series terms showing that approximation improves as N increases. This was done for $v = 5m/s$ and for 5 seconds.



2.7.3.1 Part(a)

The equation of motion is

$$
\begin{aligned}
my'' + c\left(y' - z'\right) + k\left(y - z\right) &= 0 \\
my'' + cy' + ky &= cz' + kz
\end{aligned}
\tag{2.1}
$$

From earlier, we found that fourier series approximation to $z(t)$ is

$$
\begin{aligned}
z(t) &= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(n\omega t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{i\left(n\omega t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{in\omega t} + \frac{40}{n^3\pi^3}\frac{1}{i}e^{in\omega t}\right) \\
&= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{-40}{n^3\pi^3}e^{-i\frac{n\pi}{10}}\frac{1}{i}e^{in\omega t} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}}e^{in\omega t} - \frac{2}{n^2\pi^2}e^{in\omega t} + \frac{40}{n^3\pi^3}\frac{1}{i}e^{in\omega t}\right) \\
&= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} e^{in\omega t}\left[\frac{-40}{n^3\pi^3}e^{-i\left(\frac{n\pi}{10}+\frac{\pi}{2}\right)} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}} - \frac{2}{n^2\pi^2} + \frac{40}{n^3\pi^3}e^{-i\frac{\pi}{2}}\right]\right)
\end{aligned}
$$

Let

$$
Z_n = \frac{-40}{n^3\pi^3}e^{-i\left(\frac{n\pi}{10}+\frac{\pi}{2}\right)} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}} - \frac{2}{n^2\pi^2} + \frac{40}{n^3\pi^3}e^{-i\frac{\pi}{2}}
$$

Then above can be simplified to

$$
z(t) = \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} e^{in\omega t}Z_n\right)
$$

Where $\varpi = \frac{\pi v}{2}$, hence

$$z'(t) = \text{Re}\left(\sum_{n=1}^{\infty} in\varpi e^{in\varpi t} Z_n\right)$$

Hence, let

$$y_{ss}(t) = \text{Re} \sum_{n=1}^{\infty} Y_n e^{in\varpi t}$$

Hence Eq 2.1 becomes

$$\begin{aligned}
\sum_{n=1}^{\infty} - mn^2\varpi^2 Y_n e^{in\varpi t} + \sum_{n=1}^{\infty} icn\varpi Y_n e^{in\varpi t} + \sum_{n=1}^{\infty} kY_n e^{in\varpi t} &= \sum_{n=1}^{\infty} icn\varpi e^{in\varpi t} Z_n + \frac{k}{600} + \sum_{n=1}^{\infty} ke^{in\varpi t} Z_n \\
\sum_{n=1}^{\infty} \left(-mn^2\varpi^2 + icn\varpi + k\right) Y_n e^{in\varpi t} &= \sum_{n=1}^{\infty} (icn\varpi + k) Z_n e^{in\varpi t} + \frac{k}{600} \\
\sum_{n=1}^{\infty} \left(-mn^2\varpi^2 + icn\varpi + k\right) Y_n e^{in\varpi t} &= \frac{k}{600} + \sum_{n=1}^{\infty} (icn\varpi + k) Z_n e^{in\varpi t}
\end{aligned}$$

Hence

$$Y_n = \frac{(icn\varpi + k)}{-m(n\varpi)^2 + icn\varpi + k} Z_n \tag{2.103}$$

Let

$$\begin{aligned}
D(r_n, \zeta) &= \frac{icn\varpi + k}{-m(n\varpi)^2 + icn\varpi + k} \\
&= \frac{i2\zeta m\omega_{nat} n\varpi + \omega_{nat}^2 m}{-m(n\varpi)^2 + i2\zeta m\omega_{nat} n\varpi + \omega_{nat}^2 m} \\
&= \frac{i2\zeta n \frac{\varpi}{\omega_{nat}} + 1}{-\left(n\frac{\varpi}{\omega_{nat}}\right)^2 + i2\zeta n \frac{\varpi}{\omega_{nat}} + 1} \\
&= \frac{1 + i2\zeta r_n}{\left(1 - r_n^2\right) + i2\zeta r_n}
\end{aligned}$$

Where in the above $r_n = \frac{n\varpi}{\omega_{nat}}$ where ϖ is $\frac{2\pi}{T}$ which means it is the fundamental frequency of the forcing function and ω_{nat} is the natural frequency.

Then Eq 2.111 becomes

$$Y_n = D(r_n, \zeta) Z_n$$

And the steady state solution $y_{ss}(t)$ becomes

$$y_{ss}(t) = \frac{k}{600} + \text{Re}\left(\sum_{n=1}^{\infty} D(r_n, \zeta) Z_n e^{in\varpi t}\right)$$

Now we can answer the question. When $c = 0$ then $D(r_n, \zeta)$ reduces to $\frac{k}{-m(n\varpi)^2 + k} = \frac{1}{1 - \left(n\frac{\varpi}{\omega_{nat}}\right)^2} = \frac{1}{1 - r_n^2}$, hence

$$y_{ss}(t) = \frac{k}{600} + \text{Re}\left(\sum_{n=1}^{\infty} \frac{1}{1 - r_n^2} Z_n e^{in\varpi t}\right)$$

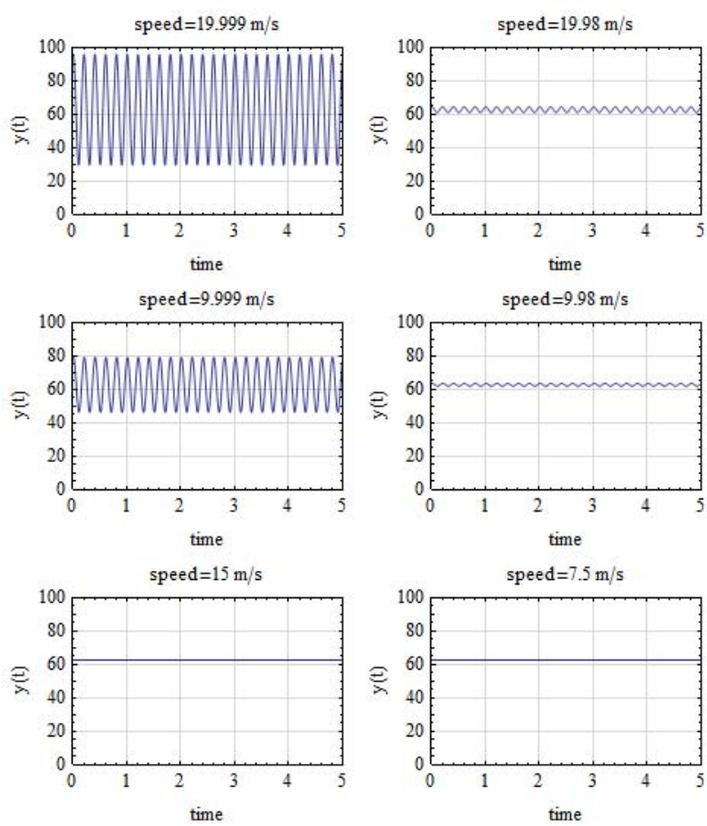
So the displacement $y_{ss}(t)$ will be resonant when $r_n = 1$ or $\frac{n\pi v}{2\omega_{nat}} = 1$ or $v = \frac{2\omega_{nat}}{n\pi}$

Hence

$$v = \frac{2(2\pi 5)}{n\pi} = \frac{20}{n}$$

Hence $v = 20, 10, 5, 2.5, 1.25, \cdots$ meter/sec will each cause resonance. To verify, here is a plot of $y_{ss}(t)$ with no damper for speed near resonance $v = 19.99$ and comparing this for speeds away from resonance speed. This plot shows that when speed v is close to any of the above speeds, then the displacement $y_{ss}(t)$ becomes very large. Once the speed is away from those values, then $y_{ss}(t)$ quickly comes down to steady state F/k value.

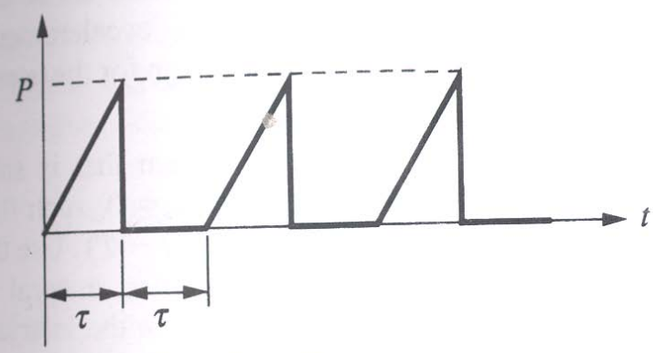
```
k = 10 π * 1200;
Grid[
 Partition[
  Plot[y[t, k, 10, #], {t, 0, 5}, PlotRange → {{0, 5}, {0, 100}},
     AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
     GridLinesStyle → LightGray,
     FrameLabel → {{"y(t)", None},
       {"time", Row[{"speed=", #, " m/s"}]}}] & /@
   {19.999, 19.98, 9.999, 9.98, 15, 7.5}, 2]]
```



2.7.4 problem 3

3.) (**20 points**) Find the steady-state response of the system in **Problems 3.45** and **3.46** from Ginsberg using FFT techniques. Perform your analysis with $\tau = \pi/(3\omega_n)$ as stated in the problem and also repeat the analysis for $\tau = 3\pi/\omega_n$. Which harmonic is dominant in the response in each case? Why? Create a plot of the steady-state displacement for each case.

3.45 A one-degree-of-freedom, underdamped system having mass m, natural frequency ω_{nat}, and critical damping ratio $\zeta = 0.04$ is subjected to cyclical triangular pulse excitation, as shown below. What is the largest harmonic in the response when $\tau = \pi/3\omega_{nat}$?



EXERCISES 3.45, 3.46

3.46 Use FFT techniques to determine and graph the steady-state displacement and acceleration of the system in Exercise 3.45 for the parameters stated there.

The function is periodic with period $T = 2\tau$

$$f(t) = \begin{matrix} \frac{P}{\tau}t & 0 < t < \tau \\ 0 & \tau < t < 2\tau \end{matrix}$$

and $f(t \pm T) = f(t)$. Let $\tilde{f}(t)$ be the Fourier series approximation to $f(t)$, hence

$$\tilde{f}(t) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right) \tag{2.104}$$

Where

$$\begin{aligned} F_n &= \frac{2}{T}\int_0^T f(t)e^{-in\frac{2\pi}{T}t}dt \\ &= \frac{2}{2\tau}\int_0^\tau \frac{P}{\tau}te^{-in\frac{\pi}{\tau}t}dt \\ &= \frac{P}{\tau^2}\int_0^\tau te^{-in\frac{\pi}{\tau}t}dt \end{aligned}$$

Using integration by parts $\int udv = uv - \int vdu$, letting $u = t$ and $dv = e^{-in\frac{\pi}{\tau}t}$ then $v =$

$\int e^{-in\frac{\pi}{\tau}t}dt = \frac{ie^{-in\frac{\pi}{\tau}t}}{n\frac{\pi}{\tau}}$hence

$$\begin{aligned}
F_n &= \frac{P}{\tau^2}\left[\left(t\frac{ie^{-in\frac{\pi}{\tau}t}}{n\frac{\pi}{\tau}}\right)_0^\tau - \frac{i}{n\frac{\pi}{\tau}}\int_0^\tau e^{-in\frac{\pi}{\tau}t}dt\right]\\
&= \frac{P}{\tau^2}\left[\left(\tau\frac{ie^{-in\frac{\pi}{\tau}\tau}}{n\frac{\pi}{\tau}}\right) - \frac{i}{n\frac{\pi}{\tau}}\left(\frac{e^{-in\frac{\pi}{\tau}t}}{-in\frac{\pi}{\tau}}\right)_0^\tau\right]\\
&= \frac{P}{\tau^2}\left[\left(\tau^2\frac{ie^{-in\pi}}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left(e^{-in\frac{\pi}{\tau}t}\right)_0^\tau\right]\\
&= \frac{P}{\tau^2}\left[\left(\tau^2\frac{ie^{-in\pi}}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left(e^{-in\pi}-1\right)\right]
\end{aligned}$$

$e^{-in\pi} = \cos(n\pi) = (-1)^n$, hence

$$F_n = \frac{P}{\tau^2}\left[\left(\tau^2\frac{i(-1)^n}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left((-1)^n-1\right)\right]$$

Hence for even n

$$\begin{aligned}
F_n &= \frac{P}{\tau^2}\left[\left(\tau^2\frac{i}{n\pi}\right)\right]\\
&= P\frac{i}{n\pi}
\end{aligned}$$

and for odd n

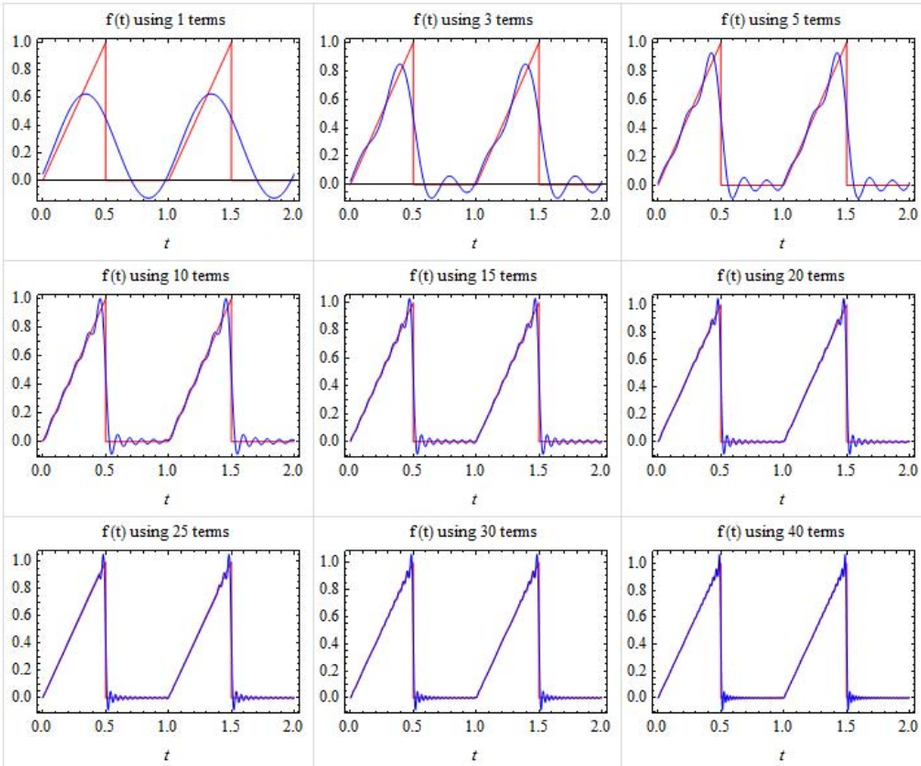
$$\begin{aligned}
F_n &= \frac{P}{\tau^2}\left[\left(-\tau^2\frac{i}{n\pi}\right) - 2\frac{\tau^2}{n^2\pi^2}\right]\\
&= -\frac{P}{n\pi}\left(\frac{2}{n\pi}+i\right)
\end{aligned}$$

$$\begin{aligned}
F_0 &= \frac{P}{\tau^2}\int_0^\tau t dt\\
&= \frac{P}{\tau^2}\left(\frac{t^2}{2}\right)_0^\tau = \frac{P}{\tau^2}\left(\frac{\tau^2}{2}\right)\\
&= \frac{P}{2}
\end{aligned}$$

Now Eq 2.117 becomes

$$\begin{aligned}
\tilde{f}(t) &= \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^N F_n e^{in\frac{2\pi}{T}x}\right)\\
&= \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{even\ n} F_n e^{in\frac{2\pi}{T}t} + \sum_{odd\ n} F_n e^{in\frac{2\pi}{T}t}\right)\\
&= \frac{p}{4} + \mathrm{Re}\left(\sum_{even\ n} P\frac{i}{n\pi}e^{in\frac{2\pi}{T}t} + \sum_{odd\ n} -\frac{P}{n\pi}\left(\frac{2}{n\pi}+i\right)e^{in\frac{2\pi}{T}t}\right)\\
&= \frac{p}{4} + \mathrm{Re}\left(\frac{P}{\pi}\sum_{even\ n}\frac{i}{n}e^{in\frac{2\pi}{T}t} - \frac{P}{\pi}\sum_{odd\ n}\frac{1}{n}\left(\frac{2}{n\pi}+i\right)e^{in\frac{2\pi}{T}t}\right)\\
&= \frac{P}{4} + \mathrm{Re}\left(\frac{P}{\pi}\sum_{even\ n}\frac{i}{n}e^{in\frac{2\pi}{T}t} - \frac{P}{\pi}\sum_{odd\ n}\left(\frac{2}{n^2\pi}+\frac{i}{n}\right)e^{in\frac{2\pi}{T}t}\right)
\end{aligned}$$

To verify, here is a plot of the above, using $P = 1$ and $\tau = 0.5$ sec for $t = 0 \cdots 2$ seconds. This shows as more terms are added, the approximation becomes very close to the function. At $N = 40$ the approximation appears very good.



Now we need to write $f(t)$ as sum of exponential to answer the question.

$$\tilde{f}(t) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)$$

where ϖ is the fundamental frequency of the force given by $\frac{2\pi}{T} = \frac{2\pi}{2\tau} = \frac{\pi}{\tau}$

Hence, let $y_{ss} = \sum_{n=-\infty}^{\infty} Y_n e^{in\varpi t}$, then

$$\mathrm{Re}\left(m\sum_{n=-\infty}^{\infty} -(n\varpi)^2 Y_n e^{in\varpi t} + c\sum_{n=-\infty}^{\infty} in\varpi Y_n e^{in\varpi t} + k\sum_{n=-\infty}^{\infty} Y_n e^{in\varpi t}\right) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)$$

$$\sum_{n=-\infty}^{\infty}\left(-m(n\varpi)^2 + icn\varpi + k\right)Y_n e^{in\varpi t} = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)$$

Hence

$$\begin{aligned} Y_n &= \frac{F_n}{k}\frac{1}{\left(1-\left(n\frac{\varpi}{\omega_{nat}}\right)^2\right) + i2\zeta n\frac{\varpi}{\omega_{nat}}} \\ &= \frac{F_n}{k}\frac{1}{\left(1-(nr)^2\right) + i2\zeta nr} \end{aligned}$$

Hence

$$y_{ss} = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

<u>Finding Y_n for $\tau = \frac{\pi}{3\omega_{nat}}$</u>

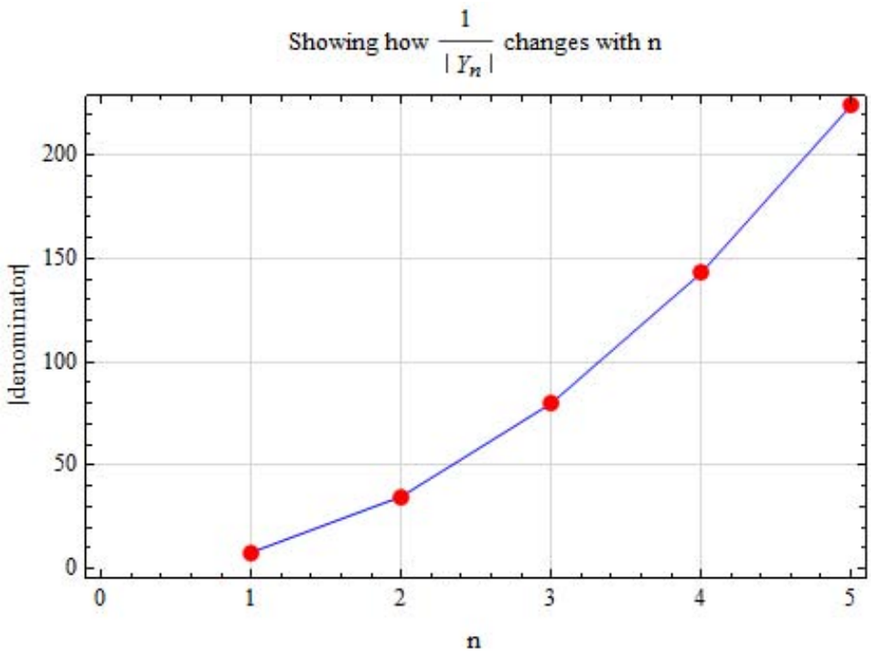
where $r = \frac{\varpi}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{\pi}{3\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{\pi}{3\omega_{nat}}\right)\omega_{nat}} = 3$,

therefore

$$
\begin{aligned}
Y_n &= \frac{F_n}{k}\frac{1}{\left(1-(3n)^2\right)+i6(0.04)n} \\
&= \frac{F_n}{k}\frac{1}{\left(1-9n^2\right)+i0.24n}
\end{aligned}
$$

The largest Y_n will occur when the denominator of the above is smallest. Plotting the modulus of the denominator $\sqrt{\left(1-9n^2\right)^2+(0.24n)^2}$ for different n values shows that $n=1$ is the values which makes it minimum.

This happens since for any $n>1$ the denominator will become larger due to n^2 and hence Y_n will become smaller. So $n=1$ will be used.



For $n=1$, we obtain

$$
Y_1 = \frac{F_1}{k}\frac{1}{(1-9)+i6(0.04)}
$$

But $F_1 = -\frac{P}{\pi}\left(\frac{2}{\pi}+i\right)$, hence

$$
\begin{aligned}
Y_1 &= \frac{-\frac{P}{\pi}\left(\frac{2}{\pi}+i\right)}{k}\frac{1}{(1-9)+i6(0.04)} = \frac{-P}{\pi k}\frac{\left(\frac{2}{\pi}+i\right)}{-8+i0.24} \\
&= \frac{P}{\pi k}\frac{\frac{2}{\pi}+i}{8-i0.24} = \frac{P}{\pi k}\frac{\left(\frac{2}{\pi}+i\right)(8+i0.24)}{(8-i0.24)(8+i0.24)} \\
&= \frac{P}{\pi k}(0.075759+0.12727i)
\end{aligned}
$$

Therefore

$$
Y_1 = \frac{P}{k}(0.024115+0.0405i)
$$

Here is a list of Y_n for $n=1\cdots 10$ with the phase and magnitude of each (this was done for $\frac{p}{k}=1$)

r=3, ζ=0.04			
n	Y_n	$\lvert Y_n \rvert$	Arg[Y_n]
1	0.0241149 + 0.0405122 i	0.0471462	59.2367
2	0.000062351 - 0.00454643 i	0.00454686	-89.2143
3	0.000269489 + 0.00132872 i	0.00135577	78.5348
4	3.73568×10^{-6} - 0.000556461 i	0.000556473	-89.6154
5	0.0000346626 + 0.000284391 i	0.000286496	83.0509
6	7.3223×10^{-7} - 0.000164243 i	0.000164245	-89.7446
7	9.00427×10^{-6} + 0.000103382 i	0.000103773	85.0223
8	2.31058×10^{-7} - 0.000069197 i	0.0000691974	-89.8087
9	3.29231×10^{-6} + 0.0000485919 i	0.0000487033	86.1239
10	9.45233×10^{-8} - 0.0000354069 i	0.000035407	-89.847

From the above we see that most of the energy in the response will be contained in Y_1 and adding more terms will not have large effect on the response shape. This is confirmed by the plot that follows.

Plot for the steady state

Since

$$y_{ss} = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\omega t}\right)$$

Where now $r = \frac{\omega}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{\pi}{3\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{\pi}{3\omega_{nat}}\right)\omega_{nat}}$

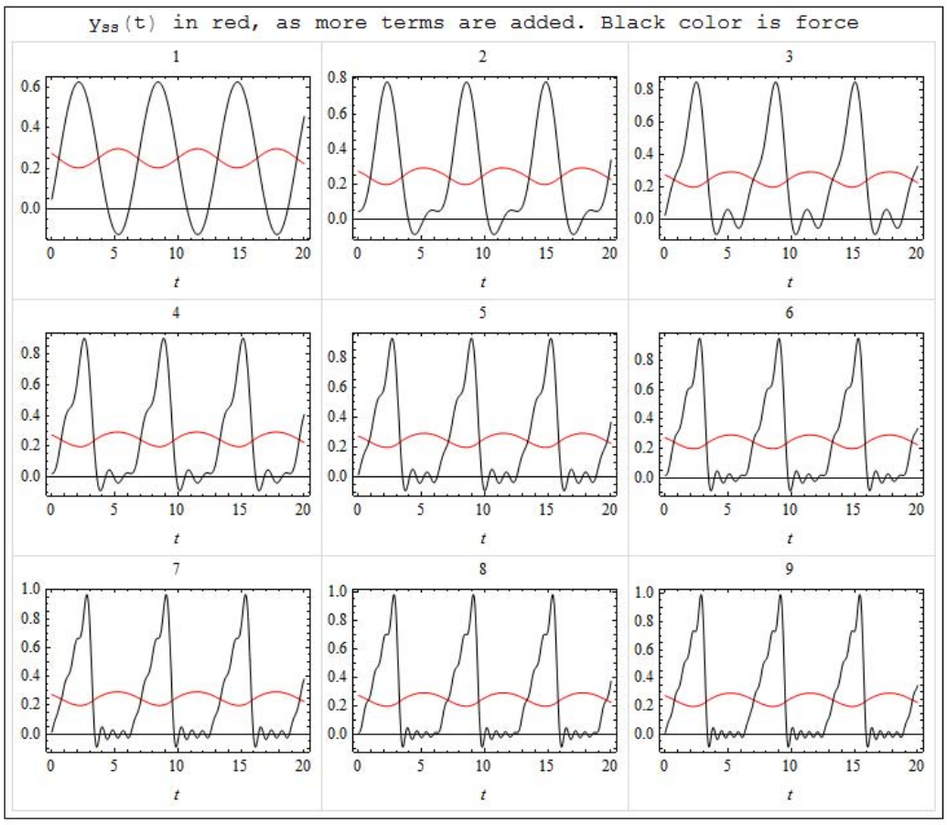
therefore $\boxed{r = 3}$

$$\begin{aligned}
y_{ss} &= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} Y_n e^{in\omega t} + \sum_{n=2,4,6\cdots}^{\infty} Y_n e^{in\omega t}\right) \\
&= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} \frac{F_{n_{odd}}}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{F_{n_{even}}}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t}\right) \\
&= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} \frac{-\frac{P}{n\pi}\left(\frac{2}{n\pi}+i\right)}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{P\frac{i}{n\pi}}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t}\right) \\
&= \frac{p}{4} + \frac{p}{k}\mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{\frac{1}{n\pi}\left(\frac{2}{n\pi}+i\right)}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{\frac{i}{n\pi}}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t}\right)
\end{aligned}$$

Now let $r = 3$, $\zeta = 0.04$. Normalizing the equation for $\omega = 1$ which implies $\tau = \pi$ and $k = 1$ and $p = 1$, then the above becomes

$$y_{ss} = \frac{1}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{\frac{1}{n\pi}\left(\frac{2}{n\pi}+i\right)}{\left(1-(3n)^2\right)+i2(0.04)3n}e^{int} + \sum_{n=2,4,6\cdots}^{\infty} \frac{\frac{i}{n\pi}}{\left(1-(3n)^2\right)+i2(0.04)3n}e^{int}\right)$$

Here is a plot of the above for $t = 0 \cdots 20$ seconds for different values of n



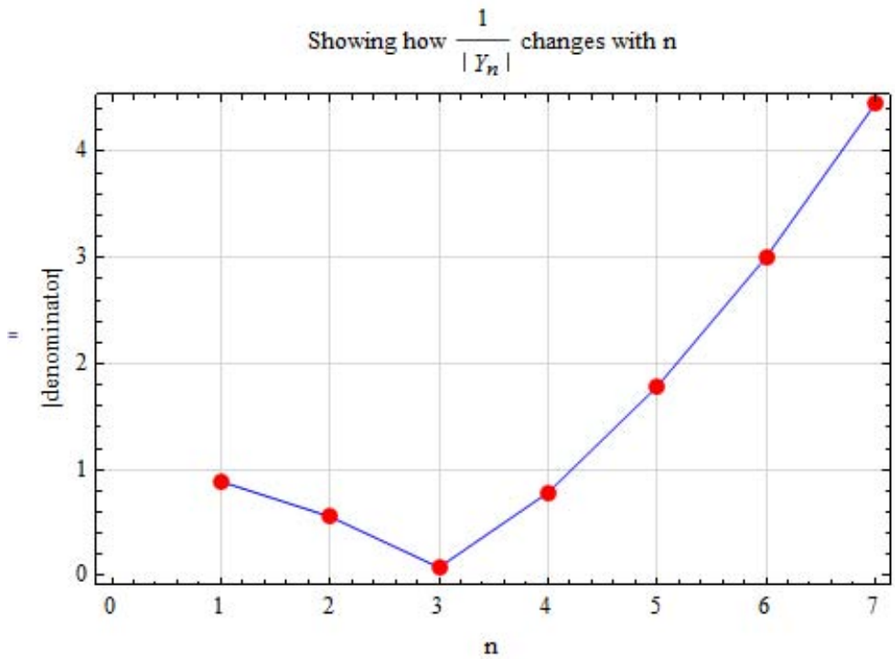
We see from the above plot, that $y_{ss}(t)$ does not change too much as more terms are added, since when $r = 3$, then Y_n for $n = 1$ contains most of the energy, hence adding more terms did not have an effect.

Repeating the calculations for $\tau = \frac{3\pi}{\omega_{nat}}$

$r = \frac{\omega}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{3\pi}{\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{3\pi}{\omega_{nat}}\right)\omega_{nat}} = \frac{1}{3}$, therefore

$$\begin{aligned}
Y_n &= \frac{F_n}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr} \\
&= \frac{F_n}{k}\frac{1}{\left(1-\left(\frac{1}{3}n\right)^2\right)+i\frac{2}{3}(0.04)n} \\
&= \frac{F_n}{k}\frac{1}{\left(1-\frac{n^2}{9}\right)+i0.0267n}
\end{aligned}$$

The largest Y_n will occur when the denominator of the above is smallest. Similar to above, we can either find n which minimizes the denominator (by taking derivative and setting it to zero and solve for n) or we can make a plot and see how the function behaves. Making a plot shows this



From the above we see that the smallest value of the denominator happens when $n = 3$. so using $n = 3$ we find

$$\begin{aligned} Y_3 &= \frac{F_3}{k}\frac{1}{\left(1-(3r)^2\right)+i2\zeta 3r} \\ &= \frac{F_3}{k}\frac{1}{\left(1-\left(3\frac{1}{3}\right)^2\right)+i2(0.04)3\frac{1}{3}} \\ &= \frac{F_3}{k}\frac{1}{i0.08} \end{aligned}$$

But $F_n = -\frac{P}{n\pi}\left(\frac{2}{n\pi}+i\right)$, hence

$$F_3 = -\frac{P}{3\pi}\left(\frac{2}{3\pi}+i\right)$$

Therefore

$$Y_3 = \frac{-\frac{P}{3\pi}\left(\frac{2}{3\pi}+i\right)}{k}\frac{1}{i0.08}$$

Hence

$$Y_3 = \frac{p}{k}(-1.3263+0.28145i)$$

Here is a list of Y_n for $n = 1\cdots 10$ with the phase and magnitude of each (this was done for $\frac{p}{k} = 1$)

r=1/3, ζ=0.04			
n	Y_n	$\lvert Y_n \rvert$	Arg[Y_n]
1	-0.238501 - 0.350944 i	0.424316	-124.2
2	0.0272508 + 0.283863 i	0.285168	84.5164
3	-1.32629 + 0.281448 i	1.35582	168.019
4	0.0137726 - 0.100425 i	0.101365	-82.191
5	0.00186323 + 0.0359496 i	0.0359979	87.0331
6	0.000940465 - 0.0176337 i	0.0176588	-86.9471
7	0.0004999 + 0.0102524 i	0.0102646	87.2085
8	0.000227012 - 0.00650296 i	0.00650692	-88.0007
9	0.000179929 + 0.00442637 i	0.00443002	87.6722
10	0.0000829696 - 0.00314593 i	0.00314703	-88.4893

We see from the above that $|Y_3|$ is the largest harmonic.

Plot for the steady state

Since

$$y_{ss} = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\omega t}\right)$$

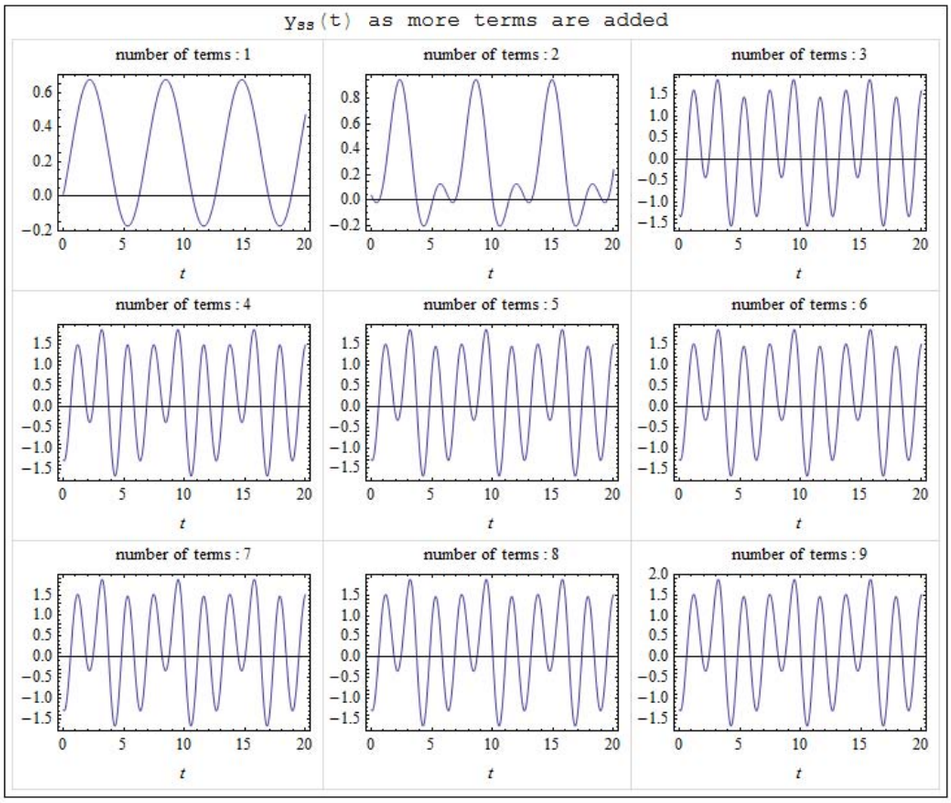
Where now $r = \frac{\omega}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{3\pi}{\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{3\pi}{\omega_{nat}}\right)\omega_{nat}} = \frac{1}{3}$, therefore from above

$$y_{ss} = \frac{p}{4} + \frac{p}{k}\mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{1}{n\pi}\left(\frac{2}{n\pi}+i\right)\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t} + \sum_{n=2,4,6\cdots}^{\infty}\frac{i}{n\pi}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\omega t}\right)$$

Now let $r = \frac{1}{3}$, $\zeta = 0.04$, and assuming $\tau = 0.5$ then $\omega = \frac{2\pi}{2\tau} = \frac{\pi}{0.5}$, and assuming $k = 1$, then the above becomes

$$y_{ss} = \frac{1}{4} + \frac{1}{k}\mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{1}{n\pi}\left(\frac{2}{n\pi}+i\right)\frac{1}{\left(1-\left(n\frac{1}{3}\right)^2\right)+i2(0.04)\frac{1}{3}n}e^{in\frac{\pi}{0.5}t}\right)$$
$$+\frac{1}{k}\mathrm{Re}\left(\sum_{n=2,4,6\cdots}^{\infty}\frac{i}{n\pi}\frac{1}{\left(1-\left(n\frac{1}{3}\right)^2\right)+i2(0.04)\frac{1}{3}n}e^{in\frac{\pi}{0.5}t}\right)$$

Here is a plot of the above for $t = 0\cdots 20$ seconds for different values of n



We see now that after $n = 3$ that the response did not change much by adding more terms, this is because more of the energy are contained in the first 3 harmonics with Y_n being the the largest.

2.7.5 Key solution for HW 6

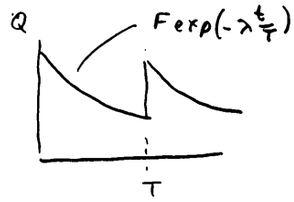
Homework #6
EMA 545, Spring 2013

1.) **Problem 3.41** in Ginsberg. Check your answer for λ=1.0 using FFT techniques with the fft_easy.m Matlab function from the course website.

2.) **Problem 3.50** in Ginsberg. DO PART (a) ONLY.

3.) (**20 points**) Find the steady-state response of the system in **Problems 3.45** and **3.46** from Ginsberg using FFT techniques. Perform your analysis with $\tau = \pi/(3\omega_n)$ as stated in the problem <u>and also repeat the analysis for</u> $\tau = 3\pi/\omega_n$. Which harmonic is dominant in the response in each case? Why? Create a plot of the steady-state displacement for each case.

Exercise 3.41



$$Q(t \pm T) = Q(t)$$

$$Q = \frac{1}{2}\sum_{n=-\infty}^{\infty} F_n \exp(2\pi i n t/\tau)$$

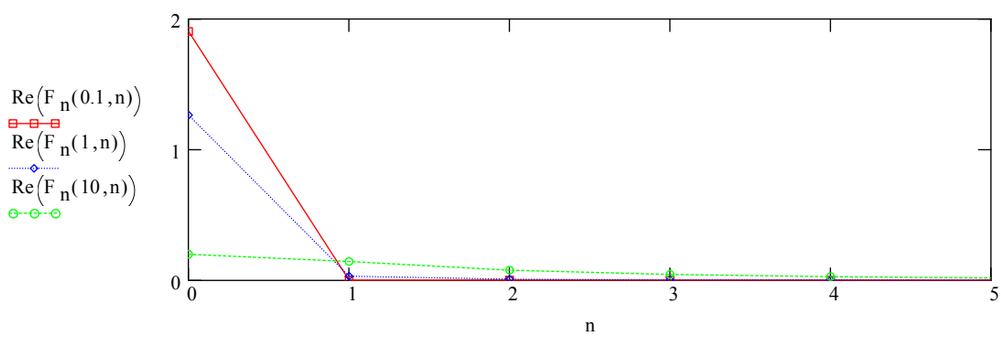
$$F_n = \frac{2}{T}\int_0^T Q \exp(-i 2\pi i n t/\tau)\,dt$$

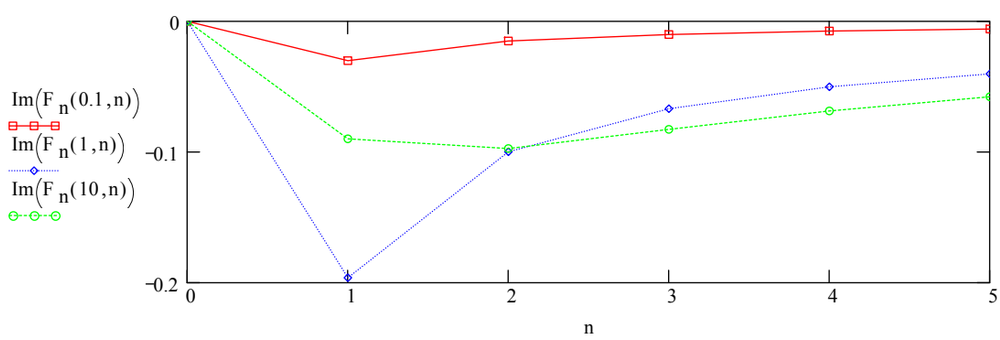
$$F_n = 2F\int_0^1 \exp[-(\lambda + 2\pi i n)t']\,dt'$$

$$= 2F\,\frac{1-\exp[-(\lambda+2\pi i n)]}{\lambda + 2\pi i n} \qquad \text{but } \exp(-i 2\pi n) = 1$$

$$\text{so } F_n = F\,\frac{2[1-\exp(-\lambda)]}{\lambda + 2\pi i n}$$

$n := 0..5$ $\qquad F_n(\lambda, n) := \dfrac{2}{(\lambda + 2i\cdot n\cdot\pi)}\cdot(1 - \exp(-\lambda))$





As λ increases, the higher harmonic amplitudes increase.

Matlab solution to HW#8, Problem 3.41

```
%% Solution to 3.41 using Matlab
% EMA 545, Spring 2011
clear all; close all

N = 2^5; % 2^13; % number of samples for FFT
tau = pi/3; % 3*pi; % pi/3;
T = 1
    w1 = 2*pi/T;
F=1; % Force will be non-dimensional
lam = 1;

dt = T/N;
ts_fft = [0:dt:(T-dt)].';

% Define Input Force in Time Domain--f(t)
ft = F*exp(-lam*ts_fft);
ft = ft(:); % make sure these are column vectors

% Use FFT easy to perform analysis
[D_fft,ws_fft] = fft_easy(ft,ts_fft);

F_fft = (2/N)*D_fft; % this only has those for positive frequencies.

ns = [0:5].';
F_fs = 2*(1-exp(-lam))./(lam+1i*2*pi*ns);

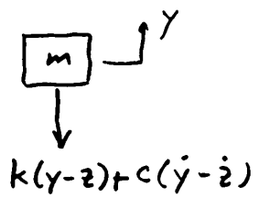
disp('n, Frequency, Fourier Coeff, FFT estimate (abs)');
[ns, ws_fft(1:6), abs(F_fs(1:6)), abs(F_fft(1:6))]
disp('Frequency, Fourier Coeff, FFT estimate (angle in deg)');
[ns, ws_fft(1:6), angle([F_fs(1:6), F_fft(1:6)])*180/pi]
```

Results with n=2^5=32

```
            n,    Frequency, Fourier Coeff, FFT estimate (abs)
ans =
            0            0       1.2642       1.2841
            1       6.2832      0.19871      0.20215
            2       12.566      0.10029      0.10252
            3        18.85     0.066976     0.069021
            4       25.133     0.050263     0.052388
            5       31.416     0.040222     0.042541
            n,    Frequency, Fourier Coeff, FFT estimate (angle in deg)
ans =
            0            0            0            0
            1       6.2832      -80.957      -75.361
            2       12.566       -85.45      -74.259
            3        18.85      -86.963      -70.177
            4       25.133      -87.721       -65.34
            5       31.416      -88.177      -60.201
```

With only 32 samples, this is already a pretty good approximation of the Fourier Coefficients. With n=2^13, the FFT is accurate to about three decimal places.

Exercise 3.50



$\omega_{nat} = 10\pi$ rad/s, $\zeta = 0.4$, $m = 1500$ kg

$m\ddot{y} + c\dot{y} + ky = c\dot{z} + kz = Q$

$z(x) = (x - 5x^2)[h(x) - h(x-0.2)]$ if $0 < x < 4$ meter

$z(x+4) = z(x)$

Substitute $x = vt \Rightarrow$ Period $= \frac{4}{v} \Rightarrow \omega_1 = \frac{2\pi}{(4/v)} = \frac{\pi v}{2}$

Resonance if any harmonic matches natural frequency

$$n\left(\frac{\pi v}{2}\right) = \omega_{nat} \Rightarrow v = \frac{2\omega_{nat}}{n\pi} = 20, 10, 6.67, .. \text{ m/s} \Leftarrow$$

Response @ $v = 5$ m/s $\Rightarrow T = 0.8$ sec

Use $N = 128 \Rightarrow t_n = (n-1)\frac{T}{N}$

Evaluate

$$z(t_n) = (vt_n - 5v^2 t_n^2)\left[h(t_n) - h\left(t_n - \frac{0.2}{v}\right)\right]$$

Take FFT $\Rightarrow Z(\omega_n)$ where $\omega_n = n\omega_1$, $\omega_1 = \frac{2\pi}{T}$

The FFT of the effective force will be

$$F_n = c[i\omega_n Z(\omega_n)] + kZ(\omega_n)$$

The DFT of the response is

$$Y_n = \frac{F_n}{k}\frac{1}{1-(nr_1)^2 + 2i\zeta n r_1}$$

where $k = m\omega_{nat}^2$, $c = 2m\omega_{nat}\zeta$, & $r_1 = \frac{\omega_n}{\omega_{nat}} = \frac{2\pi n}{\omega_{nat} T}$

Then find $y(t_n)$ from an IFFT.

$m := 1500 \qquad \omega_{nat} := 10\cdot\pi \qquad \zeta := 0.4 \qquad k := m\cdot\omega_{nat}^{2} \qquad c := 2\cdot m\cdot\omega_{nat}\cdot\zeta$

$v := 5 \qquad T := \frac{4}{v} \qquad \omega_{fund} := \frac{2\cdot\pi}{T}$

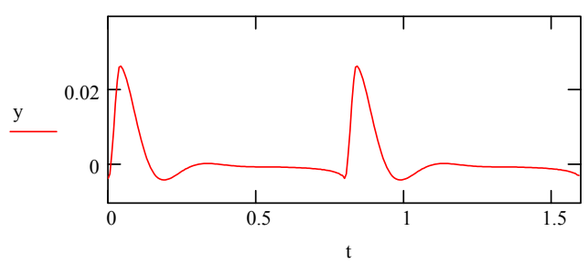
$elev(x) := \left(x - 5\cdot x^2\right)\cdot(x>0)\cdot(x<0.2)$

$N := 128 \qquad j := 1..N \qquad t_j := (j-1)\cdot\frac{T}{N} \qquad z_j := elev\left(v\cdot t_j\right) \qquad Z := FFT(z)$

$n := 1..\frac{N}{2}+1 \qquad \omega_n := (n-1)\cdot\omega_{fund} \qquad F_n := \left(i\cdot\omega_n\cdot c + k\right)\cdot Z_n \qquad r_{fund} := \frac{2\cdot\pi}{\omega_{nat}\cdot T}$

$$Y_n := \frac{F_n}{k}\cdot\frac{1}{1-\left(n\cdot r_{fund}\right)^2 + 2i\cdot\zeta\cdot n\cdot r_{fund}}$$

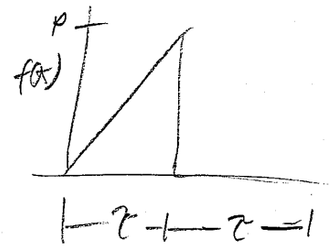
$y := IFFT(Y) \qquad y := stack(y,y) \qquad t_{j+N} := t_j + T$



3.46 Solution

$f(t)$ = Triangle pulse

$\zeta = 0.04$, ω_n



$\tau = \frac{\pi}{3\omega_n}$, $\tau = \frac{3\pi}{\omega_n}$

$$m\ddot{x} + c\dot{x} + kx = f(t) = Re(F e^{i\omega t})$$

$$x(t) = Re(X e^{i\omega t}) \longrightarrow$$

$$X = \frac{F/m}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$$

FFT ~ $T = 2\tau$

$$\omega_1 = \frac{2\pi}{T} = \frac{\cancel{2\pi}\, 3\omega_n}{\cancel{2\pi}} = 3\omega_n \quad \text{for } \tau = \frac{\pi}{3\omega_n}$$

$$\omega_1 = \frac{\cancel{2\pi}\,\omega_n}{\cancel{2}\cdot 3\cancel{\pi}} = \frac{1}{3}\omega_n \quad \text{for } \tau = \frac{\pi}{3\omega_n}$$

Nondimensionalize:

$t' = t/\omega_n$

$$x(t) = Re(X e^{i(\omega/\omega_n)t'}) = Re(X e^{irt})$$

$$\dot{x}(t) = \frac{\partial x}{\partial t'}\frac{\partial t'}{\partial t} = Re(irX e^{irt})\cdot\omega_n \longrightarrow \text{same}$$

$$X = \frac{F/(m\omega_n^2)}{1 - r^2 + i2\zeta r}$$

$$X_a = \frac{-\omega^2 F/(m\omega_n^2)}{1 - r^2 + i2\zeta r} = \frac{-r^2 F/(m\omega_n^4)}{1 - r^2 + i2\zeta r}$$

plot $X(t)\, m\omega_n^2$ and $X_a(t)\, m\omega_n^4$

$$t = \tau = \frac{t'}{\omega_n} \longrightarrow t' = \omega_n\tau \longrightarrow \tau' = \frac{\pi}{3} \text{ and } 3\pi$$

Solution: Problem 4 (3.45-46) Spring 2011

Matlab code given below, which is modified only slightly from FFT_Square_Ex_545_v2.m

```
%% Solution to P4, HW#8 (based on 3.45 and 3.46 in Ginsberg)
clear all; close all

N = 256; % number of samples for FFT
tau = pi/3; % 3*pi; % pi/3;
T = 2*tau;
    w1 = 2*pi/T;
m=1;
wn=1; % since time is non-dimensional
zt = 0.04;
P = 1; % Force will be non-dimensional

dt = T/N;
ts_fft = [0:dt:(T-dt)];

% Define Input Force in Time Domain--f(t) = mean at discontinuity
for k = 1:1:length(ts_fft)
    if ts_fft(k) < tau;
        ft(k,1) = P*ts_fft(k)/tau;
    else
        ft(k,1) = 0;
    end
end

% Use FFT easy to perform analysis
[F_fft,ws_fft] = fft_easy(ft,ts_fft);

% Each coefficient in F_fft is the complex amplitude of a harmonic
whose
% frequency is given in ws_fft.  We could scale these to obtain
estimates
% of the Fourier coefficients, but we do not need to since we are just
% going to take the IFFT later.

% Make everything a column vector:
F_fft = F_fft(:); ws_fft = ws_fft(:);

% Now form a vector of transfer function values at each frequency:
H = (1/m)./((i*ws_fft).^2 + i*ws_fft*(2*zt*wn) + wn^2); % dot or term
by term multiply
    % Same as doing a for loop over each frequency.

% Now the response is just the force times the transfer function.
X = H.*F_fft(:);

% Plot everything in the frequency domain to understand what's
happening.
figure(3)
semilogy(ws_fft,abs(F_fft),'o',ws_fft,abs(X),'*',ws_fft,abs(H),'-');
grid on;
```

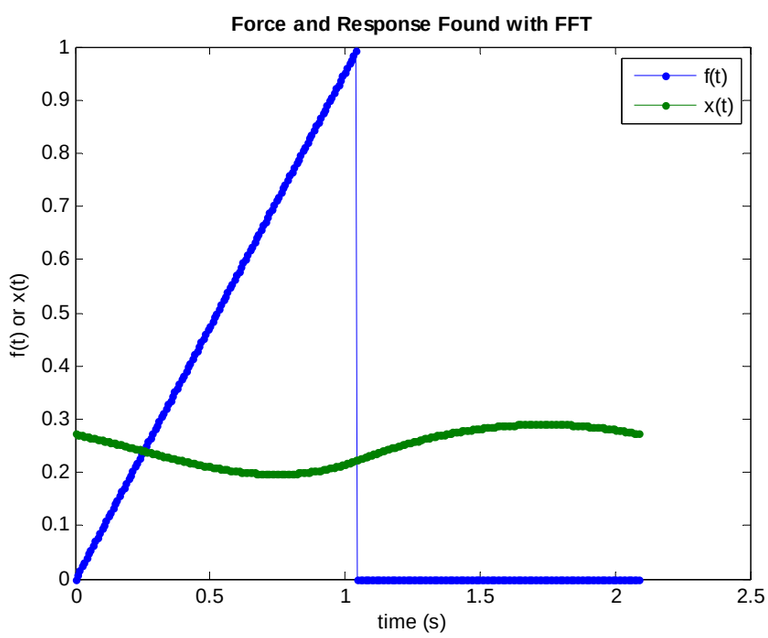
```
xlabel('\bfFrequency (rad/s))'); ylabel('\bf|X| or |F|');
legend('fft(F)','fft(X)','H(\omega)');

% take IFFT of the coefficients X to find the time signal x(t)
xt = ifft_easy(X,ws_fft);
% note ifft_easy(F_fft,ws_fft) = ft, exactly with no approximation

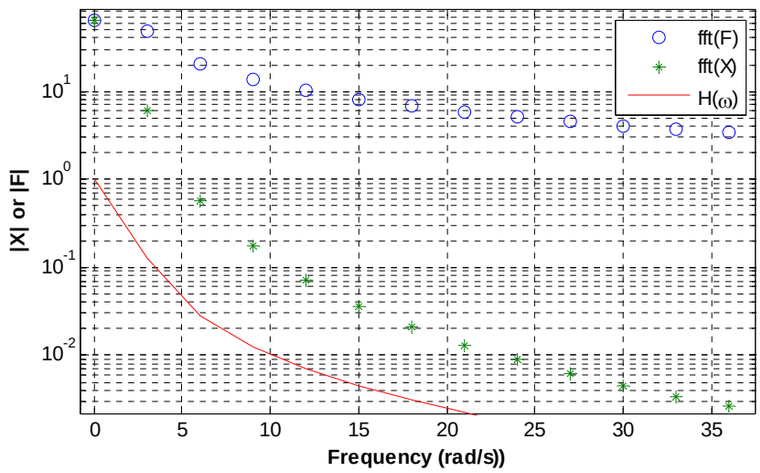
figure(4)
% [ax,h1,h2] = plotyy(ts_fft,gt,ts_fft,xt);
plot(ts_fft,ft,'.-',ts_fft,xt,'.-');
xlabel('time (s)'); ylabel('f(t) or x(t)');
legend('f(t)','x(t)');
title('\bfForce and Response Found with FFT');
```

Results:

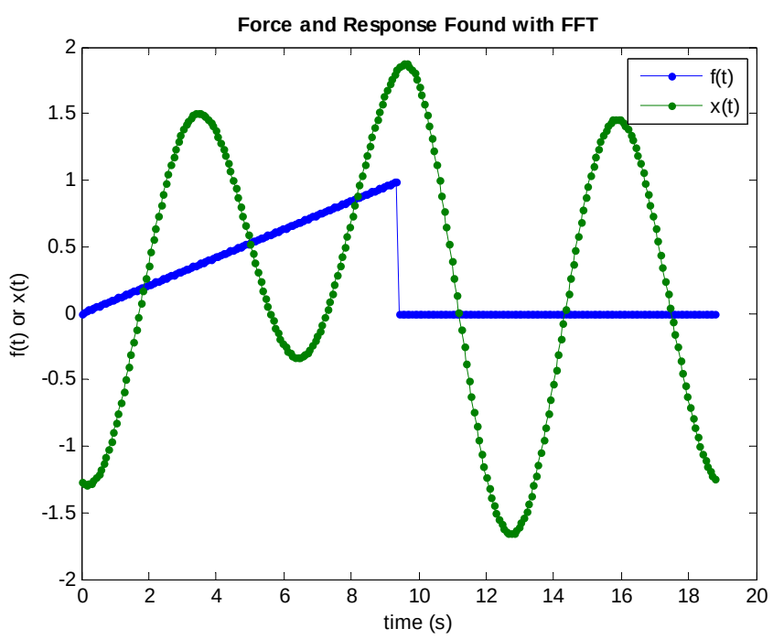
Case 1: $\tau = \pi/(3\omega_n)$ (r = 3)

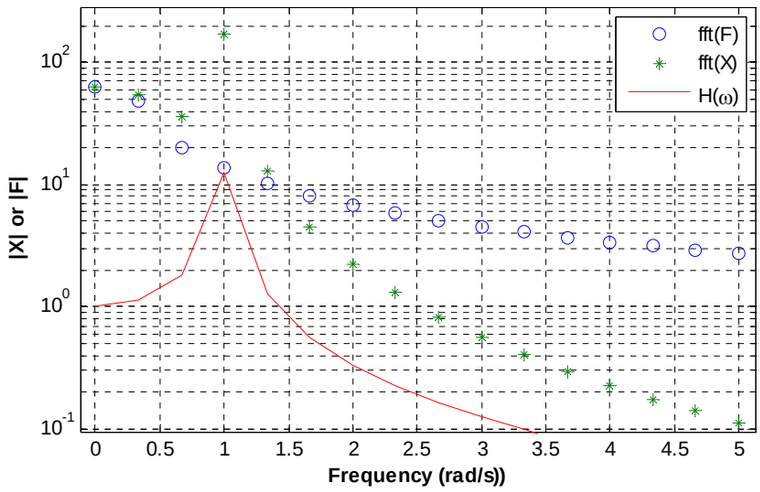


To understand this, let's look at the FFT coefficients of X, F and the transfer function H, shown in the following figure. Since the first coefficient is beyond the natural frequency, the transfer function gets smaller as omega increases. The force is also dominated by the lower frequency terms, so the low frequency terms dominate. The DC term is the largest, followed by the first harmonic (DC is Electrical Engineering terminology for Direct Current or the zero frequency). So, it shouldn't surprise us to see that the response is an offset sinusoid with low amplitude.



Case 2: $\tau = 3\pi/\omega_n$ (r = 1/3)





In this case the fundamental frequency of the sawtooth is 1/3 the natural frequency. Hence, the third harmonic is equal to the natural frequency so that harmonic is amplified in the response. So, the response looks like a 3-cycle sinusoid (per period of the force) even though the force spectrum is dominated by the lower frequency harmonics.

2.8 HW7

Local contents

2.8.1 problem description

Homework #7
EMA 545, Spring 2013

Problem 1: (**20 pts**) Use FFT techniques to find the response of your vibration isolation system to the turbulent flight profile in "FlightAccel.mat" on the course website. (To do this, you will have to assume that the aircraft experiences this exact same flight profile over and over again.)

a.) Provide at least one plot comparing the response of the aircraft to the (hopefully improved) response on the vibration isolator.

b.) Report on the values of mass, stiffness and damping that you used and how they would be realized in practice (e.g. if you use a beam as a leaf spring, what would its dimensions be?).

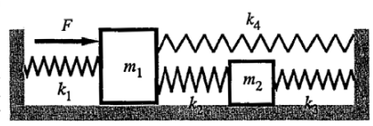
Problem 2: (**20 pts**) Do **Problem 3.60** as given in the text. Comment on how you selected adequate values for N (number of samples in the time history) and the length of the time window (in seconds).

Problem 3: Exercise 1.11 from Ginsberg. (For the proof described in (b), set m_2=0 and see what your equation of motion reduces to.)

1.11 When the system in the sketch is at its static equilibrium position, there is no axial force in each spring.

(a) Derive equations of motion for the horizontal displacements x_1 and x_2 measured from the equilibrium position.

(b) Prove that if $m_2 = 0$, the equation of motion for x_1 is the same as that obtained by replacing the four springs by a single equivalent spring. Identify the way in which the springs are connected—for example, series or parallel—according to the equivalent spring stiffness.



EXERCISE 1.11

2.8.2 problem 1

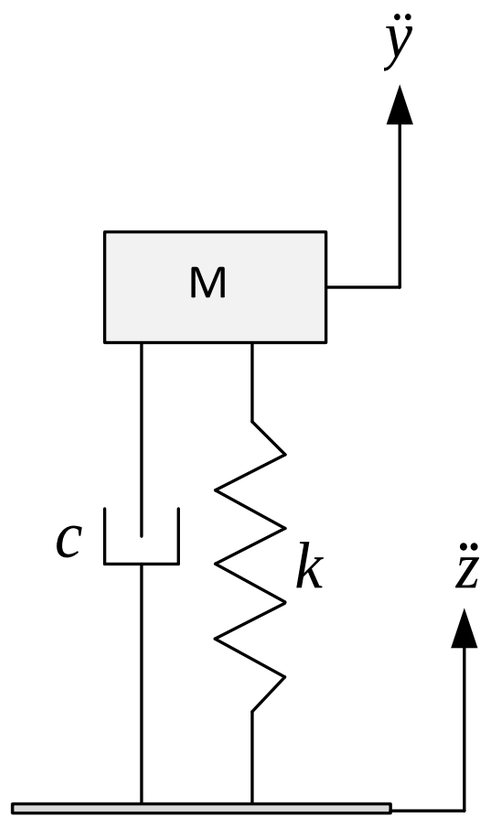
Problem 1: (**20 pts**) Use FFT techniques to find the response of your vibration isolation system to the turbulent flight profile in "FlightAccel.mat" on the course website. (To do this, you will have to assume that the aircraft experiences this exact same flight profile over and over again.)

a.) Provide at least one plot comparing the response of the aircraft to the (hopefully improved) response on the vibration isolator.

b.) Report on the values of mass, stiffness and damping that you used and how they would be realized in practice (e.g. if you use a beam as a leaf spring, what would its dimensions be?).

2.8.2.1 Part(a)

Vibration isolation was based on reducing absolute acceleration of passenger under turbulent external forces. This was done by isolating the passenger from the base motion subjected to external absolute acceleration. Hence the model is based on the following diagram



Hence EQM of motion is

$$\begin{aligned} my'' + c\left(y' - z'\right) + k\left(y - z\right) &= 0 \\ my'' + cy' + ky &= cz' + kz \end{aligned} \tag{2.105}$$

We are given the time history of the turbulent acceleration. Hence in frequency domain we can write

$$z'' = \mathrm{Re}\left\{Z_n^{acc} e^{i(\omega_1 n)t}\right\}$$

Where Z_n^{acc} is the complex amplitude of the nth harmonic component in the acceleration

data. Let $\omega_1 n \equiv \varpi_n$ then using the above, In frequency domain Eq 3.1 becomes

$$\begin{aligned}\mathrm{Re}\left\{\left(-m\varpi_n^2 + i\varpi_n c + k\right)Y_n e^{i\varpi_n t}\right\} &= \mathrm{Re}\left\{\left(c\frac{Z_n^{acc}}{i\varpi_n} + k\frac{Z_n^{acc}}{-\varpi_n^2}\right)e^{i\varpi_n t}\right\}\\ Y_n &= \left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc}\end{aligned}$$

The above gives the transfer function between the displacement of the passenger and the external acceleration. In otherwords

$$y(t) = \mathrm{Re}\left\{\left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc} e^{i(\omega_1 n)t}\right\}$$

Let

$$Y_n = \left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc}$$

then the transfer function is

$$\begin{aligned}H(\varpi_n) = \frac{Y_n}{Z_n^{acc}} &= \frac{\frac{-ic}{\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\\ &= -\frac{1}{\varpi_n^2}\frac{(k + ic\varpi_n)}{\left(k - m\varpi_n^2\right) + i\varpi_n c}\end{aligned}$$

Hence phase is

$$\arg(H(\varpi_n)) = \tan^{-1}\left(\frac{c\varpi_n}{k}\right) - \tan^{-1}\left(\frac{\varpi_n c}{k - m\varpi_n^2}\right)$$

and magnitude is

$$|H(\varpi_n)| = \left|\frac{Y_n}{Z_n^{acc}}\right| = \frac{1}{\varpi_n}\frac{\sqrt{k^2 + c^2\varpi_n^2}}{\sqrt{\left(k - m\varpi_n^2\right)^2 + (\varpi_n c)^2}}$$

These can be written in terms of ζ and ω_{nat} as follows. From $H(\varpi_n) = -\frac{1}{\varpi_n^2}\frac{(k+ic\varpi_n)}{\left(k-m\varpi_n^2\right)+i\varpi_n c}$, dividing numerator and denominator by $k = m\omega_{nat}^2$ and using $c = 2\zeta m\omega_{nat}$ then

$$H(\varpi_n) = -\frac{1}{\varpi_n^2}\frac{\left(1 + \frac{i2\zeta m\omega_{nat}\varpi_n}{m\omega_{nat}^2}\right)}{\left(1 - \frac{m\varpi_n^2}{m\omega_{nat}^2}\right) + \frac{i\varpi_n 2\zeta m\omega_{nat}}{m\omega_{nat}^2}} = -\frac{1}{\varpi_n^2}\frac{\left(1 + \frac{i2\zeta\varpi_n}{\omega_{nat}}\right)}{\left(1 - \frac{\varpi_n^2}{\omega_{nat}^2}\right) + \frac{i\varpi_n 2\zeta}{\omega_{nat}}}$$

Let $r_n = \frac{\varpi_n}{\omega_{nat}}$ then the above becomes

$$H(\varpi_n) = -\frac{1}{\varpi_n^2}\frac{(1 + i2\zeta r_n)}{\left(1 - r_n^2\right) + i2r_n\zeta}$$

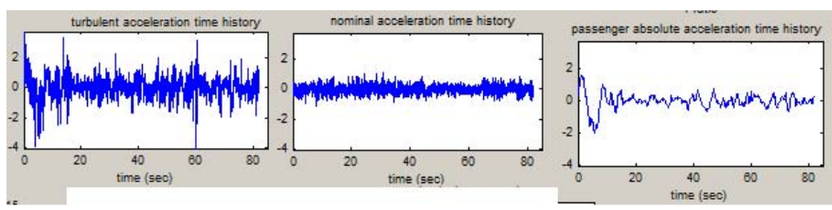
Hence

$$\begin{aligned}|H(\varpi_n)| &= \frac{1}{\varpi_n}\frac{\sqrt{1 + (2\zeta r_n)^2}}{\sqrt{\left(1 - r_n^2\right)^2 + (2r_n\zeta)^2}}\\ \arg(H(\varpi_n)) &= \tan^{-1}(2\zeta r_n) - \tan^{-1}\frac{2r_n\zeta}{1 - r_n^2}\end{aligned}$$

The following is a plot showing the passenger absolute acceleration $y''(t)$ over the period of 80 seconds against the turbulent acceleration $z''(t)$. We now see that passenger absolute acceleration is close to the nominal acceleration. This was done using the following values for the vibration isolation

M	100000 kg
ζ	0.72
k	38924 N/m
c	57746 Ns/m

The plot on the right side is the absolute acceleration of the passenger during flight in the turbulent case.



2.8.2.2 Part(b)

The length of first class cabinet was estimated to be $L = 15$ meters from looking at Boeing web page.

Using Steel, Structural ASTM-A36 I beam as a cantilever beam for the implementation, then using $k = \frac{3EI}{L^3}$ results in

$$38924 = \frac{3\left(200 \times 10^9\right)I}{15^3}$$
$$I = 2.1895 \times 10^{-4} \text{ m}^4$$

Using rectangle cross section $I = \frac{bh^3}{12}$. Letting $h = 20$ cm, then $b = \frac{\left(2.1895\times10^{-4}\right)12}{0.2^3} = 0.32843$ meter or 32 cm.

2.8.3 Problem 2

3.60 A one-degree-of-freedom system is subjected to a pulse excitation in the form of a parabola,

$$Q = 2000\frac{t(T-t)}{T^2}[h(t) - h(t-T)] \quad \text{N}$$

The system mass is 0.5 kg, and the natural frequency is 100 Hz. The pulse duration T equals the undamped period of free vibration period. The system is at rest in the equilibrium position at $t = 0$.

(a) Use FFT techniques to evaluate the response when $\zeta = 0.20$.

(b) Use FFT techniques to evaluate the response when $\zeta = 0.002$.

(c) Use superposition and Appendix B to derive the analytical solution for this pulse. Compare the analytical and FFT results for $\zeta = 0.2$ and $\zeta = 0.002$.

2.8.3.1 part(a)

$$Q = 2000t\frac{(T-t)}{T^2}[h(t) - h(t-T)]$$

$$\begin{aligned} m &= 0.5 \text{ kg} \\ \omega_n &= 2\pi f_n \\ f_n &= 100 \text{ Hz} \end{aligned}$$

Hence pulse duration is $\frac{1}{f} = 0.01$ sec.

$$my'' + cy' + ky = Q(t)$$

In the frequency domain assuming that the force $Q(t)$ can be represented in its Fourier series as

$$Q(t) = \text{Re}\left(\sum_n Q_n e^{i\omega_1 nt}\right)$$

where ω_1 is the fundamental frequency for $Q(t)$ which depends on the period we choose to select to sample over. In this example, I selected $3T$ as the overall period to sample over so that it covers the pulse duration and an additional time to show the free vibration part as well and to compare to the analytical solution. Hence the EQM becomes

$$Y_n = \frac{Q_n}{-m(n\omega_1)^2 + ic(n\omega_1) + k}$$

$k = \omega_n^2 m$ hence dividing the numerator and denominator by k we obtain

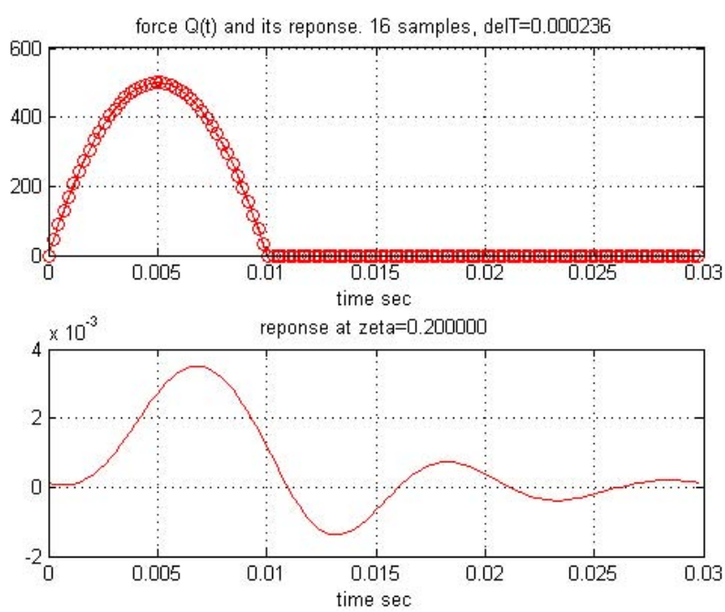
$$\begin{aligned} Y_n &= \frac{\frac{Q_n}{K}}{\left(1 - \frac{m(n\omega_1)^2}{\omega_n^2 m}\right) + \frac{ic(n\omega_1)}{\omega_n^2 m}} \\ &= \frac{1}{k}\frac{1}{\left(1 - r_n^2\right) + i2\zeta r_n} Q_n \end{aligned}$$

where $r_n = \frac{n\omega_1}{\omega_n}$.Hence response is

$$\begin{aligned} y(t) &= \mathrm{Re}\left(\sum_n Y_n e^{i\omega_1 nt}\right) \\ &= \mathrm{Re}\left(\sum_n \frac{1}{k}\frac{1}{\left(1 - r_n^2\right) + i2\zeta r_n} Q_n e^{i\omega_1 nt}\right) \end{aligned}$$

$y(t)$ is found by taking the IFFT of $\sum_n \frac{1}{k}\frac{1}{\left(1-r_n^2\right)+i2\zeta r_n} Q_n$.

Q_n values are found by taking the FFT of $Q(t)$. We start by sampling $Q(t)$. To obtain the solution for say $t = 0 \cdots 3T$, then we have to assume that the period of the signal is actually $3T$ and sample over this whole time from $0 \cdots 3T - delt$. Then we use FFT on the result. Then find the response by doing IFFT. Using $N = 128$ over $t = 0 \cdots 0.03$ seconds, the following solution was obtained



```
%by Nasser M. Abbasi, HW 7, EMA 545
close all;
T    = 0.01; %sec
duration = 3*T;  %duration to find solution over
N    = 128;
delT = duration/(N-1);
w1   = 2*pi/duration; %fundamental freq rad/sec
t    = linspace(0,(duration-delT),N);
Qt   = @(t) (2000*t.*(T-t))/T^2.*(t<=T)+0*(t>T)

subplot(2,1,1)
plot(t,Qt(t),'r-o');
hold on;
plot(0:delT:duration,Qt(0:delT:duration),'r');

title(sprintf('force Q(t) and its reponse. 16 samples, delT=%f',delT));
xlabel('time sec');
grid;

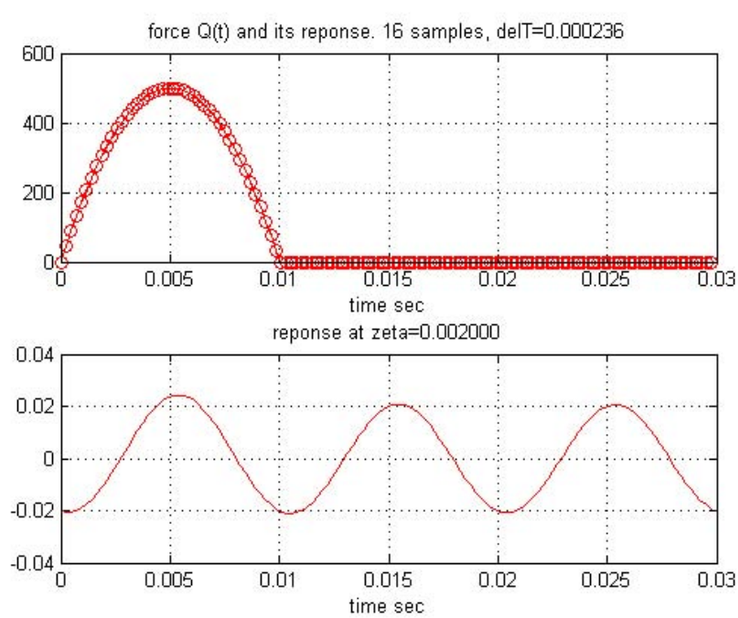
m      = 0.5; %mass kg
wn     = 2*pi*100; %natural freq
k      = wn^2*m;  %stiffness N/meter
[Q,ws] = fft_easy(Qt(t),delT);

zeta   = 0.002;
I      = sqrt(-1);
y      = ifft_easy( (Q/k)./( (1-(ws/wn).^2) + 2*I*zeta*ws/wn),ws);

subplot(2,1,2);
plot(t,y,'r');
title(sprintf('reponse at zeta=%f',zeta));
xlabel('time sec');
grid;
```

2.8.3.2 Part(b)

For $\zeta = 0.002$ the above Matlab script was modified and the following solution resulted.



Now we compare the above with the analytical solution.

2.8.3.3 Part(c)

The pulse can be written as

$$\begin{aligned} F &= Q(t)[h(t) - h(t-T)] \\ &= Q(t)h(t) - Q(t)h(t-T) \end{aligned}$$

Let $t' = t - T$, hence $t = t' + T$, therefore the above becomes

$$F = Q(t)h(t) - Q(t' + T)h(t')$$

But $Q(t) = \frac{2000t(T-t)}{T^2}$. Let $\frac{2000}{T^2} = \beta$ since it is a constant. Hence $Q(t) = \beta t(T-t)$. Now we write the above F as

$$\begin{aligned} F &= \beta t(T-t)h(t) - \beta(t'+T)(T-(t'+T))h(t') \\ &= \left(\beta Tt - \beta t^2\right)h(t) - \beta(t'+T)(-t')h(t') \\ &= \left(\beta Tt - \beta t^2\right)h(t) + \beta\left((t')^2 + Tt'\right)h(t') \\ &= \beta Tth(t) - \beta t^2 h(t) + \beta T(t')^2 + \beta Tt'h(t') \end{aligned} \tag{2.106}$$

So we see that the response to F will be the response to a unit impulse $h(t)$ with forcing basis functions that are $1, t, t^2$. Now we can use the solution from back of the book appendix B to sum the responses in order to find the final response and compare to the FFT method.

From appendix B, the response to unit ramp $th(t)$is

$$r(th(t)) = \frac{1}{m\omega_n^3}\left(\omega_n t - 2\zeta + e^{-\zeta\omega_n t}\left[2\zeta\cos\omega_d t - \left(1-2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$

and the response to quadratic $t^2h(t)$ is

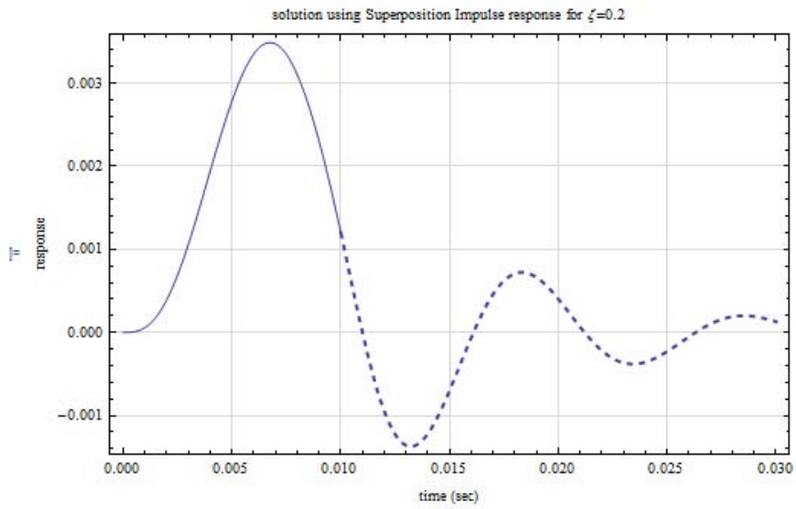
$$s\left(t^2h(t)\right) = \frac{1}{m\omega_n^4}\left((\omega_n t)^2 - 4\zeta\omega_n t - 2\left(1-4\zeta^2\right) + e^{-\zeta\omega_n t}\left[2\left(1-4\zeta^2\right)\cos\omega_d t + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t)$$

Now that we have the basis solutions, we can apply them to EQ 2.110

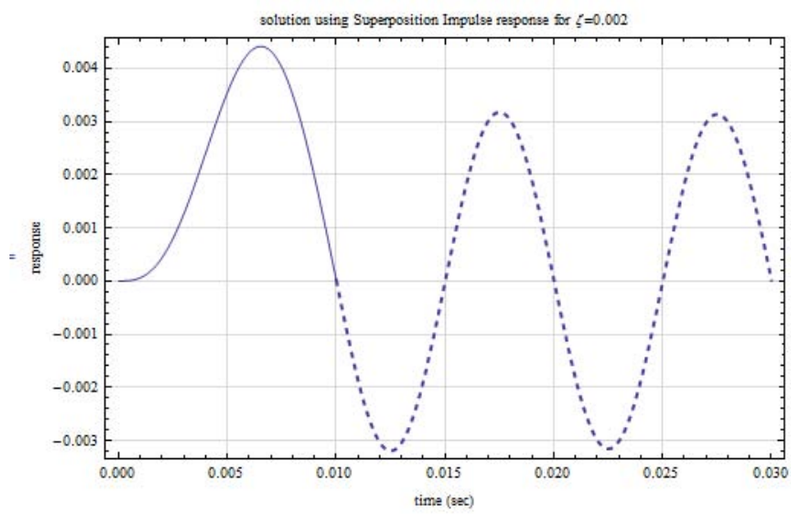
$$\begin{aligned} F &= \beta T(r(t) + r(t')) - \beta T(s(t) - s(t')) \\ &= \beta(r(t) + r(t-T)) - \beta T(s(t) - s(t-T)) \\ &= \left(\beta T\right)\frac{1}{m\omega_n^3}\left(\omega_n t - 2\zeta + e^{-\zeta\omega_n t}\left[2\zeta\cos\omega_d t - \left(1-2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t) \\ &+ \left(\beta T\right)\frac{1}{m\omega_n^3}\left(\omega_n t' - 2\zeta + e^{-\zeta\omega_n t'}\left[2\zeta\cos\omega_d t' - \left(1-2\zeta^2\right)\frac{\omega_n}{\omega_d}\sin\omega_d t'\right]\right)h(t') \\ &- \left(\beta\right)\frac{1}{m\omega_n^4}\left((\omega_n t)^2 - 4\zeta\omega_n t - 2\left(1-4\zeta^2\right) + e^{-\zeta\omega_n t}\left[2\left(1-4\zeta^2\right)\cos\omega_d t + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t\right]\right)h(t) \\ &+ \left(\beta\right)\frac{1}{m\omega_n^4}\left((\omega_n t')^2 - 4\zeta\omega_n t' - 2\left(1-4\zeta^2\right) + e^{-\zeta\omega_n t'}\left[2\left(1-4\zeta^2\right)\cos\omega_d t' + \left(6\zeta - 8\zeta^3\right)\frac{\omega_n}{\omega_d}\sin\omega_d t'\right]\right)h(t') \end{aligned}$$

In the above, $\omega_d = \omega_n\sqrt{1-\zeta^2}$. To plot this solution, the following small script was used and was run for both $\zeta = 0.2$ and $\zeta = 0.002$

For $\zeta = 0.2$



For $\zeta = 0.002$



2.8.3.4 Conclusions

The analytical solution, using superposition agreed with the FFT solution for $\zeta = 0.2$. However, for some reason which I am not able to determine why yet, the FFT solution when $\zeta = 0.002$ did not agree with the analytical solution. The analytical solution was verified to be correct using another numerical ODE solver. So the FFT method for some reason is not giving accurate result for $\zeta = 0.002$. The same Matlab script was used for both cases. I tried increasing the sampling rate but that did not change the result. Please see Appendix for verification and the code used to plot the analytical solutions.

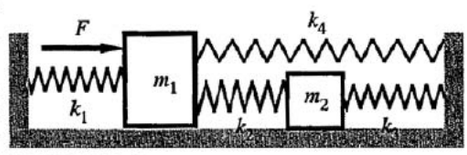
2.8.4 Problem 3

Problem 3: Exercise 1.11 from Ginsberg. (For the proof described in (b), set m_2=0 and see what your equation of motion reduces to.)

1.11 When the system in the sketch is at its static equilibrium position, there is no axial force in each spring.

(a) Derive equations of motion for the horizontal displacements x_1 and x_2 measured from the equilibrium position.

(b) Prove that if $m_2 = 0$, the equation of motion for x_1 is the same as that obtained by replacing the four springs by a single equivalent spring. Identify the way in which the springs are connected—for example, series or parallel—according to the equivalent spring stiffness.



EXERCISE 1.11

2.8.4.1 Part(a)

Let T be the kinetic energy and V be the potential energy. Then equation of motion for a generalized coordinate q_i is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Where L is the Lagrangian $L = T - V$ and Q_i is the generalized force in the q_i direction.

Assuming $x_2 > x_1$ and masses are moving to the right. For x_1 we obtain

$$\begin{aligned}
T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{x}_2^2 \\
V &= \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_4x_1^2 + \frac{1}{2}k_3x_2^2 \\
Q_1 &= F \\
Q_2 &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_1\dot{x}_2^2 - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2 + \frac{1}{2}k_4x_1^2 + \frac{1}{2}k_3x_2^2\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}_1} &= m_1\dot{x}_1 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) &= m_1\ddot{x}_1 \\
\frac{\partial L}{\partial x_1} &= -k_1x_1 - k_2(x_2 - x_1)(-1) - k_4x_1
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}_2} &= m_1\dot{x}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) &= m_1\ddot{x}_2 \\
\frac{\partial L}{\partial x_2} &= -k_2(x_2 - x_1)(1) - k_3x_2
\end{aligned}$$

Hence the 2 EOM are for x_1

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} &= F \\
m_1\ddot{x}_1 - (-k_1x_1 + k_2(x_2 - x_1) - k_4x_1) &= F \\
m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) + k_4x_1 &= F
\end{aligned}$$

Therefore EOM 1

$$m_1\ddot{x}_1 + (k_1 + k_2 + k_4)x_1 - k_2x_2 = F$$

and for x_2

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} &= 0 \\
m_1\ddot{x}_2 - (-k_2(x_2 - x_1) - k_3x_2) &= 0 \\
m_1\ddot{x}_2 + k_2(x_2 - x_1) + k_3x_2 &= 0
\end{aligned}$$

Hence EOM 2

$$m_1\ddot{x}_2 + (k_2 + k_3)x_2 - k_2x_1 = 0$$

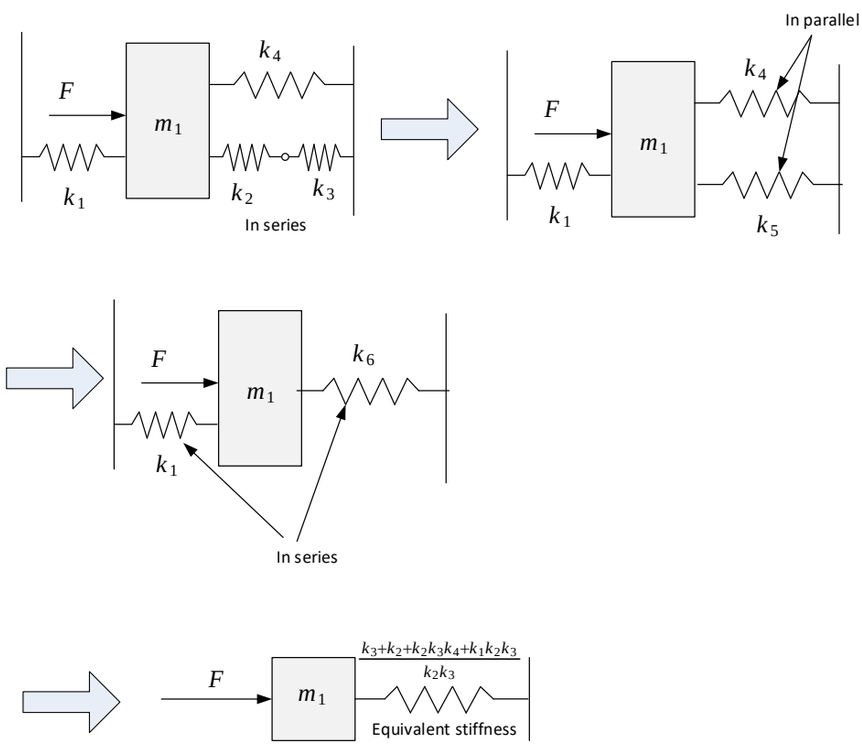
Hence in Matrix form EOM are

$$MX'' + KX = Q$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} + \begin{pmatrix} (k_1 + k_2 + k_4) & -k_2 \\ -k_2 & (k_2 + k_3) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

2.8.4.2 Part(b)

If m_2 do not exist, then this means the springs k_2 and k_3 do not have a mass between them and so these need to be replaced by single spring, say k_5 found by finding equivalent spring in series



$$\frac{1}{k_5} = \frac{1}{k_2} + \frac{1}{k_3}$$
$$k_5 = \frac{k_3 + k_2}{k_2k_3}$$

From above, EQM for m_1 becomes

$$m_1\ddot{x}_1 + \left(k_1 + \overbrace{\left(\frac{k_3 + k_2}{k_2k_3}\right)}^{k_5} + k_4\right)x_1 = F$$

So now k_4 and k_4 are in parallel, hence we replace $k_5 + k_4$ by k_6 found from

$$\begin{aligned} k_6 &= k_5 + k_4 \\ &= \left(\frac{k_3 + k_2}{k_2k_3}\right) + k_4 \\ k_6 &= \frac{k_3 + k_2 + k_2k_3k_4}{k_2k_3} \end{aligned}$$

Hence EQM for m_1 now becomes

$$m_1\ddot{x}_1 + \left(k_1 + \overbrace{\frac{k_3 + k_2 + k_2k_3k_4}{k_2k_3}}^{k_6}\right)x_1 = F$$

and finally

$$m_1\ddot{x}_1 + \frac{k_3 + k_2 + k_2k_3k_4 + k_1k_2k_3}{k_2k_3}x_1 = F$$

2.8.5 Key solution for HW 7

Homework #7
EMA 545, Spring 2013

Problem 1: (**20 pts**) Use FFT techniques to find the response of your vibration isolation system to the turbulent flight profile in "FlightAccel.mat" on the course website. (To do this, you will have to assume that the aircraft experiences this exact same flight profile over and over again.)

a.) Provide at least one plot comparing the response of the aircraft to the (hopefully improved) response on the vibration isolator.

b.) Report on the values of mass, stiffness and damping that you used and how they would be realized in practice (e.g. if you use a beam as a leaf spring, what would its dimensions be?).

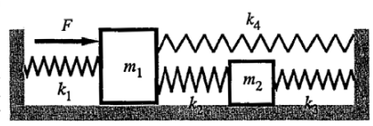
Problem 2: (**20 pts**) Do **Problem 3.60** as given in the text. Comment on how you selected adequate values for N (number of samples in the time history) and the length of the time window (in seconds).

Problem 3: Exercise 1.11 from Ginsberg. (For the proof described in (b), set m_2=0 and see what your equation of motion reduces to.)

1.11 When the system in the sketch is at its static equilibrium position, there is no axial force in each spring.

(a) Derive equations of motion for the horizontal displacements x_1 and x_2 measured from the equilibrium position.

(b) Prove that if $m_2 = 0$, the equation of motion for x_1 is the same as that obtained by replacing the four springs by a single equivalent spring. Identify the way in which the springs are connected—for example, series or parallel—according to the equivalent spring stiffness.



EXERCISE 1.11

Problem 3.60.
A SDOF system is subjected to a pulse excitation of the form:

$$Q(t) = \begin{cases} 2000t(T-t)/T^2 & 0 \le t \le T \\ 0 & t > T \end{cases}$$

We are told that m = 0.5 kg and that the natural frequency is 100Hz; i.e., the undamped period of vibration is 1/100 = 0.01 sec. Since the pulse-duration is said to be equal to an undamped period, T = 0.01 sec. We are to find the response of the system for the case of (a) ζ = 0.2 and (b) ζ = 0.002.

(a) For ζ = 0.2, the time-constant of the system is $1/\zeta\omega_n$= 1/(0.2*2*π*100) = 0.007958 sec. Four time constants would be about 0.032, so choosing a maximum time of Tmax = 0.05 sec will ensure that we allow enough time for the transient response to die down to an acceptably small value. The next parameter that we need to choose is N. Let's choose N = 256 and then check that our Nyquist critical frequency is high enough relative to the natural frequency of the system and the bandwidth of the input spectrum. Figure 1 shows the FFT of the input spectrum, the FFT of the output spectrum, and the system transfer function as a function of frequency ratio for Tmax = 0.05 sec and N = 256. Each curve is scaled so that its maximum value is unity. Although difficult to see whether there are enough points in the vicinity of the resonant peak, it is clearly the case that our Nyquist frequency is high enough. Figure 2 shows the displacement response of the system for Tmax = 0.05 sec and using 2 different values of N. Below N = 64, the errors in the response become much more noticeable. Note that the response is very small at t = 0.05 sec, indicating that "wraparound errors" are negligable.

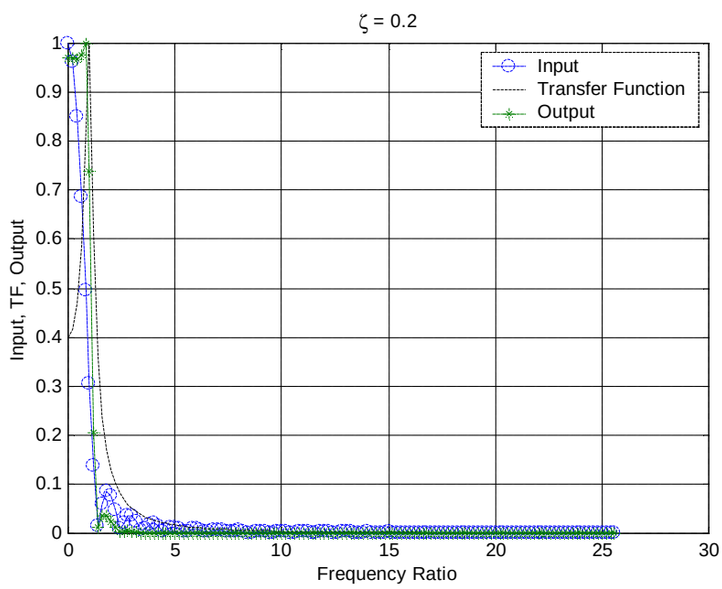


Figure 1.

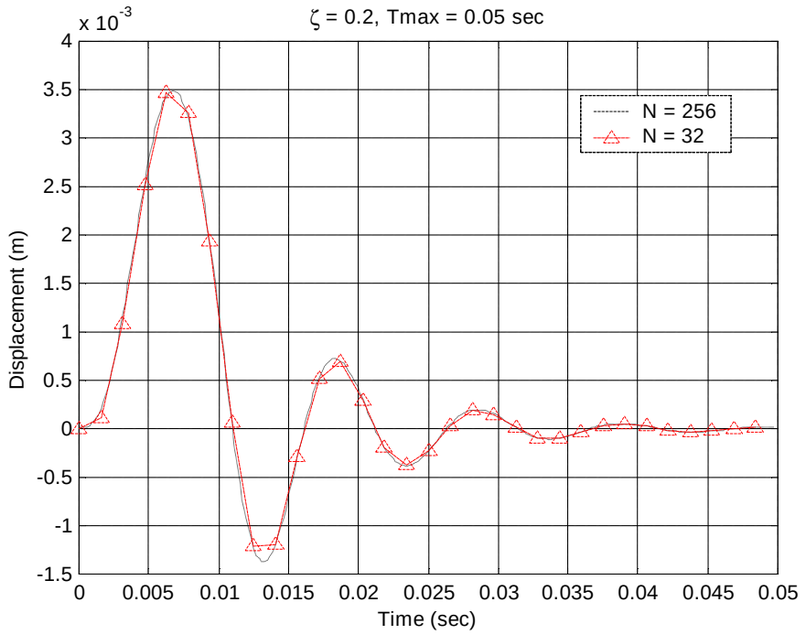


Figure 2.

Figure 3 shows the effect of varying Tmax, keeping N = 256. It is seen that as Tmax is reduced, the solution begins to degrade. In particular, examining the curve for Tmax = 0.02 sec, it is seen that the response no longer appears to begin with zero initial conditions. Thus, using a Tmax which is twice the duration of the pulse is inadequate in this case to avoid warparound errors.

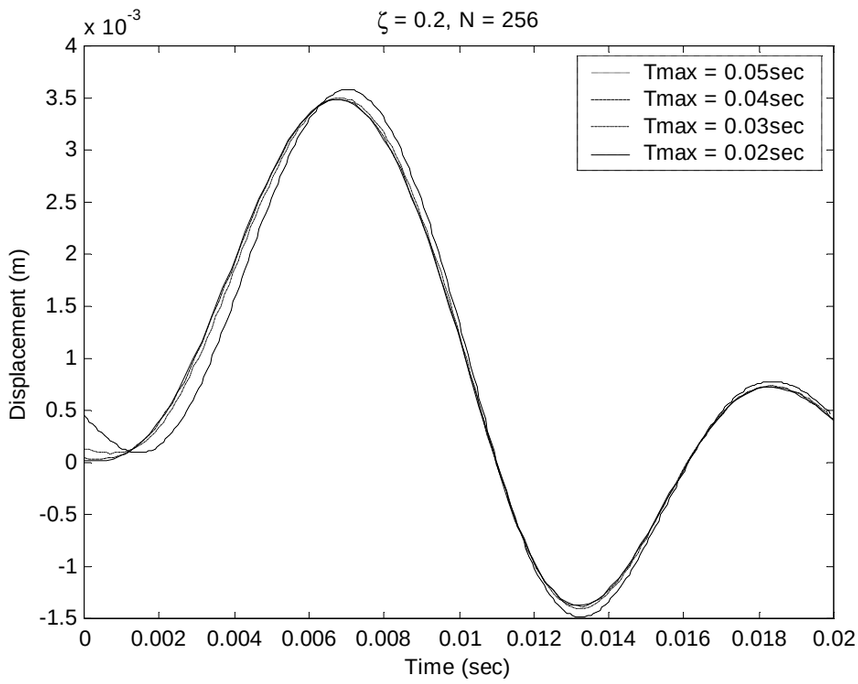


Figure 3.

(b) For $\zeta = 0.002$, the time-constant of the system is $1/\zeta\omega_n = 1/(0.2*2*\pi*100) = 0.7958$ sec. Four time constants would be about 3.2, so choosing a maximum time of Tmax = 5 sec should ensure that we allow enough time for the transient response to die down to an acceptably small value. As before, we will choose a value for N (4096) and then check that our Nyquist critical frequency is high enough relative to the natural frequency of the system and the bandwidth of the input spectrum. Figure 4 shows the FFT of the input spectrum, the FFT of the output spectrum, and the system transfer function as a function of frequency ratio for Tmax = 5 sec and N = 4096. As before, each curve is scaled so that its maximum value is unity. We observe that there appear to be enough points in the vicinity of the resonant peak and that the Nyquist frequency appears to be high enough. Figure 5 shows the displacement response of the system for Tmax = 5 sec and N = 4096. The result was checked using ode45 and found to be very close to the values produced using the FFT analysis. Figure 6 shows the first 0.01 seconds of response from Figure 5 on a larger scale. The fact that the response begins with nearly zero displacement and velocity shows that wraparound error has been avoided.

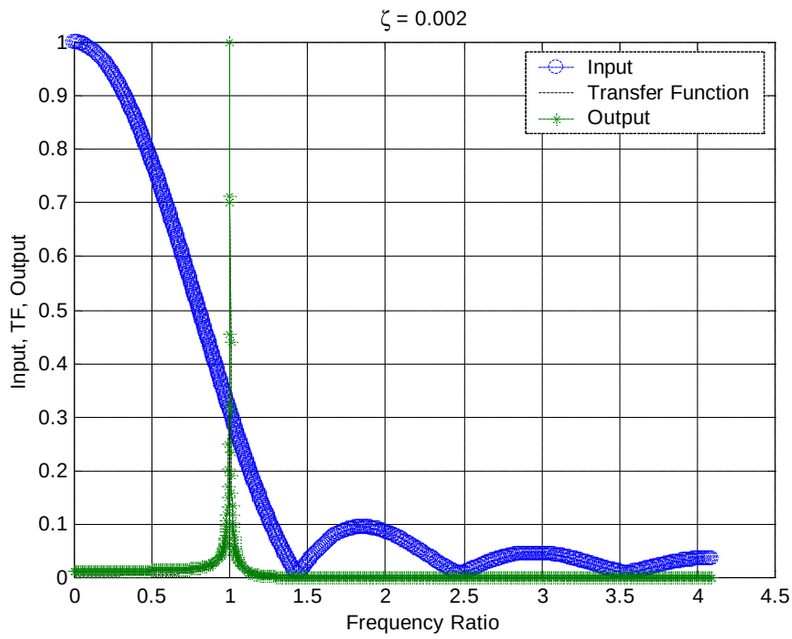


Figure 4.

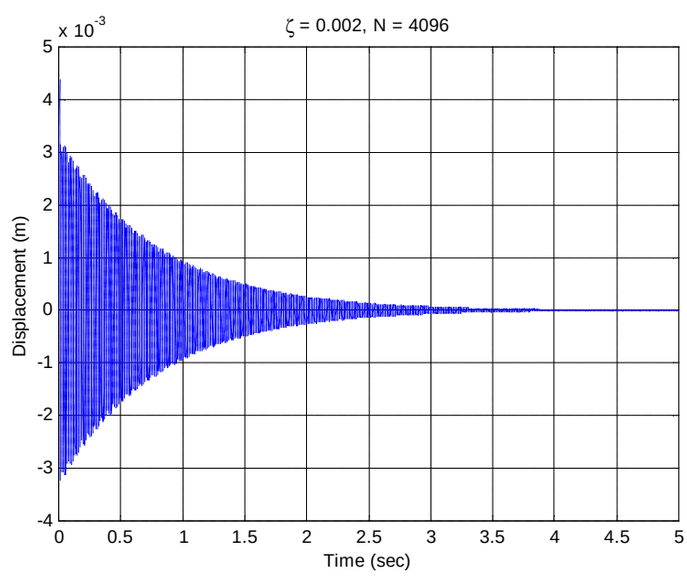


Figure 5. Tmax = 5 sec, N = 4096 points.

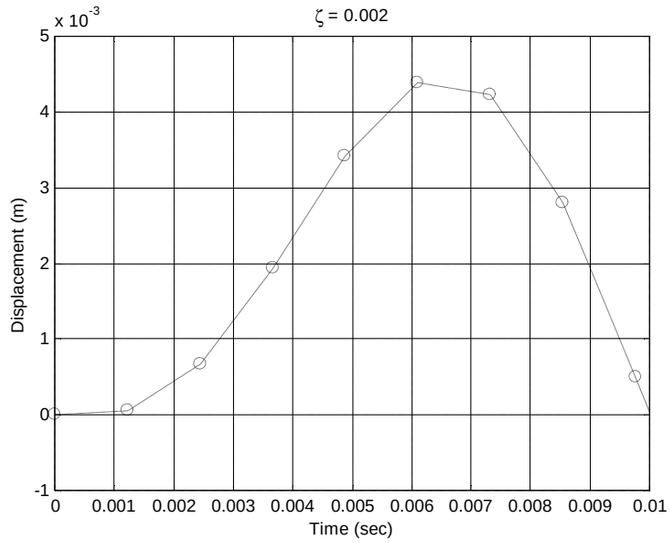


Figure 6. Tmax = 5 sec, N = 4096. Same result as shown in Figure 5, but over a smaller time range.

The Matlab code used to generate these results is shown below:

```
% ME6442, Spring 2002
% Dr. Ferri
% Problem 3.60

% Q is a short-pulse parabolic input. The system is
% a damped SDOF system with m = 0.5kg, and fn = 100 Hz. The duration of
% the pulse is Tn, an undamped period of motion.

m = 0.5 ; %kg
zeta = 0.2;  % or zeta = 0.002 for part (b)
wn = 100*2*pi; % rad/s
timeconstant = 1/(zeta*wn)
T = 2*pi/wn; % duration of pulse
Tmax = input('Enter Tmax...  ');
N = input('Enter N...  ');
w1 = 2*pi/Tmax;
delt = Tmax/N;
t = 0:delt:(N-1)*delt;

% Define input pulse

Q = zeros(1,N);
for k = 1:N;
    if t(k) > T; break; end
```

```
    Q(k) = 2000*t(k).*(T-t(k))/T^2;
end

plot(t,Q);
xlabel('Time (sec)')
ylabel('Q')
title(['N = ',num2str(N),' , \zeta = ',num2str(zeta)])
grid
pause

Q_trans = fft(Q);

r = (0:(N/2))*w1/wn;
D = 1./(1 - r.^2 + i*2*zeta*r);
Xhalf = Q_trans(1:(N/2+1)).*D/(m*wn^2);
rev_index = (N/2):-1:2;
X = [Xhalf conj(Xhalf(rev_index))];

% Plot input fft, output fft, and TF on same plot.

Qplot = Q_trans(1:(N/2+1))/max(abs(Q_trans(1:(N/2+1))));
Dplot = D/max(abs(D));
Xplot = Xhalf/max(abs(Xhalf));
plot(r,abs(Qplot),'o--',r,abs(Dplot),'k-',r,abs(Xplot),'*--')
xlabel('Frequency Ratio')
ylabel('Input, TF, Output')
title(['\zeta = ',num2str(zeta)]);
grid
legend('Input','Transfer Function','Output')
pause


xifft = ifft(X);

% Check that the ifft is "mostly real-valued." The quantity
% imag_check is the ratio of the norms of the imaginary and
% reak parts. This term should be very small (<1e-5).

imag_check = norm(imag(xifft))/norm(real(xifft))

% Assuming that imag_check is small, discard the imaginary part
% of xifft:

xifft=real(xifft);

plot(t,xifft);
xlabel('Time');
ylabel('x');
title(['N = ',num2str(N),' , \zeta = ',num2str(zeta)])
grid
zoom on
```

Exercise 3.60

Superposition solution : $t' = t - T$

$$Q(t) = 2000\left(\frac{t}{T} - \frac{t^2}{T^2}\right)h(t) - 2000\left[\frac{t'+T}{T} - \left(\frac{t'+T}{T}\right)^2\right]^2 h(t')$$

$$= \frac{2000}{T}\, t\, h(t) - \frac{2000}{T^2}\, t^2 h(t) + \frac{2000}{T}\, t' h(t') + \frac{2000}{T^2}(t')^2 h(t')$$

Let $r(t)$ and $s(t)$ denote the unit ramp and quadratic responses. Then

$$q = \frac{2000}{T}\left[r(t) + r(t-T)\right] - \frac{2000}{T^2}\left[s(t) - s(t-T)\right]$$

$M := 0.5$ $\quad \omega_{nat} := 200\cdot\pi$ $\quad \tau := \frac{2\cdot\pi}{\omega_{nat}}$ Note: Change pulse duration to τ

Superposition solution

$$\mathrm{ramp}(t,\zeta) := (t>0)\cdot\frac{1}{M\cdot\omega_{nat}^{3}}\cdot\left[\omega_{nat}\cdot t - 2\cdot\zeta + \exp\left(-\zeta\cdot\omega_{nat}\cdot t\right)\cdot\left[2\cdot\zeta\cdot\cos\left(\sqrt{1-\zeta^2}\cdot\omega_{nat}\cdot t\right) \ldots + -\frac{1-2\cdot\zeta^2}{\sqrt{1-\zeta^2}}\cdot\sin\left(\sqrt{1-\zeta^2}\cdot\omega_{nat}\cdot t\right)\right]\right]$$

$$\mathrm{quad}(t,\zeta) := (t>0)\cdot\frac{1}{M\cdot\omega_{nat}^{4}}\cdot\left[\left(\omega_{nat}\cdot t\right)^2 - 4\cdot\zeta\cdot\omega_{nat}\cdot t - 2\cdot\left(1-4\cdot\zeta^2\right) \ldots + \exp\left(-\zeta\cdot\omega_{nat}\cdot t\right)\cdot\left[2\cdot\left(1-4\cdot\zeta^2\right)\cdot\cos\left(\sqrt{1-\zeta^2}\cdot\omega_{nat}\cdot t\right) \ldots + \frac{6\cdot\zeta - 8\cdot\zeta^3}{\sqrt{1-\zeta^2}}\cdot\sin\left(\sqrt{1-\zeta^2}\cdot\omega_{nat}\cdot t\right)\right]\right]$$

$$q_{super}(t,\zeta) := \frac{2000}{\tau}\cdot\left(\mathrm{ramp}(t,\zeta) + \mathrm{ramp}(t-\tau,\zeta)\right) - \frac{2000}{\tau^2}\cdot\left(\mathrm{quad}(t,\zeta) - \mathrm{quad}(t-\tau,\zeta)\right)$$

First case $\zeta := 0.20$

$\frac{4}{\zeta \cdot \omega_{nat}} = 0.032$ This is greater than $\tau = 0.01$, so let $T' = 4/\zeta^*\omega_{nat}$ $T' := \frac{4}{\zeta \cdot \omega_{nat}}$

Because damped period is larger than the undamped period, select $\Delta' > \pi/2\omega_{nat}$ $\Delta' := \frac{\pi}{8 \cdot \omega_{nat}}$

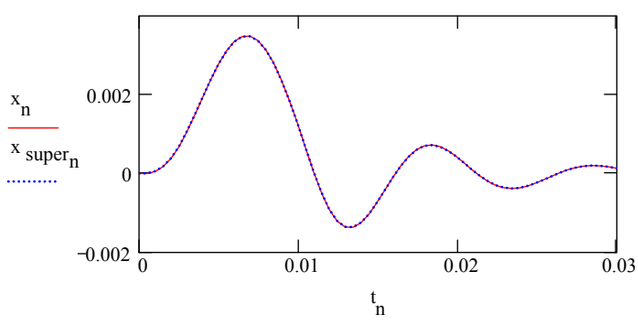
$N' := \frac{2 \cdot T'}{\Delta'}$ $N' = 101.859$ $N := 128$ $T := 2 \cdot T'$ $\Delta := \frac{T}{N}$

$j := 1 .. N$ $t_j := (j - 1) \cdot \Delta$ $Q_j := 2000 \cdot \frac{t_j \cdot (\tau - t_j)}{\tau^2} \cdot (t_j < \tau)$

$F := FFT(Q)$ $r_{fund} := \frac{2 \cdot \pi}{\omega_{nat} \cdot T}$ $K := M \cdot \omega_{nat}^2$

$n := 1 .. \frac{N}{2} + 1$ $X_n := \frac{F_n}{K} \cdot \frac{1}{1 - \left[(n - 1) \cdot r_{fund}\right]^2 + 2i \cdot \zeta \cdot \left[(n - 1) \cdot r_{fund}\right]}$ $x := IFFT(X)$

$x_{super_n} := q_{super}(t_n, \zeta)$



$\tau = 0.01$

Second case $\zeta := 0.0020$

$\frac{4}{\zeta \cdot \omega_{nat}} = 3.183$ This is greater than τ = 0.01, so let T' = 4/ζ*ω_{nat} $T' := \frac{4}{\zeta \cdot \omega_{nat}}$

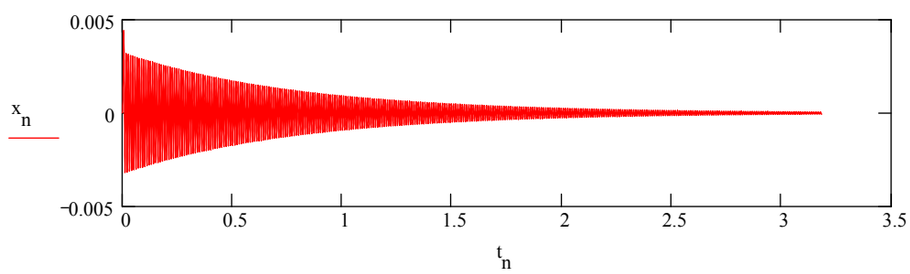
Because damped is larger than the undamped period, select $\Delta' > \pi/2\omega_{nat}$ $\Delta' := \frac{\pi}{8 \cdot \omega_{nat}}$

$N' := \frac{2 \cdot T'}{\Delta'}$ $N' = 1.019 \cdot 10^4$ $N := 1024 \cdot 16$ $T := 2 \cdot T'$ $\Delta := \frac{T}{N}$

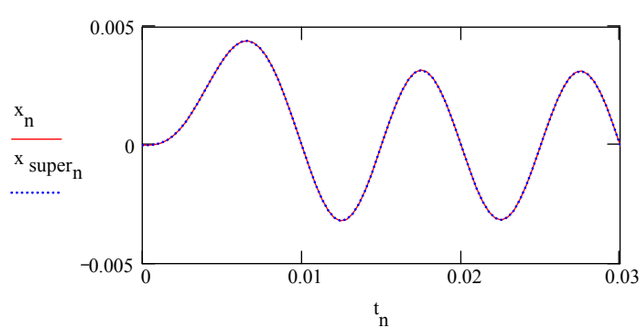
$j := 1 .. N$ $t_j := (j - 1) \cdot \Delta$ $Q_j := 2000 \cdot \frac{t_j \cdot (\tau - t_j)}{\tau^2} \cdot (t_j < \tau)$

$F := FFT(Q)$ $r_{fund} := \frac{2 \cdot \pi}{\omega_{nat} \cdot T}$ $K := M \cdot \omega_{nat}^2$

$n := 1 .. \frac{N}{2} + 1$ $X_n := \frac{F_n}{K} \cdot \frac{1}{1 - \left[(n - 1) \cdot r_{fund}\right]^2 + 2i \cdot \zeta \cdot \left[(n - 1) \cdot r_{fund}\right]}$ $x := IFFT(X)$



$x_{super_n} := q_{super}(t_n, \zeta)$



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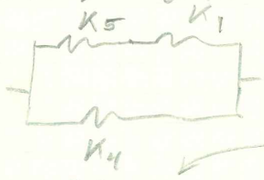
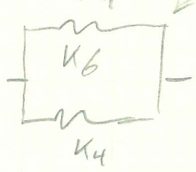
P1 (1.1) eq. to K_1



$K_5 = K_2 + K_3$ (parallel)

K_1, K_5 in series

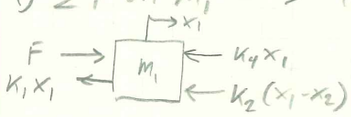


$K_6 = \frac{K_5 K_1}{K_5 + K_1}$

$\Rightarrow \boxed{K_{eq} = \frac{(K_2+K_3)K_1}{K_1+K_2+K_3} + K_4}$

P2 (1.11)

a) ΣF on m_1 $\longrightarrow$ let x_1, x_2 = disp. of m_1, m_2 to right

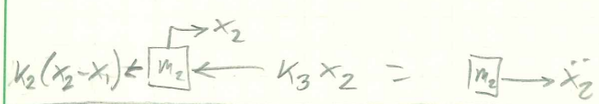


$= \boxed{m_1} \longrightarrow m_1 \ddot{x}_1$

Check, $x_1 > x_2$, spring K_2 pushes left ✓

$F - K_4 x_1 - K_1 x_1 - K_2(x_1 - x_2) = m_1 \ddot{x}_1$

$$\boxed{m_1 \ddot{x}_1 + (K_4 + K_1 + K_2)x_1 - K_2 x_2 = F} \quad (1)$$



$= \boxed{m_2} \longrightarrow \ddot{x}_2$

$$\boxed{m_2 \ddot{x}_2 + (K_2 + K_3)x_2 - K_2 x_1 = 0} \quad (2)$$

✓ 2 masses, so we have 2 EOM

b) set $m_2 = 0$ in (2), solve for $K_2 x_2$ $\longrightarrow$

$K_2 x_2 = \frac{K_2^2 x_1}{(K_2 + K_3)}$ $\longrightarrow$ plug into (1)

$m_1 \ddot{x}_1 + (K_4 + K_1 + K_2)x_1 - \frac{K_2^2 x_1}{(K_2+K_3)} = F$

$m_1 \ddot{x}_1 + (K_4 + K_1)x_1 + \frac{K_2(K_2+K_3) - K_2^2}{(K_2+K_3)} x_1 = F$

$$\boxed{m_1 \ddot{x}_1 + \left(K_4 + K_1 + \frac{K_2 K_3}{(K_2+K_3)}\right) x_1 = F}$$

(Identical to system with K_1, K_4 and K_5 in //

with $K_5 = \frac{K_2 K_3}{(K_2+K_3)}$ = K_2 and K_3 in series. ✓

2.8.6 appendix

Appendix, HW7, EMA 545, spring 2013

- **Analytical (superposition) solution for HW7 EMA 545**

```
T = 0.01;
wn = 2.0 * Pi * 100;
m = 0.5;
```

In[108]:=
$$\text{wd}[\zeta\_] := \text{wn}\sqrt{1-\zeta^2};$$
$$\beta = \frac{2000}{T^2};$$
$$\text{tp} = t - T;$$
$$\text{f1}[t\_, \zeta\_] := \beta\, T \frac{1}{m\,\text{wn}^3}\left(\text{wn}\, t - 2\,\zeta + \text{Exp}[-\zeta\,\text{wn}\, t]\left(2\,\zeta\,\text{Cos}[\text{wd}[\zeta]\, t] - \left(1-2\,\zeta^2\right)\frac{\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\, t]\right)\right);$$
$$\text{f2}[t\_, \zeta\_] := \beta\frac{1}{m\,\text{wn}^4}\left((\text{wn}\, t)^2 - 4\,\zeta\,\text{wn}\, t - 2\left(1-4\,\zeta^2\right) + \text{Exp}[-\zeta\,\text{wn}\, t]\left(2\left(1-4\,\zeta^2\right)\text{Cos}[\text{wd}[\zeta]\, t] + \left(6\,\zeta - 8\,\zeta^3\right)\frac{\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\, t]\right)\right);$$
$$\text{f3}[t\_, \zeta\_] := 2\,\beta\, T\frac{1}{m\,\text{wn}^3}\left(\text{wn}\,\text{tp} - 2\,\zeta + \text{Exp}[-\zeta\,\text{wn}\,\text{tp}]\left(2\,\zeta\,\text{Cos}[\text{wd}[\zeta]\,\text{tp}] - \left(1-2\,\zeta^2\right)\frac{\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\,\text{tp}]\right)\right);$$
$$\text{f4}[t\_, \zeta\_] := \beta\frac{1}{m\,\text{wn}^4}\left((\text{wn}\,\text{tp})^2 - 4\,\zeta\,\text{wn}\,\text{tp} - 2\left(1-4\,\zeta^2\right) + \text{Exp}[-\zeta\,\text{wn}\,\text{tp}]\left(2\left(1-4\,\zeta^2\right)\text{Cos}[\text{wd}[\zeta]\,\text{tp}] + \left(6\,\zeta - 8\,\zeta^3\right)\frac{\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\,\text{tp}]\right)\right);$$
$$\text{f5}[t\_, \zeta\_] := 2\,\beta\, T^2\frac{1}{m\,\text{wn}^2}\left(1 - \text{Exp}[-\zeta\,\text{wn}\,\text{tp}]\left(\text{Cos}[\text{wd}[\zeta]\,\text{tp}] + \frac{\zeta\,\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\,\text{tp}]\right)\right);$$
$$\text{f6}[t\_, \zeta\_] := \beta\, T\frac{1}{m\,\text{wn}^3}\left(\text{wn}\,\text{tp} - 2\,\zeta + \text{Exp}[-\zeta\,\text{wn}\,\text{tp}]\left(2\,\zeta\,\text{Cos}[\text{wd}[\zeta]\,\text{tp}] - \left(1-2\,\zeta^2\right)\frac{\text{wn}}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\,\text{tp}]\right)\right);$$
$$\text{freeResponse}[t\_, q0\_, v0\_, \zeta\_] := \text{Exp}[-\zeta\,\text{wn}\, t]\left(q0\,\text{Cos}[\text{wd}[\zeta]\, t] + \frac{v0 + \zeta\,\text{wn}\, q0}{\text{wd}[\zeta]}\,\text{Sin}[\text{wd}[\zeta]\, t]\right);$$

In[118]:=
```
impulseResponse[t_, ζ_] := f1[t, ζ] UnitStep[t] - f2[t, ζ] UnitStep[t] - f3[t, ζ] UnitStep[tp] +
    f4[t, ζ] UnitStep[tp] + f5[t, ζ] UnitStep[tp] - f6[t, ζ] UnitStep[tp];
```

- **case ζ=0.2**

In[121]:=
```
ζ = 0.2;
p1 = Plot[impulseResponse[t, ζ] (UnitStep[t] - UnitStep[tp]), {t, 0, T}];
```

- **Evaluate IC at end of impulse to use for free vibration response**

In[123]:=
```
q0 = (f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ]) /. t → T
```

Out[123]= 0.00120186

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2 | *problem_2.nb*

In[124]:=
```
v0 = D[f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ], t] /. t → T
```

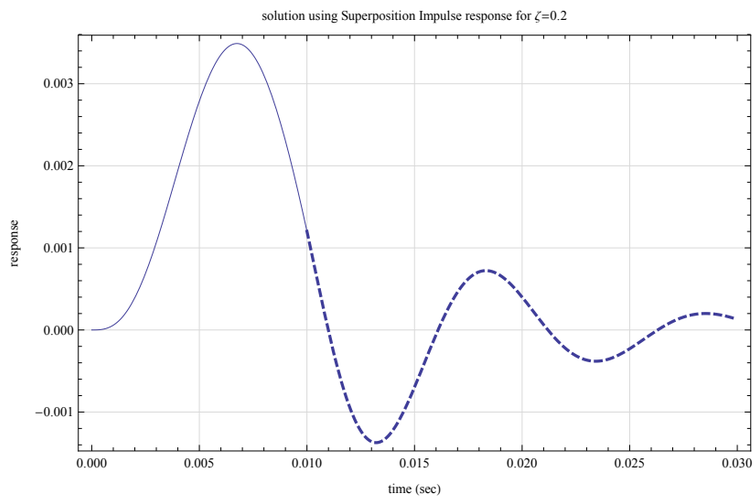
Out[124]= -1.21014

In[125]:=
```
p2 = Plot[freeResponse[tp, q0, v0, ζ] UnitStep[tp],
    {t, T, 3 * T}, Exclusions → None, PlotStyle → {Dashed, Thick}];
```

- **Plot the complete solution by combining the above 2 plots**

In[126]:=
```
Show[p1, p2, PlotRange → All, Frame → True, FrameLabel → {{"response", None},
    {"time (sec)", Row[{"solution using Superposition Impulse response for ζ=", ζ}]}},
 GridLines → Automatic, GridLinesStyle → LightGray, Axes → False, ImageSize → 500]
```

Out[126]=



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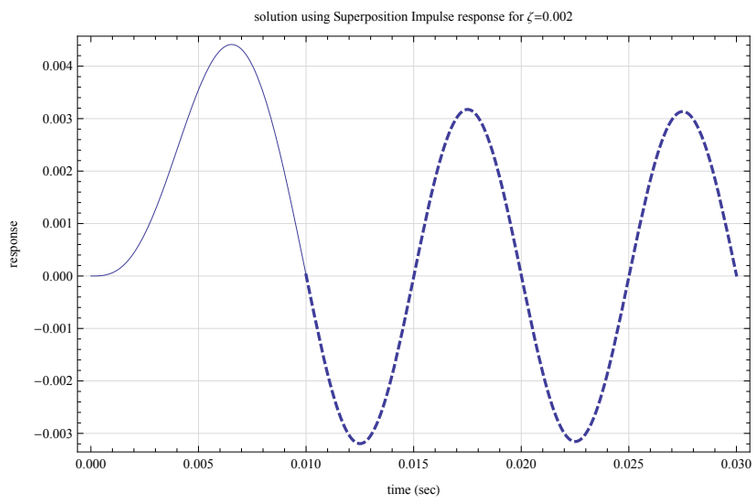
- **case ζ=0.002**

In[137]:=
```
ζ = 0.002;
p1 = Plot[impulseResponse[t, ζ] (UnitStep[t] - UnitStep[tp]), {t, 0, T}];
q0 = (f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ]) /. t → T;
v0 = D[f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ], t] /. t → T;
p2 = Plot[freeResponse[tp, q0, v0, ζ] UnitStep[tp],
   {t, T, 3 * T}, Exclusions → None, PlotStyle → {Dashed, Thick}];
Show[p1, p2, PlotRange → All, Frame → True, FrameLabel → {{"response", None},
   {"time (sec)", Row[{"solution using Superposition Impulse response for ζ=", ζ}]}},
 GridLines → Automatic, GridLinesStyle → LightGray, Axes → False, ImageSize → 500]
```

Out[142]=



- **Verify the solutions using Numerical DE solver**

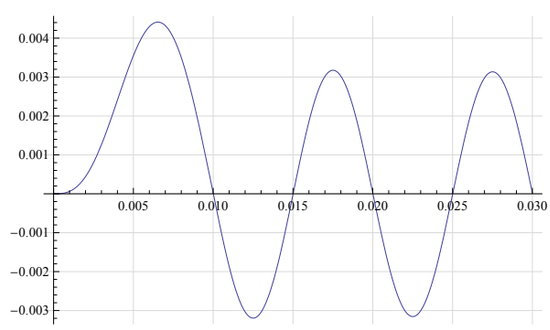
In[143]:=
```
ζ = 0.002;
eq = y''[t] + 2 ζ wn y'[t] + wn^2 y[t] == f[t] / m
```

Out[144]= $394\,784.\ y[t] + 2.51327\ y'[t] + y''[t] == 4.\times 10^{7}\ (0.01 - t)\ t\ (-\text{UnitStep}[-0.01 + t] + \text{UnitStep}[t])$

In[145]:=
```
sol = First@NDSolve[{eq, y'[0] == 0, y[0] == 0}, y[t], {t, 0, 3 T}];
```

In[147]:=
```
Plot[y[t] /. sol, {t, 0, 3 T}, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[147]=



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The above shows that the analytical solution using superposition is correct.

2.9 HW8

Local contents

2.9.1 problem description

Homework #8
EMA 545, Spring 2013

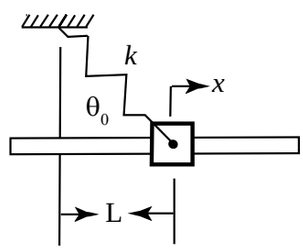
Problem 1: (40 points)

a.) Find the nonlinear equation of motion for the system pictured below. The block has mass *m* and the guide can be approximated as frictionless. In the position shown the spring is unstretched and the angle between the spring and guide bar is θ_0.

b.) Linearize your equation of motion for small deflections from the position shown (i.e. using a Taylor series expansion on $k(x)$ about x=0). Use a computer to plot k(x) versus the linear approximation for L=1 m, k=1000 N/m and θ_0 = 45 degrees for x ranging from -1 m to +1 m.

c.) Find the equations of motion for the system using the stiff spring approximation and assuming small displacements from an equilibrium position defined by L=1 m, k=1000 N/m and θ_0 = 45 degrees. Compare your result with your linearized result from part (b).

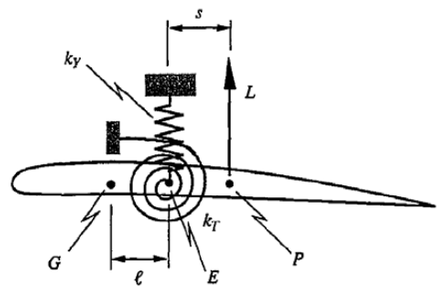
d.) Using m=1, find the response of the nonlinear system (in part a) using ode45 and plot the displacement of the mass over a few cycles when it is released from rest at x(0)=0.1 and also at x(0)=0.5 meters. Overlay both curves on the same set of axes. How does the period of the response compare with the linearized natural frequency in each case? In what other way(s) does the nonlinearity manifest itself in the response of the system when x(0)=0.5?



Problem 2: Exercise 1.27 from Ginsberg.

A standard model for a wing has a translational spring k_y and a torsional spring k_T representing the elastic rigidity. Point E represents the elastic center because static application of a vertical force at that point results in upward displacement without an associated rotation. The design of the wing is such that horizontal movement of point E is negligible. The lift force L acts at point P, which is called the center of pressure. The lift force may be treated as known. When the wing is in its static equilibrium position, points G, E and P form a horizontal line. Point G is the center of mass, and the radius of gyration of the wing about that point is r_G. Denote the mass of the wing m. Derive the equations of motion for the wing, assuming small displacements (and small rotational displacements). Put the equations in matrix form and check the units and sign of each

term in your EOM. (Hint: use the displacement of the center of gravity and the rotation of the wing as generalized coordinates.)

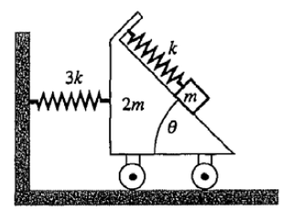


EXERCISE 1.27

Problem 3: Use the power balance method and the stiff spring approximation to find the equation of motion of the system pictured in Problem 1.16.

Problem 4: Exercise 1.33 from Ginsberg: (be very careful to write a correct expression for the acceleration of the small block.) Check the unit and sign of each term in your EOM.

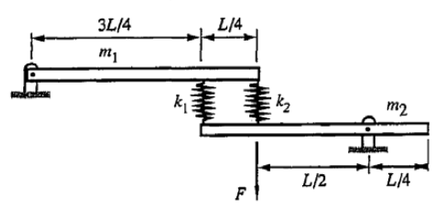
1.33 Determine the equations of motion governing a pair of generalized coordinates that locate the position of the cart and the sliding block. Friction is negligible.



EXERCISE 1.33

Problem 5: Exercise 1.30 from Ginsberg: Use the stiff spring approximation and assume small deflections of both bars. Check the units and sign of each term in your EOM. Gravity acts downward (same direction as the force, F).

1.30 Both bars in the linkage are horizontal, as shown, when the system is in static equilibrium. Determine the linearized equations of motion for this system.



EXERCISE 1.30

Problem 6: Exercise 4.1 in Ginsberg. Solve the eigenvalue problem by hand to get the natural frequencies and mode shapes. You may check your answers with Matlab.

2.9.2 problem 1

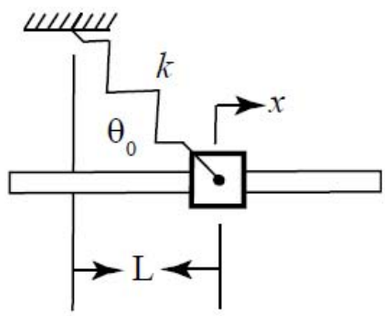
Problem 1: (40 points)

a.) Find the nonlinear equation of motion for the system pictured below. The block has mass m and the guide can be approximated as frictionless. In the position shown the spring is unstretched and the angle between the spring and guide bar is θ_0.

b.) Linearize your equation of motion for small deflections from the position shown (i.e. using a Taylor series expansion on $k(x)$ about x=0). Use a computer to plot k(x) versus the linear approximation for L=1 m, k=1000 N/m and θ_0 = 45 degrees for x ranging from -1 m to +1 m.

c.) Find the equations of motion for the system using the stiff spring approximation and assuming small displacements from an equilibrium position defined by L=1 m, k=1000 N/m and θ_0 = 45 degrees. Compare your result with your linearized result from part (b).

d.) Using m=1, find the response of the nonlinear system (in part a) using ode45 and plot the displacement of the mass over a few cycles when it is released from rest at x(0)=0.1 and also at x(0)=0.5 meters. Overlay both curves on the same set of axes. How does the period of the response compare with the linearized natural frequency in each case? In what other way(s) does the nonlinearity manifest itself in the response of the system when x(0)=0.5?



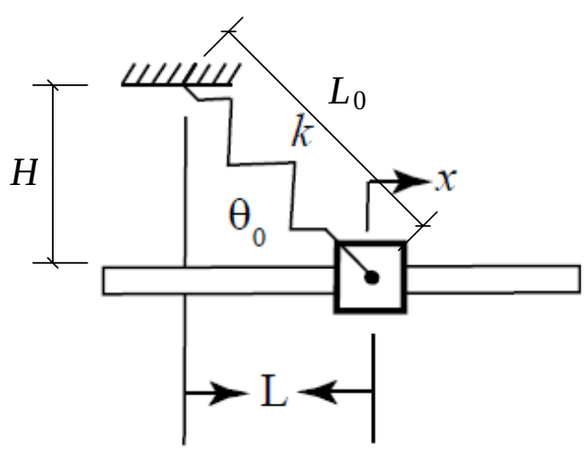
2.9.2.1 part(a)

Let initial length of the spring (un stretched length) be L_0 and when the mass m has moved to the right by an amount x then let the current length be L_{cur}.

Therefore the stretch in the spring is

$$\Delta = L_{cur} - L_0$$

Let the height of the bar by H, where $\tan\theta_0 = \frac{H}{L}$ or $H = L\tan\theta_0$



Hence from the above diagram we see that $L_0 = \sqrt{H^2 + L^2}$ and $L_{cur} = \sqrt{H^2 + (L + x)^2}$,

therefore

$$\Delta = \sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}$$
$$\Delta^2 = \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2$$

Now we can derive the equation of motion using energy methods.

Let T be the current kinetic energy in the system, and let V be the current potential energy. This system is one degree of freedom, since we only need one generalized coordinate to determine the position of the mass m. This coordinate is x.

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ V &= \frac{1}{2}k\Delta^2 \\ &= \frac{1}{2}k\left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2 \end{aligned}$$

Hence the Lagrangian Φ is

$$\begin{aligned} \Phi &= T - V = T - V \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2 \end{aligned}$$

Now the equation of motion for coordinate x is (using the standard Lagrangian form)

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} = Q_x$$

But Q_x, then generalized force, is zero since there is no external force and no damping. Now we just need to evaluate each part of the above expression to obtain the EOM.

$$\begin{aligned} \frac{\partial \Phi}{\partial \dot{x}} &= m\dot{x} \\ \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) &= m\ddot{x} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2\right) \\ &= -k\left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)\frac{1}{2}\left(H^2 + (L+x)^2\right)^{\frac{-1}{2}} 2(L+x) \\ &= -k\left(\frac{\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}}{\sqrt{H^2 + (L+x)^2}}\right)(L+x) \end{aligned}$$

Hence EOM becomes

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} = 0$$
$$m\ddot{x} + k\left(\frac{\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}}{\sqrt{H^2 + (L+x)^2}}\right)(L+x) = 0$$

2.9.2.2 part(b)

For small x we need to expand $f(x) = k\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)$ around $x=0$ in Taylor series and let higher powers of x go to zero.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + HOT.$$

$$\begin{aligned}
f(0) &= k\left(\frac{\sqrt{H^2+(L+0)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+0)^2}}\right)(L+0)\\
&= k\left(\frac{\sqrt{H^2+L^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+L^2}}\right)L\\
&= 0
\end{aligned}$$

and now for $f'(0)$

$$\begin{aligned}
f'(0) &= \frac{d}{dx}f(x)_{x=0}\\
&= k\frac{d}{dx}\left[\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)\right]_{x=0}\\
&= k\left((L+x)\frac{d}{dx}\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)+\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)\frac{d}{dx}(L+x)\right)_{x=0}\\
&= k\left((L+x)\left(\sqrt{H^2+L^2}\,\frac{L+x}{\left(H^2+L^2+2Lx+x^2\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)\right)_{x=0}
\end{aligned}$$

Now we evaluate it at $x=0$

$$\begin{aligned}
f'(0) &= k\left((L+0)\left(\sqrt{H^2+L^2}\,\frac{L+0}{\left(H^2+L^2+2L0+0\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+(L+0)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+0)^2}}\right)\right)\\
&= k\left(L\left(\sqrt{H^2+L^2}\,\frac{L}{\left(H^2+L^2\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+L^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+L^2}}\right)\right)\\
&= k\left(L^2\left(\frac{\left(H^2+L^2\right)^{\frac{1}{2}}}{\left(H^2+L^2\right)^{\frac{3}{2}}}\right)\right)\\
&= k\left(\frac{L^2}{\left(H^2+L^2\right)}\right)
\end{aligned}$$

Therefore, EOM of motion becomes (notice we ignored higher order terms, which contains x^2 in them)

$$m\ddot{x} + \left(f(0) + xf'(0)\right) = 0$$

Hence the linearized EOM is

$$\boxed{m\ddot{x} + k\frac{L^2}{\left(H^2+L^2\right)}x = 0}$$

Or in terms of θ_0 the EOM can be written as

$$m\ddot{x} + k\frac{L^2}{\left((L\tan\theta_0)^2 + L^2\right)}x = 0$$
$$m\ddot{x} + k\frac{1}{1+\tan^2\theta_0}x = 0$$

This is the linearized EOM around $x = 0$. Using numerical values given in the problem $L = 1, m = 1, k = 1000N/m, \theta_0 = \frac{\pi}{4}$, it becomes

$$\ddot{x} + 1000\frac{1}{1+\left(\tan\frac{\pi}{4}\right)^2}x = 0$$
$$\ddot{x} + 500x = 0$$

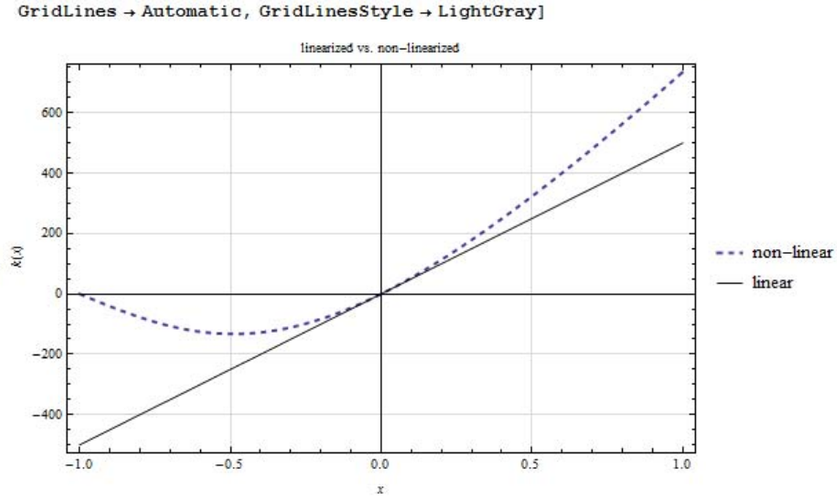
Therefore the $\boxed{\text{linearized stiffness is } 500x}$ while the nonlinearized stiffness is

$$\left[k\left(\frac{\sqrt{H^2+(L+x)^2} - \sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)\right]_{L=1,\theta=45^0}$$
$$= 1000\left(\frac{\sqrt{\left(\tan\frac{\pi}{4}\right)^2+(1+x)^2} - \sqrt{\left(\tan\frac{\pi}{4}\right)^2+1}}{\sqrt{\left(\tan\frac{\pi}{4}\right)^2+(1+x)^2}}\right)(1+x)$$
$$= 1000\left(\frac{\sqrt{(x+1.0)^2+1.0} - 1.4142}{\sqrt{(x+1.0)^2+1.0}}\right)(1+x)$$

Here is a plot of linearized vs. non-linearized stiffness for $x = -1 \cdots 1$

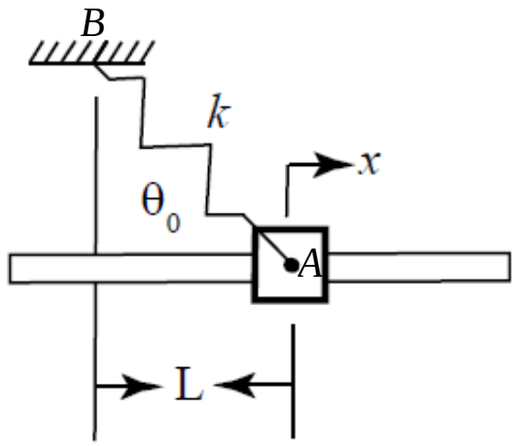
```
In[13]:= nonlinear[x_] := k (L + x) (Sqrt[h^2 + (L + x)^2] - Sqrt[h^2 + L^2]) / Sqrt[h^2 + (L + x)^2];
         linear[x_] := x 500;
         values = {L → 1, θ → 45 Degree, h → Tan[45 Degree], k → 1000};
         Plot[{nonlinear[x] /. values, linear[x] /. values}, {x, -1, 1},
          PlotStyle → {{Dashed, Thick}, Black}, Frame → True, PlotLegends → {"non-linear", "linear"},
          FrameLabel → {{k[x], None}, {x, "linearized vs. non-linearized"}}, ImageSize → 500,
          GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[16]=



2.9.2.3 part(c)

The spring extension Δ is first found by assuming there is a point A at $x = 0$ and point B where the spring is attached to the ceiling. Hence



$$
\begin{aligned}
\dot{\Delta} &= (\dot{u}_A - \dot{u}_B)e_{A/B} \\
&= \left(\dot{x}\hat{\imath} - 0\hat{\jmath}\right) \cdot \left(\cos\theta_0\hat{\imath} - \sin\theta_0\hat{\jmath}\right) \\
&= \dot{x}\cos\theta_0
\end{aligned}
$$

Therefore

$$
\Delta = x\cos\theta_0
$$

Now we repeat the same calculations but using $\Delta = x\cos\theta_0$ for the spring extension.

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 \\
V &= \frac{1}{2}k\Delta^2 \\
&= \frac{1}{2}k(x\cos\theta_0)^2
\end{aligned}
$$

Hence the Lagrangian Φ is

$$
\begin{aligned}
\Phi &= T - V \\
&= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x\cos\theta_0)^2
\end{aligned}
$$

Now the equation of motion for coordinate x is (using the standard Lagrangian form)

$$
\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) - \frac{\partial\Phi}{\partial x} = 0
$$

It is equal to zero above, since there is no generalized force associated with coordinate x. Now we just need to evaluate each part of the above expression to obtain the EOM.

$$
\begin{aligned}
\frac{\partial\Phi}{\partial\dot{x}} &= m\dot{x} \\
\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) &= m\ddot{x}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial\Phi}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x\cos\theta_0)^2\right) \\
&= -k(x\cos\theta_0)\cos\theta_0 \\
&= -kx\cos^2\theta_0
\end{aligned}
$$

Hence EOM becomes

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} = 0$$
$$m\ddot{x} + kx\cos^2\theta_0 = 0$$

But $\cos\theta_0 = \frac{L}{\sqrt{H^2+L^2}}$ hence

$$\boxed{m\ddot{x} + kx\frac{L^2}{H^2+L^2} = 0}$$

This is the same as the EOM for the linearized case found in part(c)

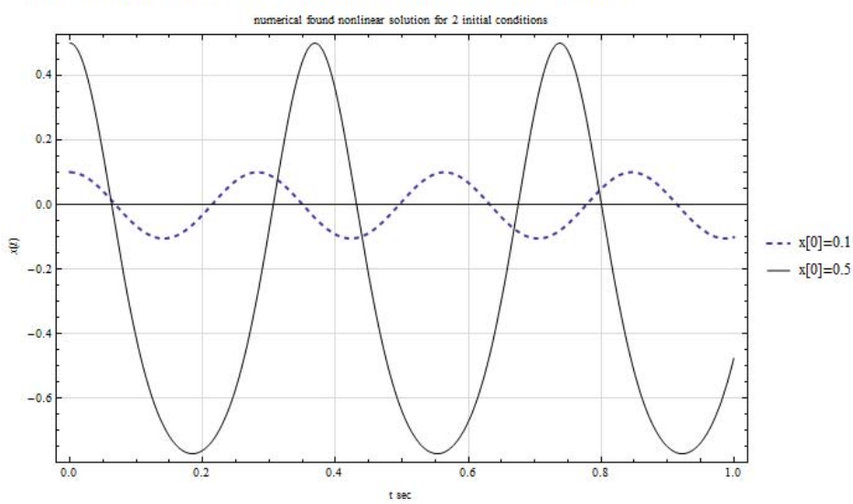
2.9.2.4 part(d)

Now we need to solve numerically the nonlinear EOM found in part(a) which is

$$m\ddot{x} + k\left(\frac{\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}}{\sqrt{H^2 + (L+x)^2}}\right)(L+x) = 0$$

using $m = 1, k = 1000, L = 1, \theta_0 = 45^0$. For IC we use $x(0) = 0.1, x'(0) = 0$ for first case, and for second case using $x(0) = 0.5, x'(0) = 0$. This is a plot showing both responses on same diagram

```
eq = m x''[t] + k (L + x[t]) (Sqrt[h^2 + (L + x[t])^2] - Sqrt[h^2 + L^2]) / Sqrt[h^2 + (L + x[t])^2]) = 0;
ic = {{x[0] = 0.1, x'[0] = 0}, {x[0] = 0.5, x'[0] = 0}};
values = {L → 1, h → Tan[45 Degree], m → 1, k → 1000};
sol = First@NDSolve[{Evaluate[eq /. values], #}, x[t], {t, 0, 1}] & /@ ic;
Plot[Evaluate[x[t] /. sol], {t, 0, 1}, PlotStyle → {{Dashed, Thick}, Black}, Frame → True,
 PlotLegends → {"x[0]=0.1", "x[0]=0.5"},
 FrameLabel → {{x[t], None}, {"t sec", "numerical found nonlinear solution for 2 initial conditions"}},
 ImageSize → 600, GridLines → Automatic, GridLinesStyle → LightGray]
```



The period for the response for case of IC given by x(0) = 0.5 is seen to be about 0.375 seconds and for the case $x(0) = 0.1$ it is 0.275 sec.

The linearized EOM is $\ddot{x} + 500x = 0$ and hence $\omega_n^2 = 500$ or $\omega_n = \sqrt{500} = 22.361$ rad/sec, hence $T = \frac{2\pi}{\omega_n} = \frac{2\pi}{22.361} = \boxed{0.281 \text{ sec}}$.

We notice this agrees well with the period of the response of the nonlinear equation for only the case $x = 0.1$.This is because $x = 0.1$ is very close to $x = 0$ the point at which the linearization happened. Therefore, the linearized EOM gave an answer of 0.281 sec that is very close the more exact value of 0.275 seconds. But when the initial conditions changed to $x(0) = 0.5$, then T found from linearized EOM does not agree with the exact value of 0.375 seconds.

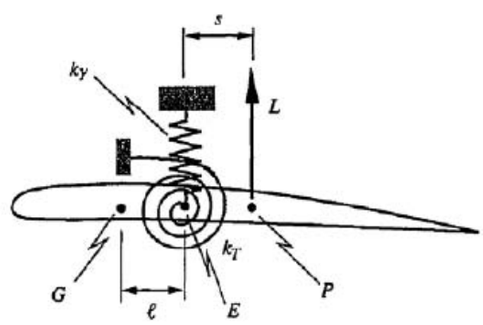
This is because $x = 0.5$ is far away from the point $x = 0$ where the linearized was done. Hence the linearized EOM can be used for only initial conditions that are close to the point where the linearization was done.

Additionally, the nonlinearity manifests itself in the response of the system by noticing that the frequency of the free vibration response has actually changed depending on initial conditions. In a linear system, only the phase and amplitude of the free vibration response will change as initial conditions is changed, while the natural frequency of vibrations does not change.

2.9.3 problem 2

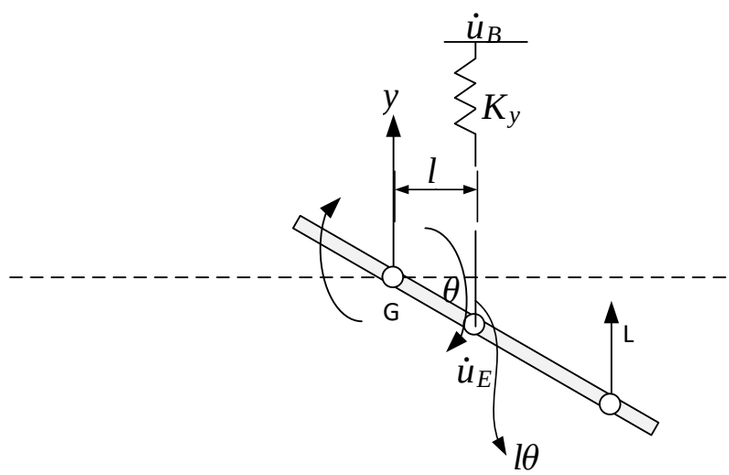
Problem 2: Exercise 1.27 from Ginsberg.

A standard model for a wing has a translational spring k_y and a torsional spring k_T representing the elastic rigidity. Point E represents the elastic center because static application of a vertical force at that point results in upward displacement without an associated rotation. The design of the wing is such that horizontal movement of point E is negligible. The lift force L acts at point P, which is called the center of pressure. The lift force may be treated as known. When the wing is in its static equilibrium position, points G, E and P form a horizontal line. Point G is the center of mass, and the radius of gyration of the wing about that point is r_G. Denote the mass of the wing m. Derive the equations of motion for the wing, assuming small displacements (and small rotational displacements). Put the equations in matrix form and check the units and sign of each term in your EOM. (Hint: use the displacement of the center of gravity and the rotation of the wing as generalized coordinates.)



EXERCISE 1.27

Use y and θ as generalized coordinates as shown in this diagram in the positive direction



Using Lagrangian method, we start by finding the kinetic energy of the system, then the potential energy.

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(mr_G^2\right)\dot{\theta}^2$$

For the potential energy, there will be potential energy due to k_y spring extension and due to k_T spring angle of rotation in system. From the diagram above, we see that, for small angle θ

$$V = \frac{1}{2}k_y\Delta^2 + \frac{1}{2}k_T\theta^2$$

To find Δ we use the stiff spring approximation. Let the point the spring is attached at

the top be B, then

$$\begin{aligned}\dot{\Delta} &= (\dot{u}_E - \dot{u}_B)e_{E/B}\\ &= \left(\left(l\dot{\theta} - \dot{y}\right)\hat{\jmath} - 0\right)\cdot\left(-\hat{\jmath}\right)\\ &= \left(l\dot{\theta} - \dot{y}\right)\hat{\jmath}\cdot\left(-\hat{\jmath}\right)\\ &= \left(\dot{y} - l\dot{\theta}\right)\end{aligned}$$

Hence

$$\Delta = y - l\theta$$

Therefore, the potential energy now can be found to be

$$V = \frac{1}{2}k_y\left(y - l\theta\right)^2 + \frac{1}{2}k_T\theta^2$$

Therefore, the Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V\\ &= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(mr_G^2\right)\dot{\theta}^2 - \frac{1}{2}k_y\left(y - l\theta\right)^2 - \frac{1}{2}k_T\theta^2\end{aligned}$$

We now find the equations for each coordinate. For y

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{y}} &= m\dot{y}\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{y}} &= m\ddot{y}\\ \frac{\partial\Phi}{\partial y} &= -k_y\left(y - l\theta\right)\end{aligned}$$

Hence EOM is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{y}} - \frac{\partial\Phi}{\partial y} &= Q_y\\ m\ddot{y} + k_y\left(y - l\theta\right) &= Q_y\end{aligned}$$

We just need to find Q_y the generalized force in the y direction. Using virtual work, we make small virtual displacement δy in positive y direction while fixing all other generalized coordinates from moving (in this case θ) and then find out the work done by external forces. In this case, there is only one external force which is L. Hence

$$\delta W = L\delta y$$

Therefore $Q_y = L$ since that is the force that is multiplied by δy. Hence EOM for y is now found

$$\boxed{m\ddot{y} + k_y\left(y - l\theta\right) = L}$$

verification: As L increases, then we see that y'' gets larger. This makes sense since y is upwards acceleration, so wing accelerates in the same direction.

Now we find EOM for θ

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{\theta}} &= mr_G^2\dot{\theta}\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} &= mr_G^2\ddot{\theta}\\ \frac{\partial\Phi}{\partial\theta} &= k_y l\left(y - l\theta\right) - k_T\theta\end{aligned}$$

Therefore the EOM is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\theta}} - \frac{\partial \Phi}{\partial \theta} &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l\left(y - l\theta\right) + k_T\theta &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l y + k_y l^2\theta + k_T\theta &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l y + \left(k_T + k_y l^2\right)\theta &= Q_\theta
\end{aligned}$$

We just need to find Q_θ the generalized force in the θ direction. Using virtual work, we make small virtual displacement $\delta\theta$ in positive θ direction (i.e. anticlock wise) while fixing all other generalized coordinates from moving (in this case y) and then find out the work done by external forces. In this case, there is only one external force which is L. When we make $\delta\theta$ rotation in the positive θ direction, the displacement where the force L acts is $(l+s)\delta\theta$ for small angle. But this displacement is in the downward direction, hence it is negative, since we are using y as positive upwards. Hence

$$\delta W = -L(l+s)\delta\theta$$

Therefore $Q_\theta = -L(l+s)$ since that is the force that is multiplied by $\delta\theta$. Hence EOM for θ is now found

$$mr_G^2\ddot{\theta} - k_y l y + \left(k_T + k_y l^2\right)\theta = -L(l+s)$$

Verification: As L gets larger, then $\ddot{\theta}$ gets negative (since L has negative sign). This makes sense, since as L gets larger, the rotation as shown in the positive direction will change sign and the wing will now swing the opposite direction (i.e. anticlockwise).

Now we can make the matrix of EOM

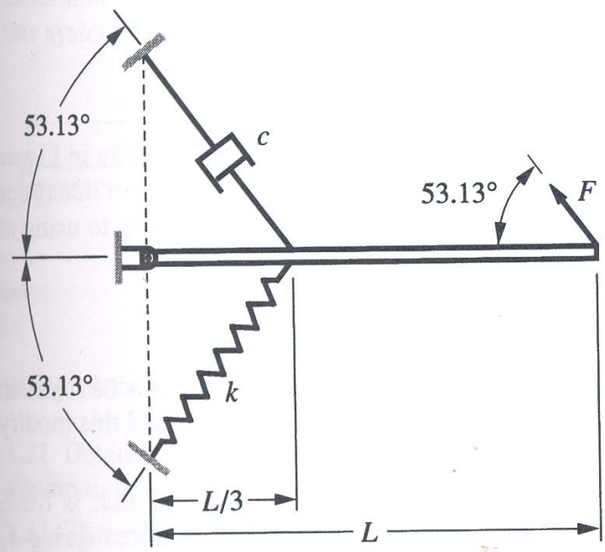
$$MX'' + kX = Q$$

$$\begin{pmatrix} m & 0 \\ 0 & mr_G^2 \end{pmatrix}\begin{pmatrix} \ddot{y} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} k_y & -lk_y \\ -lk_y & k_T + k_y l^2 \end{pmatrix}\begin{pmatrix} y \\ \theta \end{pmatrix} = \begin{pmatrix} L \\ -L(l+s) \end{pmatrix}$$

Notice that for $[k]$ the matrix is symmetric as expected, and also positive on the diagonal as expected. The mass matrix $[m]$ is symmetric and positive definite as well.

2.9.4 problem 3

1.16 The bar executes small rotations in the vertical plane relative to the static equilibrium position depicted in the sketch. Let the rotation of the bar be the generalized coordinate. Determine the damping coefficient C_{11}.



EXERCISE 1.16

Problem 3: Use the power balance method and the stiff spring approximation to find the equation of motion of the system pictured in Problem 1.16.

Let θ be the small angle of rotation that the rod rotates by in the anti clockwise direction. Let the point the spring is fixed be B and the moving point where the spring is attached to the rod be A.To find spring extension Δ we use the stiff spring approximation. Let the angle $\alpha = 53.13^0$, hence

$$\begin{aligned}\dot{\Delta} &= (\dot{u}_A - \dot{u}_B)\cdot \mathbf{e}_{A/B}\\ &= \left(\left(\frac{L}{3}\dot{\theta}\right)\hat{\jmath} - 0\right)\cdot\left(\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}\right)\\ &= \frac{L}{3}\dot{\theta}\sin\alpha\end{aligned}$$

Hence

$$\Delta = \frac{L}{3}\theta\sin\alpha$$

Using Lagrangian method, we start by finding the kinetic energy of the system, then the potential energy. θ is the only generalized coordinate. Assume bar has mass m and hence $I = \frac{mL^2}{3}$

$$T = \frac{1}{2}I\dot{\theta}^2$$

For the potential energy, there will be potential energy due to k spring extension. From the diagram above, we see that

$$V = \frac{1}{2}k\left(\frac{L}{3}\theta\sin\alpha\right)^2$$

Therefore, the Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V\\ &= \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}k\frac{L^2}{9}\theta^2\sin^2\alpha\end{aligned}$$

Now we find EOM for θ

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{\theta}} &= I\dot{\theta}\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} &= I\ddot{\theta}\\ \frac{\partial\Phi}{\partial\theta} &= -\frac{kL^2\sin^2\alpha}{9}\theta\end{aligned}$$

Therefore the EOM is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} - \frac{\partial\Phi}{\partial\theta} &= Q_\theta\\ I\ddot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= Q_\theta\end{aligned}$$

We now need to find the generalized force due to virtual $\delta\theta$ rotation using the virtual work method. There are 2 external forces, the damping force which will have negative sign since it takes energy away from the system, and the external force F which will add energy hence will have positive sign.

We start by making $\delta\theta$ and then find the work done by these 2 forces.

Work done by F is $FL\delta\theta$ since the displacement is $L\delta\theta$ for small angle. Now the work done by damping is$\left(c\frac{L}{3}\dot{\theta}\right)\frac{L}{3}\delta\theta$ hence total work is

$$\begin{aligned}\delta W &= FL\delta\theta - \left(c\frac{L}{3}\dot{\theta}\right)\frac{L}{3}\delta\theta\\ &= \left(FL - c\frac{L^2}{9}\dot{\theta}\right)\delta\theta\end{aligned}$$

Notice that work due to damping was added with negative sign since damping removes energy from the system.

Hence $Q_\theta = \left(FL - c\frac{L^2}{9}\dot{\theta}\right)$ therefore the EOM is

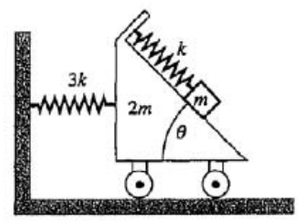
$$\begin{aligned}I\ddot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= FL - c\frac{L^2}{9}\dot{\theta}\\ I\ddot{\theta} + c\frac{L^2}{9}\dot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= FL\end{aligned}$$

Hence the damping coefficient is $c\frac{L^2}{9}$.

2.9.5 problem 4

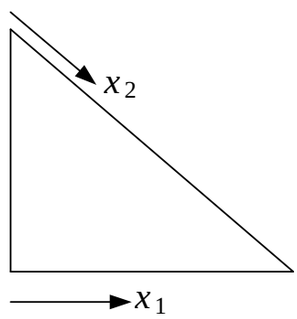
Problem 4: Exercise 1.33 from Ginsberg: (be very careful to write a correct expression for the acceleration of the small block.) Check the unit and sign of each term in your EOM.

1.33 Determine the equations of motion governing a pair of generalized coordinates that locate the position of the cart and the sliding block. Friction is negligible.



EXERCISE 1.33

Let x_1 and x_2 be the generalized coordinates as shown in this diagram

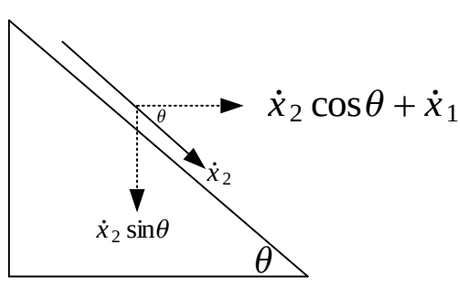


Let mass of cart be m_1 and mass of small sliding block be m_2 (at the end, they will be replaced by values given). Let k for spring attached to wall be k_1 and k for spring for small block be k_2.We start by finding the kinetic energy of the system

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2v^2$$

where v is the velocity of the block. To find this v it is easier to resolve components on the x and y direction. Therefore we find that

$$\vec{v} = \dot{x}_2 \sin\theta j + (\dot{x}_2 \cos\theta + \dot{x}_1)i$$



Hence

$$\begin{aligned}
\left|\vec{v}\right|^2 &= (\dot{x}_2 \sin\theta)^2 + (\dot{x}_2 \cos\theta + \dot{x}_1)^2 \\
&= \left(\dot{x}_2^2 \sin^2\theta\right) + \left(\dot{x}_2^2 \cos^2\theta + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta\right) \\
&= \dot{x}_2^2\left(\sin^2\theta + \cos^2\theta\right) + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta \\
&= \dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta
\end{aligned}$$

Therefore

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta\right)$$

Now we find the potential energy.

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 - m_2g(x_2\sin\theta)$$

There are no external forces, hence generalized forces Q_{x_1}, Q_{x_2} are zero. The Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V\\ &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta\right) - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 + m_2g(x_2\sin\theta)\end{aligned}$$

Now we find EOM for x_1 is

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_1\dot{x}_1 + m_2\dot{x}_1 + m_2\dot{x}_2\cos\theta\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_1\ddot{x}_1 + m_2\ddot{x}_1 + m_2\ddot{x}_2\cos\theta\\ &= (m_1+m_2)\ddot{x}_1 + m_2\ddot{x}_2\cos\theta\\ \frac{\partial\Phi}{\partial x_1} &= -k_1x_1\end{aligned}$$

Therefore the EOM for x_1 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} - \frac{\partial\Phi}{\partial x_1} &= 0\\ (m_1+m_2)\ddot{x}_1 + m_2\ddot{x}_2\cos\theta + k_1x_1 &= 0\end{aligned}$$

Now we replace the actual values for $m_1 = 2m, m_2 = m, k_1 = 3k$ hence

$$\boxed{3m\ddot{x}_1 + m\ddot{x}_2\cos\theta + 3kx_1 = 0}$$

Now we find EOM for x_2 is

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{x}_2} &= m_2(\dot{x}_2 + \dot{x}_1\cos\theta)\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_2(\ddot{x}_2 + \ddot{x}_1\cos\theta)\\ &= m_2\cos\theta\ddot{x}_1 + m_2\ddot{x}_2\\ \frac{\partial\Phi}{\partial x_2} &= -k_2x_2 + m_2g\sin\theta\end{aligned}$$

Therefore the EOM for x_2 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_2} - \frac{\partial\Phi}{\partial x_2} &= 0\\ m_2\cos\theta\ddot{x}_1 + m_2\ddot{x}_2 + k_2x_2 - m_2g\sin\theta &= 0\end{aligned}$$

Now we replace the actual values for $m_1 = 2m, m_2 = m, k_2 = k$ hence

$$m\cos\theta\ddot{x}_1 + m\ddot{x}_2 + kx_2 = m_2g\sin\theta$$

Now we can make the matrix of EOM

$$MX'' + kX = Q$$

$$\begin{pmatrix} 3m & m\cos\theta \\ m\cos\theta & m \end{pmatrix}\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} + \begin{pmatrix} 3k & 0 \\ 0 & k\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix} 0 \\ m_2g\sin\theta\end{pmatrix}$$

Hence

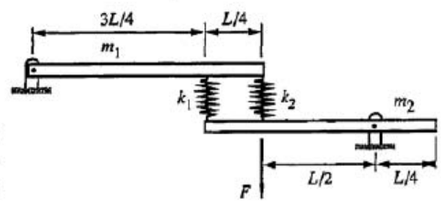
$$\boxed{m\begin{pmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + k\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ m_2 g \sin\theta \end{pmatrix}}$$

Notice that there zeros now off diagonal in the $[K]$ matrix, which means the springs are not coupled. (which is expected, as motion of one is not affected by the other). But mass matrix $[m]$ has non-zeros off the diagonal. So the masses are coupled. i.e. EOM is coupled. This means we can't solve on EOM on its own and both have to be solved simultaneously.

2.9.6 problem 5

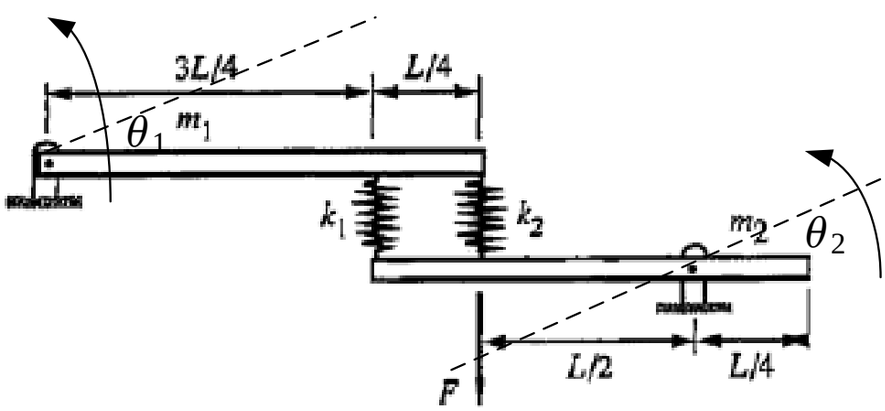
Problem 5: Exercise 1.30 from Ginsberg: Use the stiff spring approximation and assume small deflections of both bars. Check the units and sign of each term in your EOM. Gravity acts downward (same direction as the force, F).

1.30 Both bars in the linkage are horizontal, as shown, when the system is in static equilibrium. Determine the linearized equations of motion for this system.



EXERCISE 1.30

There are 2 degrees of freedom, θ_1 and θ_2 as shown in this diagram, using anticlock wise rotation as positive



The Lagrangian $\Phi = T - V$ where

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

Where $I_1 = \frac{m_1 L^2}{3}$ and $I_2 = \frac{m_2 L^2}{12} + m_2\left(\frac{L}{4}\right)^2$ (using parallel axis theorem). Hence $I_2 = \frac{m_2 L^2}{12} + m_2\frac{L^2}{16} = \frac{7}{48}L^2 m_2$

Now we find the potential energy, assuming springs remain straight (stiff spring assumption) and assuming small angles

$$V = \frac{1}{2}k_1\left(\overbrace{\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2}^{\Delta_1}\right)^2 + \frac{1}{2}k_2\left(\overbrace{L\theta_1 + \frac{L}{2}\theta_2}^{\Delta_2}\right)^2 + m_1 g\frac{L}{2}\theta_1 - m_2 g\frac{L}{4}\theta_2$$

Hence

$$\begin{aligned}\Phi &= T - V\\ &= \left(\frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2\right) - \left(\frac{1}{2}k_1\left(\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2\right)^2 + \frac{1}{2}k_2\left(L\theta_1 + \frac{L}{2}\theta_2\right)^2 + m_1 g\frac{L}{2}\theta_1 - m_2 g\frac{L}{4}\theta_2\right)\end{aligned}$$

Now we find EOM for θ_1

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{\theta}_1} &= I\dot{\theta}_1\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_1} &= I\ddot{\theta}_1\\ \frac{\partial\Phi}{\partial\theta_1} &= -k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) - k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)(L) - m_1 g\frac{L}{2}\\ &= -\frac{3L}{4}k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right) - k_2 L\left(\frac{L}{2}\theta_2 + L\theta_1\right) - m_1 g\frac{L}{2}\end{aligned}$$

Therefore the EOM for θ_1 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_1} - \frac{\partial\Phi}{\partial\theta_1} &= Q_{\theta_1}\\ I_1\ddot{\theta}_1 + \frac{3L}{4}k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right) + k_2 L\left(\frac{L}{2}\theta_2 + L\theta_1\right) + m_1 g\frac{L}{2} &= 0\end{aligned}$$

The generalized force is zero, since there is no direct external force acting on top rod.

Hence EOM for θ_1 is from above

$$\frac{m_1 L^2}{3}\ddot{\theta}_1 + \theta_1\left(k_1\left(\frac{3L}{4}\right)^2 + k_2 L^2\right) + \theta_2\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2}\right) = -m_1 g\frac{L}{2}$$

Now we find EOM for θ_2

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{\theta}_2} &= I\dot{\theta}_2\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_2} &= I\ddot{\theta}_2\\ \frac{\partial\Phi}{\partial\theta_2} &= -k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) - k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)\left(\frac{L}{2}\right) + m_2 g\frac{L}{4}\end{aligned}$$

Therefore the EOM for θ_2 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_2} - \frac{\partial\Phi}{\partial\theta_2} &= Q_{\theta_2}\\ I_2\ddot{\theta}_2 + k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) + k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)\left(\frac{L}{2}\right) - m_2 g\frac{L}{4} &= Q_{\theta_1}\end{aligned}$$

Now $Q_{\theta_2}$is found by virtual work. Making a virtual displacement $\delta\theta_2$ while fixing θ_1 and finding the work done by all external forces.

$$\delta W = F\frac{L}{2}\delta\theta_2$$

Hence $Q_{\theta_2} = F\frac{L}{2}$ with positive sign since it add energy to the system. Hence EOM for θ_2 is

$$\frac{7}{48}L^2 m_2\ddot{\theta}_2 + \theta_1\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2}\right) + \theta_2\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\left(\frac{L}{2}\right)^2\right) = m_2 g\frac{L}{4} + F\frac{L}{2}$$

Now we can make the matrix of EOM

$$MX'' + kX = Q$$

$$\begin{pmatrix} \frac{m_1 L^2}{3} & 0 \\ 0 & \frac{7}{48}L^2 m_2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} k_1\frac{9L^2}{16} + k_2 L^2 & k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2} \\ k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2} & k_1\left(\frac{3L}{4}\right)^2 + k_2\left(\frac{L}{2}\right)^2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -m_1 g\frac{L}{2} \\ m_2 g\frac{L}{4} + F\frac{L}{2} \end{pmatrix}$$

The matrix $[k]$ is coupled but the mass matrix $[m]$ is not.

2.9.7 problem 6

The inertia and stiffness matrices for a system are $[M] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ kg, $[K] = \begin{bmatrix} 200 & 200 \\ 200 & 800 \end{bmatrix}$ N/m. determine the corresponding natural frequencies and modes of free vibration.

$$\Big[[k] - \omega^2[M]\Big]\{\Phi\} = \{0\}$$

Solving for eigenvalues

$$\det\left(\begin{bmatrix} 200 & 200 \\ 200 & 800 \end{bmatrix} - \omega^2\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 200 - 4\omega^2 & 200 \\ 200 & 800 - 2\omega^2 \end{bmatrix} = 0$$

$$\left(200 - 4\omega^2\right)\left(800 - 2\omega^2\right) - 200^2 = 0$$

$$8\omega^4 - 3600\omega^2 + 120000 = 0$$

Hence, taking the positive square root only we find

$$\omega_1 = 20.341 \text{ rad/sec}$$
$$\omega_2 = 6.0211 \text{ rad/sec}$$

When $\boxed{\omega = \omega_1}$

$$\begin{bmatrix} 200 - 4\omega_1^2 & 200 \\ 200 & 800 - 2\omega_1^2 \end{bmatrix}\begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} 200 - 4\omega_1^2 & 200 \\ \times & \times \end{bmatrix}\begin{Bmatrix} 1 \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$200 - 4\omega_1^2 + 200\Phi_{21} = 0$$

$$\Phi_{21} = \frac{-200 + 4\omega_1^2}{200} = \frac{-200 + 4(20.341)^2}{200} = 7.2751$$

Hence the first mode associated with $\omega = 20.341$ rad/sec is

$$\begin{Bmatrix} 1 \\ 7.2751 \end{Bmatrix}$$

When $\boxed{\omega = \omega_2}$

$$\begin{bmatrix} 200 - 4\omega_2^2 & 200 \\ 200 & 800 - 2\omega_2^2 \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} 200 - 4\omega_2^2 & 200 \\ \times & \times \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$200 - 4\omega_2^2 + 200\Phi_{22} = 0$$

$$\Phi_{22} = \frac{-200 + 4\omega_2^2}{200} = \frac{-200 + 4(6.0211)^2}{200} = -0.27493$$

Hence the first mode associated with $\omega = 6.0211$ rad/sec is

$$\begin{Bmatrix} 1 \\ -0.27493 \end{Bmatrix}$$

Summary

ω_n (rad/sec)	mode shape
6.0211	$\begin{Bmatrix} 1 \\ -0.27493 \end{Bmatrix}$
20.341	$\begin{Bmatrix} 1 \\ 7.2751 \end{Bmatrix}$

Verification using Matlab:

```
EDU>> M=[4 0;0 2]; K=[200 200;200 800];
EDU>> [phi,omega]=eig(K,M);
EDU>> sqrt(omega)

    6.0211         0
         0   20.3407
EDU>> phi(:,1)/abs(phi(1,1))

   -1.0000
    0.2749

EDU>> phi(:,2)/abs(phi(1,2))

    1.0000
    7.2749
```

Which matches the result derived. One mode shape has both displacement in phase, and the other mode shape shows the displacements to be out of phase.

2.9.8 Key solution for HW 8

Homework #8
EMA 545, Spring 2013

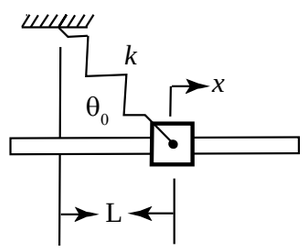
Problem 1: (40 points)

a.) Find the nonlinear equation of motion for the system pictured below. The block has mass *m* and the guide can be approximated as frictionless. In the position shown the spring is unstretched and the angle between the spring and guide bar is θ_0.

b.) Linearize your equation of motion for small deflections from the position shown (i.e. using a Taylor series expansion on $k(x)$ about x=0). Use a computer to plot k(x) versus the linear approximation for L=1 m, k=1000 N/m and θ_0 = 45 degrees for x ranging from -1 m to +1 m.

c.) Find the equations of motion for the system using the stiff spring approximation and assuming small displacements from an equilibrium position defined by L=1 m, k=1000 N/m and θ_0 = 45 degrees. Compare your result with your linearized result from part (b).

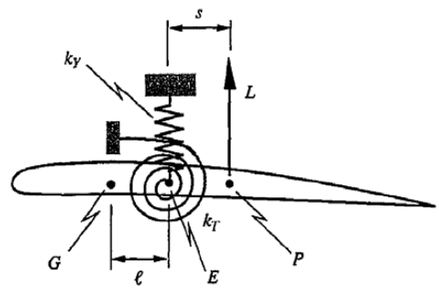
d.) Using m=1, find the response of the nonlinear system (in part a) using ode45 and plot the displacement of the mass over a few cycles when it is released from rest at x(0)=0.1 and also at x(0)=0.5 meters. Overlay both curves on the same set of axes. How does the period of the response compare with the linearized natural frequency in each case? In what other way(s) does the nonlinearity manifest itself in the response of the system when x(0)=0.5?



Problem 2: Exercise 1.27 from Ginsberg.

A standard model for a wing has a translational spring k_y and a torsional spring k_T representing the elastic rigidity. Point E represents the elastic center because static application of a vertical force at that point results in upward displacement without an associated rotation. The design of the wing is such that horizontal movement of point E is negligible. The lift force L acts at point P, which is called the center of pressure. The lift force may be treated as known. When the wing is in its static equilibrium position, points G, E and P form a horizontal line. Point G is the center of mass, and the radius of gyration of the wing about that point is r_G. Denote the mass of the wing m. Derive the equations of motion for the wing, <u>assuming small displacements (and small rotational displacements)</u>. Put the equations in matrix form and <u>check the units and sign of each</u>

term in your EOM. (Hint: use the displacement of the center of gravity and the rotation of the wing as generalized coordinates.)

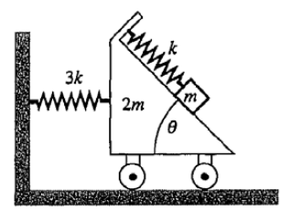


EXERCISE 1.27

Problem 3: Use the power balance method and the stiff spring approximation to find the equation of motion of the system pictured in Problem 1.16.

Problem 4: Exercise 1.33 from Ginsberg: (be very careful to write a correct expression for the acceleration of the small block.) Check the unit and sign of each term in your EOM.

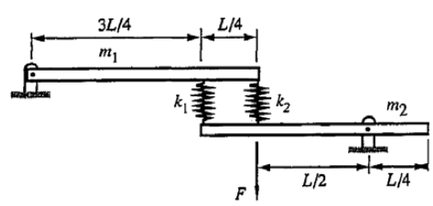
1.33 Determine the equations of motion governing a pair of generalized coordinates that locate the position of the cart and the sliding block. Friction is negligible.



EXERCISE 1.33

Problem 5: Exercise 1.30 from Ginsberg: Use the stiff spring approximation and assume small deflections of both bars. Check the units and sign of each term in your EOM. Gravity acts downward (same direction as the force, F).

1.30 Both bars in the linkage are horizontal, as shown, when the system is in static equilibrium. Determine the linearized equations of motion for this system.

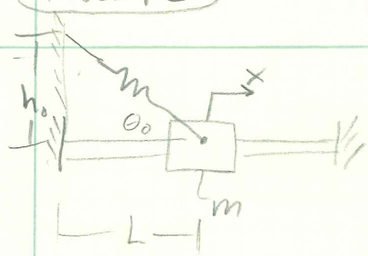


EXERCISE 1.30

Problem 6: Exercise 4.1 in Ginsberg. Solve the eigenvalue problem by hand to get the natural frequencies and mode shapes. You may check your answers with Matlab.

HW1 P5 | Solution - EMA545 | 1/12/2011 | 4

a)



$$|\vec{F}_{spr}| = K\Delta$$

$$= \square \rightarrow m\ddot{x}$$

$$\Sigma \vec{F}\cdot\hat{i} = -|\vec{F}_{spr}|\cos\theta = m\ddot{x}$$

$$h = L\tan\theta_0$$

Length $= L_{spr}$ $\quad L_{spr}^2 = h^2 + (L+x)^2$

$$\cos\theta = \frac{L+x}{\sqrt{h^2+(L+x)^2}}$$

$$\Delta = (L_{curr} - L_0) = \sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}$$

So, everything is known, and eom becomes

$$-K\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)\left(\frac{L+x}{\sqrt{h^2+(L+x)^2}}\right) = m\ddot{x}$$

or: $\boxed{m\ddot{x} + K(x) = 0,\ K(x) = K\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)\left(\frac{L+x}{\sqrt{h^2+(L+x)^2}}\right)}$

b) Linearize: expand $K(x)$ in a taylor series

$$K(x) = K(0) + \frac{\partial K}{\partial x}\Big|_{x=0}\Delta x + \text{H.O.T.}$$

$$K(x) \approx 0 + +K\Big[\underbrace{\left(h^2+(L+x)^2\right)^{-1/2}\left(\cancel{\tfrac{1}{2}}\right)\left(\cancel{2}(L+x)\right)}_{0\ @x=0}\left(\frac{L+x}{(h+(L+x)^2)^{1/2}}\right)$$

$$+\underbrace{\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)}_{0\ @x=0}\frac{L}{\sqrt{h^2+(L+x)^2}} +$$

$$+\left(\sqrt{h^2+(L+x)^2} - \sqrt{h^2+L^2}\right)(L+x)\left(h+(L+x)^2\right)^{-3/2}\left(-\tfrac{1}{2}\right)\left(\cancel{2}(L+x)\right)\Big]\Delta x$$

eval at $x=0$

$$K(x) \approx +K\left[\frac{(L+x)^2}{h^2+(L+x)^2}\right]\Big|_{x=0}\Delta x = \boxed{+K\left(\frac{L^2}{h^2+L^2}\right)\Delta x \approx K(x)}$$

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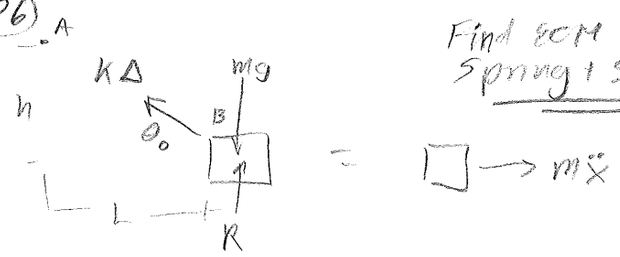
(P6)



Find EOM Assuming stiff spring + small deformations!

$$\Delta = u_B - \cancel{u_A}^{\,0} \qquad (e_{B/A})$$
$$= (x\hat{i})\cdot(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) = x\cos\theta$$

$$\Sigma F\cdot\hat{i} = -(Kx\cos\theta)\cos\theta = m\ddot{x}$$

$$\boxed{m\ddot{x} + K\cos^2\theta\, x = 0}$$

Compare with HW # 1 →

$$\cos\theta = \frac{L}{\sqrt{h^2+L^2}} \rightarrow m\ddot{x} + \frac{KL^2}{h^2+L^2}x = 0 \quad ✓$$

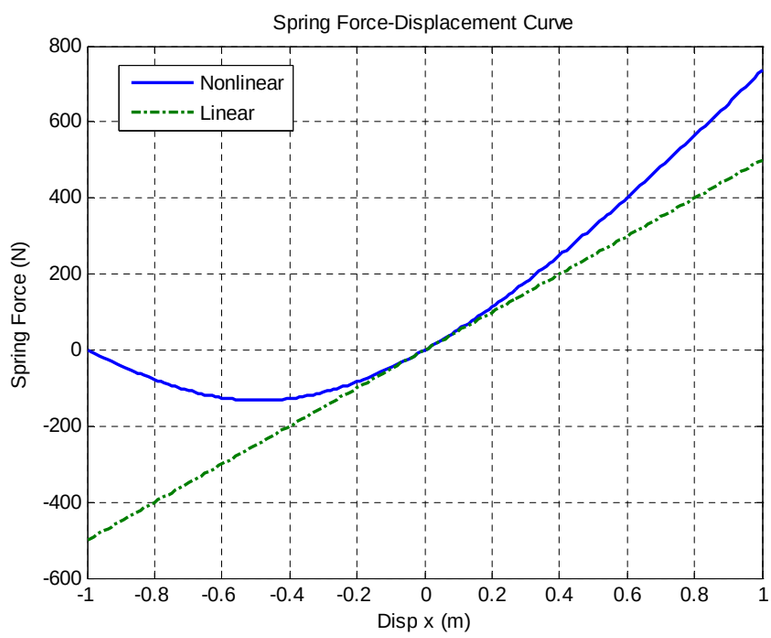
```
% Part (b)
% Plot nonlinear k(x) for large deformations of spring-mass system.
% M.S. Allen, Spring 2011, EMA 545

k=1000; %N/m
L = 1; %m
theta = 45*pi/180; % rad
xs = [-1:0.01:1]; % m
h = L*tan(theta);

kx = k*((h^2+(L+xs).^2).^(1/2)-
sqrt(h^2+L^2)).*((L+xs)./(h^2+(L+xs).^2).^(1/2));

klin = k*(L^2/(h^2+L^2));

figure(1)
plot(xs,kx,xs,klin*xs,'-.'); set(get(gca,'Children'),'LineWidth',2);
grid on;
xlabel('Disp x (m)'); ylabel('Spring Force (N)');
title('Spring Force-Displacement Curve');
legend('Nonlinear','Linear');
```

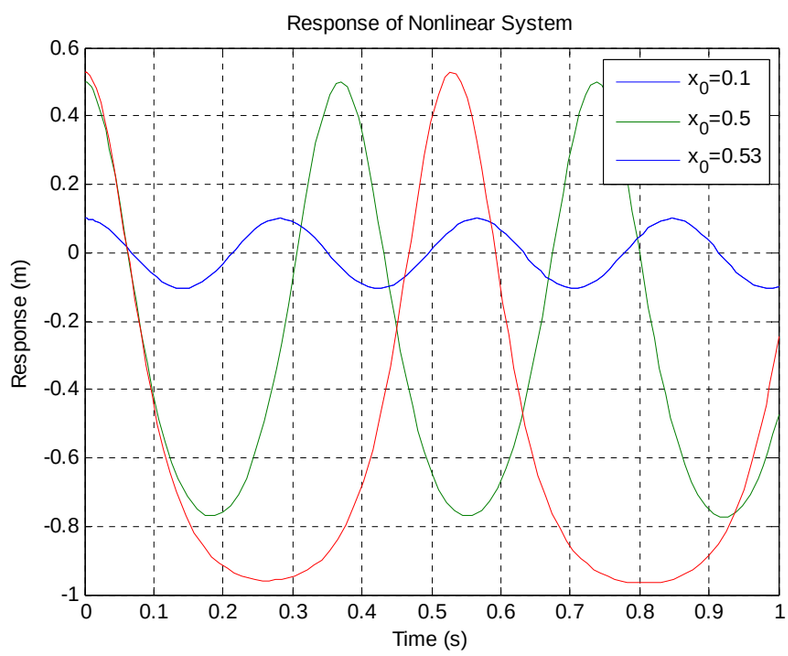



```
% Part (d)
% Find response to a small disturbance.
m=1;
wn_lin = sqrt(klin/m)
```

```
eom = @(t,x) [x(2)+0*t; -(1/m)*(k*((h^2+(L+x(1)).^2).^(1/2)-
sqrt(h^2+L^2)).*((L+x(1))./(h^2+(L+x(1)).^2).^(1/2)))]

[ts1,y1]=ode45(eom,[0,1],[0.1; 0]);
[ts2,y2]=ode45(eom,[0,1],[0.5; 0]);

figure(2)
plot(ts1,y1(:,1),ts2,y2(:,1)); hold on; grid on;
xlabel('Time (s)'); ylabel('Response (m)');
title('Response of Nonlinear System');
```

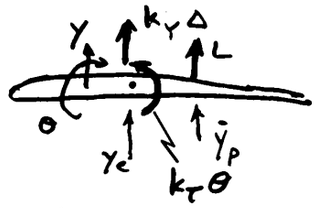


The period of the nonlinear response in each case is given below (found using ginput on the plot). The linearized natural frequency is 22.36 rad/s and the corresponding period is 0.281 seconds.

Initial Displacement	**Period (s)**
x_0=0.1	0.2823
x_0=0.5	0.3698
x_0=0.53	0.5311

The behavior of the system is quite peculiar. The period becomes longer (frequency lower) as the system approaches the region where the stiffness vanishes. As shown, with a slightly larger initial displacement of 0.53, the mass almost comes to rest as the mass approaches x=-1, which is the other equilibrium position. Incidentally, the body panels of a hypersonic aircraft, which I am studying as part of an Air Force grant, can behave very similarly. They buckle due to thermal expansion and then as they vibrate they may jump between two equilibria.

Exercise 1.27



$q_1 = y$ of G, $q_2 = \theta$ (cw)

$\Delta = -y_e$

$y_e = y - l\theta$, $\dot{y}_p = \dot{y} - (l+s)\dot{\theta}$

$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} I_G \dot{\theta}^2$

$I_G = m r_G^2$

$$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2}(m r_G^2)\dot{\theta}^2 = \frac{1}{2}[M_{11}\dot{y}^2 + M_{22}\dot{\theta}^2 + 2M_{12}\dot{y}\dot{\theta}]$$

$M_{11} = m$, $M_{22} = m r_G^2$, $M_{12} = 0$

$$V = \frac{1}{2} k_y \Delta^2 + \frac{1}{2} k_T \theta^2 = \frac{1}{2} k_y (y - l\theta)^2 + \frac{1}{2} k_T \theta^2$$

$$= \frac{1}{2}[k_y y^2 + (k_y l^2 + k_T)\theta^2 - 2k_y l y\theta]$$

$$= \frac{1}{2}[K_{11} y^2 + K_{22}\theta^2 + 2K_{12} y\theta]$$

$K_{11} = k_y$, $K_{22} = k_y l^2 + k_T$, $K_{12} = -k_y l$

$P_{dis} = 0$, $P_{in} = L\dot{y}_p = L[\dot{y} - (l+s)\dot{\theta}] = Q_1\dot{y} + Q_2\dot{\theta}$

$Q_1 = L$, $Q_2 = -L(l+s)$

$[M]\{\ddot{q}\} + [K]\{q\} = \{Q\}$

$\Rightarrow$ where $\{q\} = \{y \;\; \theta\}^T$, $[M] = \begin{bmatrix} m & 0 \\ 0 & m r_G^2 \end{bmatrix}$

$\Rightarrow$ $[K] = \begin{bmatrix} k_y & -k_y l \\ -k_y l & (k_y l^2 + k_T) \end{bmatrix}$, $\{Q\} = \begin{Bmatrix} L \\ -L(l+s) \end{Bmatrix}$

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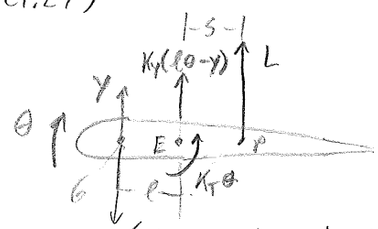
P1 (1.27)



$\cancel{mg}$, leave off - write about static eq.

$$\Sigma M_g = -I\ddot{\theta}$$

$$K_T\theta + lK_y(l\theta - y) + L(s+l) = -I_g\ddot{\theta}$$

$$\boxed{m r_g^2 \ddot{\theta} + K_T\theta + K_y l^2\theta - lK_y y = -(s+l)L}$$

$$\Sigma F_y = K_y(l\theta - y) + L = m\ddot{y}$$

$$\boxed{m\ddot{y} + K_y y - lK_y\theta = L}$$

Grader, -1 or -2 pts if no check at EOM!

✓ +L causes $\ddot{\theta} < 0$, $\ddot{y} > 0$

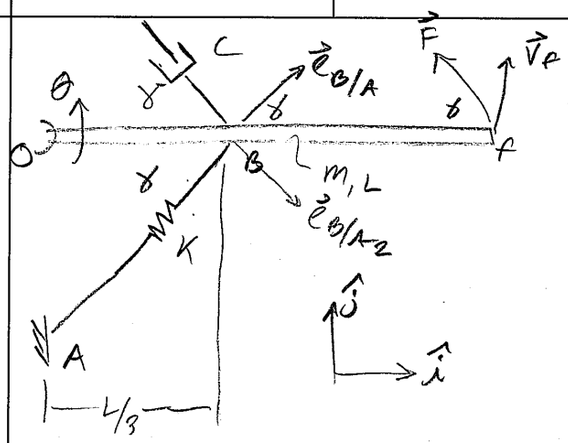
✓ Spring stiffnesses + on diagonal, neg off diagonal

* Note, bend and twist are coupled (as in a real wing) since the elastic center ≠ center of mass.

$$\boxed{\begin{bmatrix} I_g & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} (K_T + K_y l^2) & -lK_y \\ -lK_y & K_y \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} -(s+l)L \\ L \end{Bmatrix}}$$

units - $1 Kg\text{-}m^2 \cdot rad/s^2$, N-m/rad ✓, N/m·m·m ✓

1.16



Find EOM using P.B.

$T = \frac{1}{2} I_o \dot{\theta}^2 \rightarrow$

$m_{11} = I_o = \frac{1}{3} m L^2$

Use stiff spring approx.

$\Delta \approx u_B - u_A$, $\dot{u}_B = \vec{V}_B \cdot \vec{e}_{B/A}$

$\vec{V}_B = \frac{L}{3} \dot{\theta} \hat{j}$

$\vec{e}_{B/A} = \cos\gamma \hat{i} + \sin\gamma \hat{j}$

$\dot{u}_B = \frac{L}{3} \dot{\theta} \sin\gamma$

$\Delta = \left(\frac{L}{3} \theta \sin\gamma - \overset{u_A = 0}{0} \right)$

$V = \frac{1}{2} K \Delta^2 = \frac{1}{2} K \frac{L^2}{9} \sin^2\gamma \, \theta^2 \Rightarrow K_{11} = \frac{KL^2}{9} \sin^2\gamma$

$P_{dis} = c \dot{\Delta}^2$ $\qquad \dot{\Delta} = \vec{V}_B \cdot (\vec{e}_{B/A})_2 = \frac{L}{3} \dot{\theta} \hat{j} \cdot (\cos\gamma \hat{i} - \sin\gamma \hat{j})$

$= c \left(-\frac{L}{3} \dot{\theta} \sin\gamma \right)^2 = \underbrace{\frac{cL^2}{9} \sin^2\gamma}_{c_{11}} \dot{\theta}^2 \Rightarrow c_{11} = \frac{cL^2}{9} \sin^2\gamma$

$P_{in} = F \cdot \vec{V}_f = F \sin\gamma \, L \dot{\theta}$

$P_{in} = FL \sin\gamma \, \dot{\theta} = Q_1 \dot{q}_1 \rightarrow Q_1 = FL \sin\gamma$

EOM:

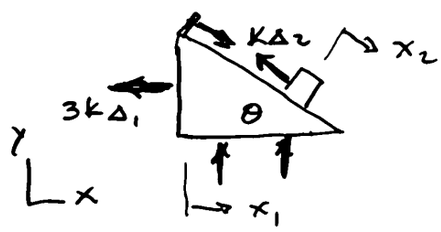
$$\boxed{\frac{1}{3} m L^2 \ddot{\theta} + \frac{cL^2}{9} \sin^2\gamma \, \dot{\theta} + \frac{KL^2}{9} \sin^2\gamma = FL \sin\gamma}$$

✓ signs ~ +F causes pos $\ddot{\theta}$ or pos. θ, etc...

$\sin\gamma \sim \gamma \rightarrow 0$, no contribution ✓

Units ✓ all moments

Exercise 1.33



$q_1 = x_1,\ q_2 = x_2$

$\bar{v}_{cart} = \dot{x}_1 \bar{i}$

$\bar{v}_{bl} = \bar{v}_{cart} + \bar{v}_{bl/cart}$

$= \dot{x}_1 \bar{i} + \dot{x}_2 (\cos\theta \bar{i} - \sin\theta \bar{j})$

$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2} m \bar{v}_{bl} \cdot \bar{v}_{bl}$

$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2} m[(\dot{x}_1 + \dot{x}_2 \cos\theta)^2 + (\dot{x}_2 \sin\theta)^2]$

$= \frac{1}{2}[3m\dot{x}_1^2 + m\dot{x}_2^2 + 2m\dot{x}_1\dot{x}_2 \cos\theta]$

$= \frac{1}{2}[M_{11}\dot{x}_1^2 + M_{22}\dot{x}_2^2 + 2M_{12}\dot{x}_1\dot{x}_2]$

$M_{11} = 3m,\ M_{22} = M,\ M_{12} = m\cos\theta$

$V = \frac{1}{2}(3k)\Delta_1^2 + \frac{1}{2}k\Delta_2^2$ {gravity only affects

$\Delta_1 = x_1,\ \Delta_2 = x_2$ {equil position of block)

$V = \frac{1}{2}[K_{11}x_1^2 + K_{22}x_2^2 + 2K_{12}x_1x_2]$

$K_{11} = 3k,\ K_{22} = k,\ K_{12} = 0$

$[M] = m\begin{bmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{bmatrix},\ [K] = k\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

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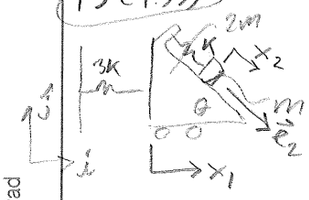
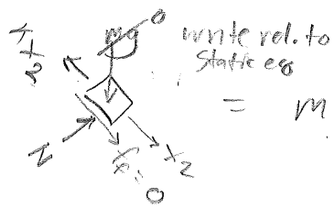
P3 (1.33)



- let x_2 denote disp. of block relative to the large cart relative to static equilibrium.

$$\vec{a}_2 = \vec{a}\big|_{relative} + \vec{a}\big|_{cart}$$
$$= \ddot{x}_2 \vec{e}_2 + \ddot{x}_1 \hat{i}$$

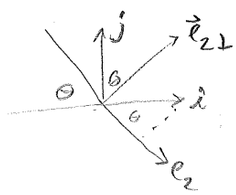


$$= m\vec{a}_2 \qquad \Sigma F\cdot\vec{e}_2 = -Kx_2 = m(\ddot{x}_2 + \ddot{x}_1(\vec{e}_2\cdot\hat{i}))$$

$$e_2 = \cos\theta\,\hat{i} - \sin\theta\,\hat{j}$$

$$-Kx_2 = m\ddot{x}_2 + m\cos\theta\,\ddot{x}_1$$

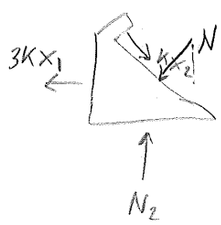
$$\boxed{m\ddot{x}_2 + m\cos\theta\,\ddot{x}_1 + Kx_2 = 0}$$



if $x_2 = 0$, $\ddot{x}_2 = -\cos\theta\,\ddot{x}_1$ ✓ units ✓

$$\Sigma F\cdot\vec{e}_{2\perp} = N = m(\ddot{x}_2\vec{e}_2 + \ddot{x}_1\hat{i})\cdot\vec{e}_{2\perp} = m\ddot{x}_1\vec{e}_{2\perp}\cdot\hat{j}$$

$$N = m\ddot{x}_1(\cos\theta\,\hat{j} + \sin\theta\,\hat{i})\cdot\hat{i} = m\ddot{x}_1\sin\theta = N$$



$$= 2m\ddot{x}_1 \rightarrow$$

$$\Sigma F\cdot\hat{i} = -3Kx_1 + Kx_2(\cos\theta\,\hat{i} - \sin\theta\,\hat{j})\cdot\hat{i} - N\sin\theta$$
$$= 2m\ddot{x}_1$$

$$2m\ddot{x}_1 + (m\ddot{x}_1\sin\theta)\sin\theta - Kx_2\cos\theta + 3Kx_1 = 0$$

$$\begin{bmatrix} 2m + m\sin^2\theta & 0 \\ m\cos\theta & m \end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix} 3K & -K\cos\theta \\ 0 & K\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Note, EOM are non-symmetric. Can happen sometimes when using this Newton-Euler method.

$\cos\theta\cdot(2) + (1) \longrightarrow$ replace w/ (1)

$$3m = \begin{bmatrix} 2m + m(\sin^2\theta + \cos^2\theta) & m\cos\theta \\ m\cos\theta & m \end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix} 3K & 0 \\ 0 & K\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix} 3m & m\cos\theta \\ m\cos\theta & m \end{bmatrix}\begin{bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{bmatrix} + \begin{bmatrix} 3K & 0 \\ 0 & K\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

✓ springs are uncoupled ~ x_2 doesn't affect spring (3K)
✓ Mass coupled, checked above, $\ddot{x}_2 = 0 \rightarrow 3m\ddot{x}_1 + 3Kx_1 = 0$ ✓

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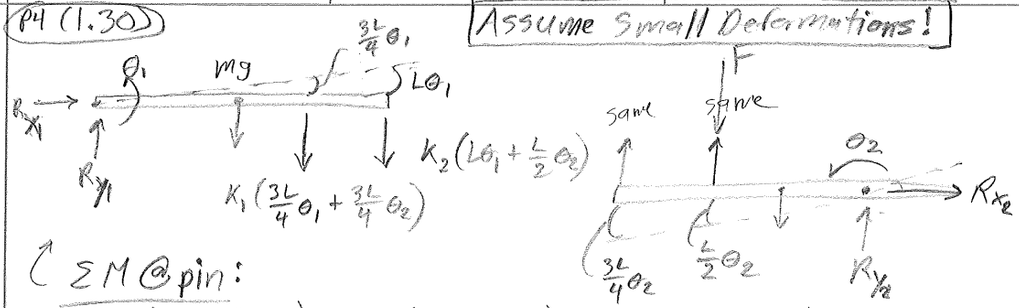
P4 (1.30) $\boxed{\text{Assume Small Deformations!}}$



$\circlearrowleft \Sigma M$ @ pin:

$$-\left(\frac{3L}{4}\right)K_1\left(\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2\right) - LK_2\left(L\theta_1 + \frac{L}{2}\theta_2\right) - mg\frac{L}{2}\cos\theta_1 = \frac{1}{3}m_1L^2\ddot{\theta}_1$$

$$\boxed{\frac{1}{3}m_1L^2\ddot{\theta}_1 + \left(\frac{9L^2}{16}K_1 + L^2K_2\right)\theta_1 + \left(\frac{9L^2}{16}K_1 + \frac{L^2}{2}K_2\right)\theta_2 = -mg\frac{L}{2}\cos\theta_1} \quad (1)$$

static

*Last term falls out → it is a constant moment for small θ_1 and must be zero in static equilibrium.

Bar 2: ΣM @ pin

static, no stiffness effect for small θ_2

$$-K_1\frac{3L}{4}(\theta_1 + \theta_2)\left(\frac{3L}{4}\right) - K_2\left(L\theta_1 + \frac{L}{2}\theta_2\right)\left(\frac{L}{2}\right) + mg\frac{L}{4}\cos\theta_2 + \left(F\frac{L}{2}\right) = I_2\ddot{\theta}_2$$

$$I_2 = \frac{1}{12}mL^2 + m\left(\frac{L}{4}\right)^2 = \left(\frac{1}{12} + \frac{1}{16}\right)mL^2\ddot{\theta}_2$$

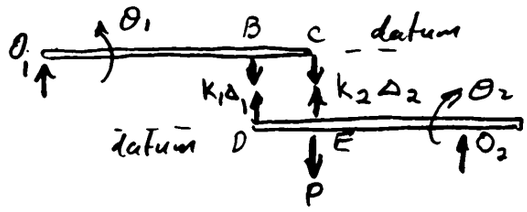
$$\boxed{\left(\frac{1}{12} + \frac{1}{16}\right)mL^2\ddot{\theta}_2 + \left(\frac{9L^2}{16}K_1 + \frac{L^2}{2}K_2\right)\theta_1 + \left(\frac{9L^2}{16}K_1 + \frac{L^2}{4}K_2\right)\theta_2 = \frac{FL}{2}} \quad (2)$$

✓ units kg·m² rad/s² ✓, N/m·m²·m·1 ✓

✓ $\theta_1 = 0$, $\theta_2 > 0$, $\ddot{\theta}_1 < 0$ ✓ (in EOM 1)

✓ $\theta_2 = 0$, $\theta_1 > 0$, $\ddot{\theta}_2 < 0$ ✓ (in EOM 2) $F > 0 \rightarrow \ddot{\theta}_2 > 0$ ✓

Exercise 1.30



Given equil position is horizontal

Find eqs of motion

Solution: $q_1 = \theta_1$, $q_2 = \theta_2$

Both bars rotate about fixed points $\Rightarrow \omega_1 = \dot{\theta}_1$, $\omega_2 = \dot{\theta}_2$

$$T = \frac{1}{2} I_{O1}\dot{\theta}_1^2 + \frac{1}{2} I_{O2}\dot{\theta}_2^2$$

where $I_{O1} = \frac{1}{3} m_1 L^2$, $I_{O2} = \frac{1}{12} m_2 L^2 + m\left(\frac{L}{4}\right)^2 = \frac{7}{48} m_2 L^2$

Match to standard form: $T = \frac{1}{2}\left[(M_{11}\dot{\theta}_1^2 + M_{22}\dot{\theta}_2^2) + 2M_{12}\dot{\theta}_1\dot{\theta}_2\right]$

Thus $M_{11} = \frac{1}{3} m_1 L^2$, $M_{22} = \frac{7}{48} m_2 L^2$, $M_{12} = 0$

$$V = V_{sp} + V_{gr}$$

Points on bar 1 displace upward by $r\theta_1$
Point on bar 2 displace upward by $r\theta_2$ } r = distance from pivot

$\Delta_1 = \frac{3L}{4}\theta_1 - \frac{3L}{4}\theta_2$, $\Delta_2 = L\theta_1 - \frac{L}{2}\theta_2$

$$V_{sp} = \frac{1}{2} k_1 \Delta_1^2 + \frac{1}{2} k_2 \Delta_2^2$$
$$= \frac{1}{2} k_1 \left(\frac{3L}{4}\theta_1 - \frac{3L}{4}\theta_2\right)^2 + \frac{1}{2} k_2 \left(L\theta_1 - \frac{L}{2}\theta_2\right)^2$$
$$= \frac{1}{2}\left[\left(\frac{9}{16}k_1 L^2 + k_2 L^2\right)\theta_1^2 + \left(\frac{9}{16}k_1 L^2 + \frac{1}{4}k_2 L^2\right)\theta_2^2 - 2\left(\frac{9}{16}k_1 L^2 + \frac{1}{2}k_2 L^2\right)\theta_1\theta_2\right]$$

Standard: $V = \frac{1}{2}\left[(K_{11}\theta_1^2 + K_{22}\theta_2^2) + 2K_{12}\theta_1\theta_2\right]$

Thus $(K_{11})_{sp} = \frac{9}{16}k_1 L^2 + k_2 L^2$, $(K_{22})_{sp} = \frac{9}{16}k_1 L^2 + \frac{1}{4}k_2 L^2$

$$(K_{12})_{sp} = -\left(\frac{9}{16}k_1 L^2 + \frac{1}{2}k_2 L^2\right)$$

$V_{gr} = m_1 g h_1 + m_2 g h_2 \Rightarrow$ datum for each bar at its pin

so $h_1 = \frac{L}{2}\sin\theta_1$, $h_2 = \frac{L}{4}\sin\theta_2 \Rightarrow V_{gr} = m_1 g \frac{L}{2}\sin\theta_1 + m_2 g \frac{L}{4}\sin\theta_2$

$\frac{\partial^2 V_{gr}}{\partial\theta_1^2} = -m_1 g \frac{L}{2}\sin\theta_1$, $\frac{\partial^2 V_{gr}}{\partial\theta_2^2} = -m_2 g \frac{L}{4}\sin\theta_2$, $\frac{\partial^2 V_{gr}}{\partial\theta_1\partial\theta_2} = 0$

Evaluate at $\theta_1 = \theta_2 = 0 \Rightarrow (K_{11})_{gr} = (K_{22})_{gr} = (K_{12})_{gr} = 0$

Thus $K_{11} = (K_{11})_{sp}$, $K_{22} = (K_{22})_{sp}$, $K_{12} = (K_{12})_{sp}$

No dashpots $\Rightarrow [C] = [0]$

$\mathcal{P}_{in} = P v_E$, but $v_E = \frac{L}{2}\dot{\theta}_2$ upward $\Rightarrow \mathcal{P}_{in} = -P\frac{L}{2}\dot{\theta}_2$

Standard $\mathcal{P}_{in} = Q_1\dot{\theta}_1 + Q_2\dot{\theta}_2$

Thus $Q_1 = 0$, $Q_2 = -P\frac{L}{2}$

Standard eq of motion : $[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q\}$

$$\begin{bmatrix} \frac{1}{3}m_1L^2 & 0 \\ 0 & \frac{7}{48}m_2L^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (\frac{9}{16}k_1L^2 + k_2L^2) & -(\frac{9}{16}k_1L^2 + \frac{1}{2}k_2L^2) \\ -(\frac{9}{16}k_1L^2 + \frac{1}{2}k_2L^2) & (\frac{9}{16}k_1L^2 + \frac{1}{4}k_2L^2) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -P\frac{L}{2} \end{Bmatrix} \quad \Leftarrow$$

Exercise 4.1

$$\left[[K]-\omega^2[M]\right]\{\phi\} = \begin{bmatrix} 200-4\omega^2 & 200 \\ 200 & 800-2\omega^2 \end{bmatrix}\{\phi\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$|[K]-\omega^2[M]| = (200-4\omega^2)(800-2\omega^2)-(200)^2 = 0$$

$$8\omega^4 - 3600\omega^2 + 120000 = 0 \Rightarrow$$

$$\omega_1^2 = 36.25, \quad \omega_2^2 = 413.75$$

Then $\begin{bmatrix} 200-4\omega_j^2 & 200 \\ \times & \times \end{bmatrix}\begin{Bmatrix} 1 \\ \phi_{2j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$

$$\phi_{2j} = -\frac{200-4\omega_j^2}{200}$$

First mode $\omega_1 = 6.021$ rad/s, $\{\phi_1\} = \begin{Bmatrix} 1 \\ -0.275 \end{Bmatrix}$ $\Leftarrow$

Second mode $\omega_2 = 20.341$ rad/s, $\{\phi_2\} = \begin{Bmatrix} 1 \\ 7.275 \end{Bmatrix}$

2.10 HW9

Local contents

2.10.1 problem description

Homework #9
EMA 545, Spring 2013

For the following problems, you may have the equations of motion for some of these systems in your past homework assignments or in the solutions to those that were posted online, so you may use those if you wish.

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

1.) **Problem 4.3** in Ginsberg. Sketch the deformation of the system when it moves in each of the modes. (Notice that you can pull out factors such as *k* and *m* so that only numbers remain in the mass and stiffness matrices. Then it is possible to check your answers using Matlab.)

2.) **Problem 4.7** as given in the text. You may use the following equations of motion:

$$mL^2\begin{bmatrix} 4/3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + mgL\begin{bmatrix} 2\beta - 3/2 & -\beta \\ -\beta & \beta - 1/2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3.) **Problem 4.11** as given in the text. Hint: normal modes = mass normalized modes

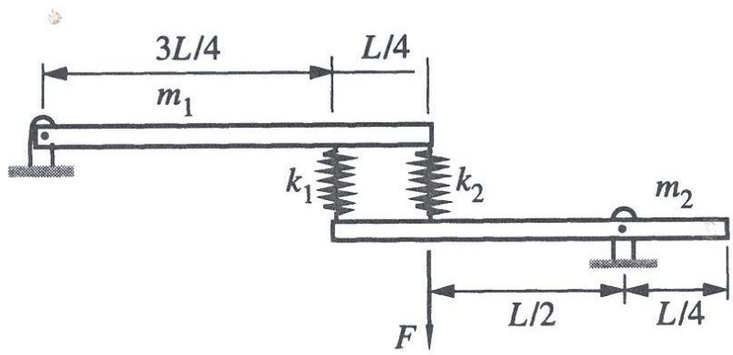
4.) **Problem 4.29** as given in the text. Also, plot the motion of the automobile as a function of time. Is the response a pure-sinusoid? Why or why not? Note: The answer provided by the book is incorrect. The correct answer is:

$y_1(t) = 0.16\cos(1.5t') + 0.84\cos(2.0t')$
$y_2(t) = 0.45\cos(1.5t') - 0.45\cos(2.0t')$

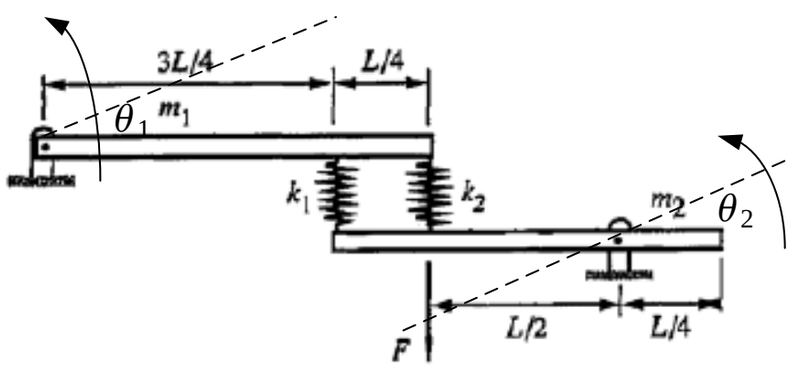
2.10.2 problem 1

After solving problem 4.3 in text, Sketch the deformation of the system when it moves in each of the modes.

4.3 Determine the natural frequencies and modes of free vibration for this system of bars and springs for the case where $m_1 = m$, $m_2 = 2m$, $k_1 = k$, and $k_2 = k/2$, where m and k are basic units of mass and stiffness.



There are 2 degrees of freedom, θ_1 and θ_2 as shown in this diagram, using anticlock wise rotation as positive



We solved this problem in HW8, using classical Lagrangian method. This problem will now be solved using power balance method. The static equilibrium position must be chosen so that all generalized coordinates have value zero. Hence, using the above diagram as the static equilibrium, we take $\theta_1 = \theta_2 = 0$ in this position.

Now, as in Lagrangian method, we always start by finding kinetic energy T

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

Where $\boxed{I_1 = \frac{m_1 L^2}{3}}$ and $I_2 = \frac{m_2 L^2}{12} + m_2 \left(\frac{L}{4}\right)^2$ (using parallel axis theorem). Hence $I_2 = \frac{m_2 L^2}{12} + m_2 \frac{L^2}{16} = \boxed{\frac{7}{48} L^2 m_2}$

Now we compare the above expression to the quadratic form

$$T = \frac{1}{2}\left(M_{11}\dot{\theta}_1^2 + M_{22}\dot{\theta}_2^2 + 2M_{12}\dot{\theta}_1\dot{\theta}_2\right)$$

Hence we see that $M_{11} = I_1, M_{22} = I_2, M_{12} = M_{21} = 0$, therefore the mass matrix is

$$[M] = \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}$$

We now find the potential energy due to springs. For this, we need to write down the relative displacement between end points of each spring. Let Δ_1 be the relative displacement in the first spring k_1 and let Δ_2 be the relative displacement in the second spring k_2. Hence (and assuming springs remain straight, since we are assuming very stiff springs and small angles) then

$$\begin{aligned}\Delta_1 &= \frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2 \\ \Delta_2 &= L\theta_1 + \frac{L}{2}\theta_2\end{aligned}$$

Then

$$\begin{aligned}V_{spring} &= \frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_1\Delta_2^2 \\ &= \frac{1}{2}k_1\left(\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2\right)^2 + \frac{1}{2}k_2\left(L\theta_1 + \frac{L}{2}\theta_2\right)^2 \\ &= \frac{1}{2}k_1\left(\frac{9}{16}L^2\theta_1^2 + \frac{9}{8}L^2\theta_1\theta_2 + \frac{9}{16}L^2\theta_2^2\right) + \frac{1}{2}k_2\left(L^2\theta_1^2 + L^2\theta_1\theta_2 + \frac{1}{4}L^2\theta_2^2\right) \\ &= \frac{9}{32}L^2\theta_1^2k_1 + \frac{1}{2}L^2\theta_1^2k_2 + \frac{9}{32}L^2\theta_2^2k_1 + \frac{1}{8}L^2\theta_2^2k_2 + \frac{9}{16}L^2\theta_1\theta_2k_1 + \frac{1}{2}L^2\theta_1\theta_2k_2 \\ &= \left(\frac{9}{32}L^2k_1 + \frac{1}{2}L^2k_2\right)\theta_1^2 + \left(\frac{9}{32}L^2k_1 + \frac{1}{8}L^2k_2\right)\theta_2^2 + \left(\frac{9}{16}L^2k_1 + \frac{1}{2}L^2k_2\right)\theta_1\theta_2\end{aligned}$$

Now we compare the above to quadratic form

$$V_{spring} = \frac{1}{2}\left(K_{11}\theta_1^2 + K_{22}\theta_2^2 + 2K_{12}\theta_1\theta_2\right)$$

We see that

$$\begin{aligned}K_{11} &= \frac{9}{16}L^2k_1 + L^2k_2 \\ K_{22} &= \frac{9}{16}L^2k_1 + \frac{1}{4}L^2k_2 \\ K_{12} &= \frac{9}{16}L^2k_1 + \frac{L^2}{2}k_2\end{aligned}$$

Now we need to find $V_{gravity}$.Taking the static equilibrium position as the datum, then upward displacement of center of gravity will be positive and downward displacement is negative. This means the left bar will add positive potential energy due to gravity and the right bar will add negative potential energy, hence

$$V_g = m_1g\frac{L}{2}\sin\theta_1 - m_2g\frac{L}{4}\sin\theta_2$$

Now we need to find the components of the gravity potential energy stiffness matrix. Notice that each term is evaluated at static equilibrium

$$\begin{aligned}V_{g_{11}} &= \left(\frac{\partial V_g^2}{\partial\theta_1^2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = \left(-m_1g\frac{L}{2}\sin\theta_1\right)_{\theta_1=0} = 0 \\ V_{g_{22}} &= \left(\frac{\partial V_g^2}{\partial\theta_2^2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = \left(-m_1g\frac{L}{2}\sin\theta_2\right)_{\theta_2=0} = 0 \\ V_{g_{12}} &= \left(\frac{\partial V_g^2}{\partial\theta_1\partial\theta_2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = 0\end{aligned}$$

Hence, no contribution from gravity is added to the stiffness matrix. All contribution comes from the springs potential energy. Therefore, the stiffness matrix is

$$[K] = \begin{pmatrix} \frac{9}{16}L^2k_1 + L^2k_2 & \frac{9}{16}L^2k_1 + \frac{L^2}{2}k_2 \\ \frac{9}{16}L^2k_1 + \frac{L^2}{2}k_2 & \frac{9}{16}L^2k_1 + \frac{1}{4}L^2k_2 \end{pmatrix}$$

Now since there is no damping, then $P_{disp} = 0$. To find P_{in} we need to find

$$P_{in} = Q_1\theta_1 + Q_2\theta_2$$

The only external force is F which generates a torque $F\frac{L}{2}\theta_2$, hence by comparing to the above

$$P_{in} = F\frac{L}{2}\theta_2$$
$$Q_2 = F\frac{L}{2}$$

Now we can make the matrix of EOM

$$MX'' + kX = Q$$

$$\begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} \frac{9}{16}L^2k_1 + L^2k_2 & \frac{9}{16}L^2k_1 + \frac{L^2}{2}k_2 \\ \frac{9}{16}L^2k_1 + \frac{L^2}{2}k_2 & \frac{9}{16}L^2k_1 + \frac{1}{4}L^2k_2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F\frac{L}{2} \end{pmatrix}$$

$$L^2\begin{pmatrix} \frac{m_1}{3} & 0 \\ 0 & \frac{7}{48}m_2 \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + L^2\begin{pmatrix} \frac{9}{16}k_1 + k_2 & \frac{9}{16}k_1 + \frac{1}{2}k_2 \\ \frac{9}{16}k_1 + \frac{1}{2}k_2 & \frac{9}{16}k_1 + \frac{1}{4}k_2 \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F\frac{L}{2} \end{pmatrix}$$

Now we can solve the problem given.

When $m_1 = m, m_2 = 2m, k_1 = k, k_2 = \frac{k}{2}$ we obtain

$$mL^2\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{7}{24} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + kL^2\begin{pmatrix} \frac{17}{16} & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ F\frac{L}{2} \end{pmatrix}$$

To find modes of free vibration, let the RHS $\{0\}$ then we write

$$\left[[K] - \omega^2\frac{m}{k}[M]\right]\{\Phi\} = \{0\}$$

Let $\lambda = \omega^2\frac{m}{k}$, hence

$$[[K] - \lambda[M]]\{\Phi\} = \{0\}$$

Solving for eigenvalues

$$\det\left[\begin{pmatrix} \frac{17}{16} & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} \end{pmatrix} - \lambda\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{7}{24} \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} \frac{17}{16} - \frac{1}{3}\lambda & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} - \frac{7}{24}\lambda \end{bmatrix} = 0$$

$$9.7222 \times 10^{-2}\lambda^2 - 0.53906\lambda + 7.0313 \times 10^{-2} = 0$$

Hence, taking the positive square root only we find

$$\lambda_1 = 0.13366$$
$$\lambda_2 = 5.4110$$

When $\lambda_1 = 0.13366$

$$[[k] - \lambda_1[M]]\{\Phi\}_1 = \{0\}$$
$$\begin{bmatrix} \frac{17}{16} - \frac{1}{3}\lambda_1 & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} - \frac{7}{24}\lambda_1 \end{bmatrix} \begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} \frac{17}{16} - \frac{1}{3}\lambda_1 & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} - \frac{7}{24}\lambda_1 \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$
$$\frac{17}{16} - \frac{1}{3}\lambda_1 + \frac{13}{16}\Phi_{21} = 0$$
$$\Phi_{21} = \frac{16}{13}\left(\frac{1}{3}\lambda_1 - \frac{17}{16}\right) = \frac{16}{13}\left(\frac{1}{3}(0.137) - \frac{17}{16}\right) = -1.253$$

Hence the first mode associated with $\lambda_1 = 0.13366$ is

$$\begin{Bmatrix} 1 \\ -1.253 \end{Bmatrix}$$

When $\lambda_2 = 5.4110$

$$[[k] - \lambda_1[M]]\{\Phi\}_1 = \{0\}$$
$$\begin{bmatrix} \frac{17}{16} - \frac{1}{3}\lambda_2 & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} - \frac{7}{24}\lambda_2 \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} \frac{17}{16} - \frac{1}{3}\lambda_2 & \frac{13}{16} \\ \frac{13}{16} & \frac{11}{16} - \frac{7}{24}\lambda_2 \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$
$$\frac{17}{16} - \frac{1}{3}\lambda_2 + \frac{13}{16}\Phi_{22} = 0$$
$$\Phi_{22} = \frac{16}{13}\left(\frac{1}{3}\lambda_2 - \frac{17}{16}\right) = \frac{16}{13}\left(\frac{1}{3}(5.411) - \frac{17}{16}\right) = 0.912$$

Hence the second mode associated with $\lambda_2 = 5.411$ is

$$\begin{Bmatrix} 1 \\ 0.912 \end{Bmatrix}$$

Summary

ω (rad/sec)	mode shape
$\lambda = \omega^2 \frac{m}{k} \Rightarrow \omega = \sqrt{\frac{k}{m}}\sqrt{0.137} = 0.366\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} 1 \\ -1.253 \end{Bmatrix}$
$\lambda = \omega^2 \frac{m}{k} \Rightarrow \omega = \sqrt{\frac{k}{m}}\sqrt{5.411} = 2.326\sqrt{\frac{k}{m}}$	$\begin{Bmatrix} 1 \\ 0.912 \end{Bmatrix}$

2.10.2.1 verification using Matlab

```
EDU>> M=[1/3 0;0 7/24]; K=[17/16 13/16;13/16 11/16];
EDU>> [phi,omega]=eig(K,M);
EDU>> sqrt(omega)

    0.3656         0
         0    2.3262

EDU>> phi(:,1)/abs(phi(1,1))

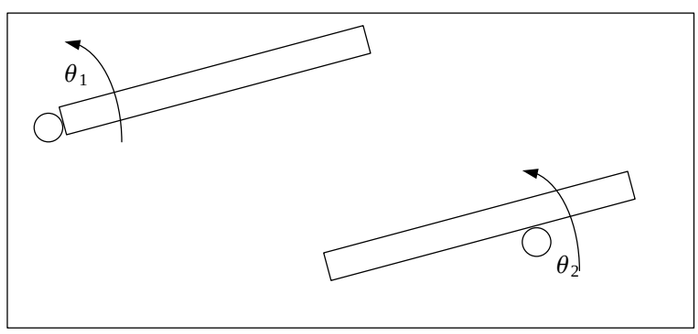
    1.0000
   -1.2529

EDU>> phi(:,2)/abs(phi(1,2))

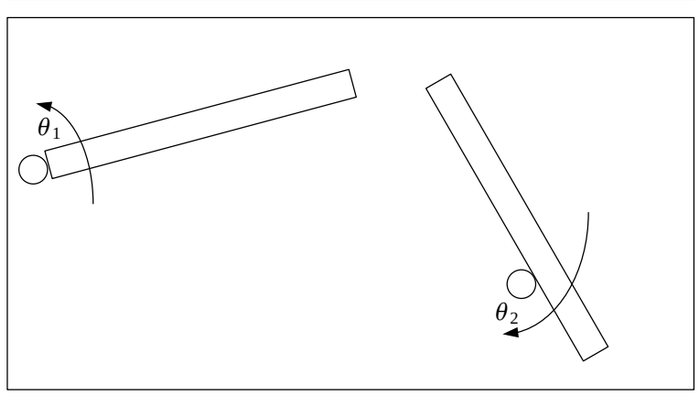
   -1.0000
   -0.9122
```

2.10.2.2 Sketch of each mode

$\begin{Bmatrix} 1 \\ 0.912 \end{Bmatrix}$ means that θ_1 and θ_2 are in phase, and for each 1 unit rotation of θ_1 there will be 0.912 units of rotation of θ_2, while $\begin{Bmatrix} 1 \\ -1.253 \end{Bmatrix}$ means that θ_1 and θ_2 are out of phase, and for each 1 unit rotation of θ_1 there will be 1.253 units of rotation of θ_2 but in the opposite direction. This is a sketch of both modes



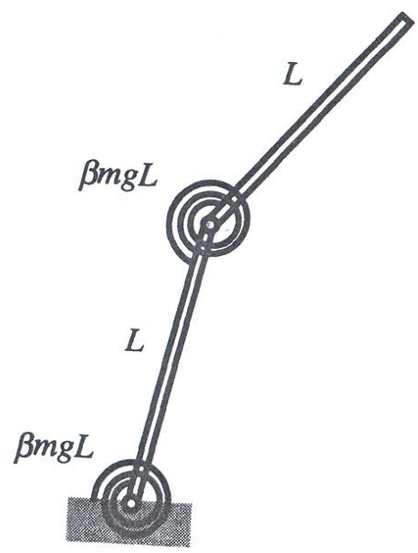
$$\begin{Bmatrix} 1 \\ 0.912\,29 \end{Bmatrix}$$



$$\begin{Bmatrix} 1 \\ -1.\,2529 \end{Bmatrix}$$

2.10.3 problem 2

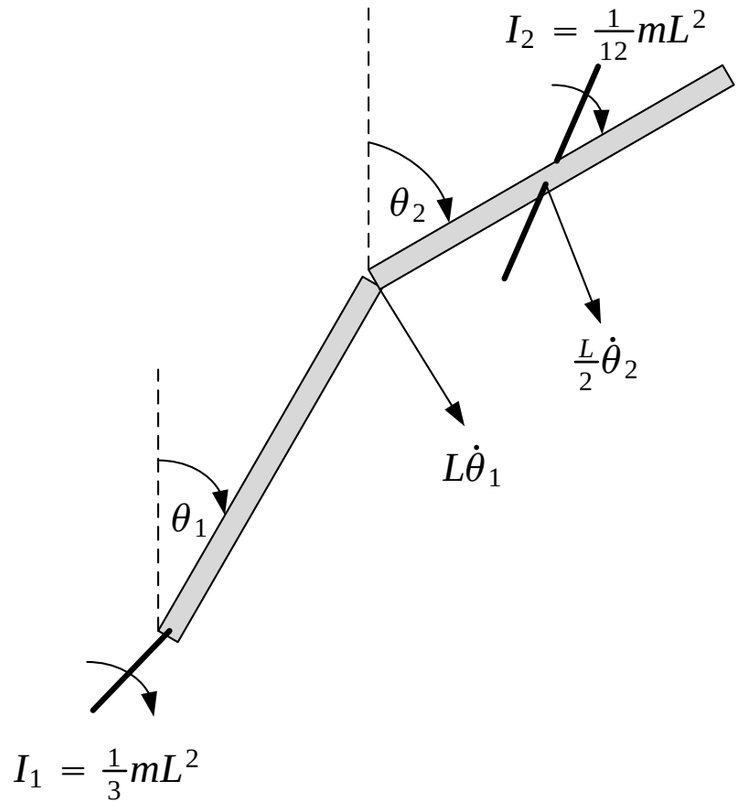
4.7 The linkage shown consists of two rigid bars having length L and mass m. The torsional springs, whose stiffness is βmgL (β is a nondimensional parameter), are undeformed when the bars are vertical. For the case where $\beta = 4$, determine the natural frequencies and modes of free vibration. Then determine the natural frequencies when $\beta = 2$. Explain the significance of the result of the second case.



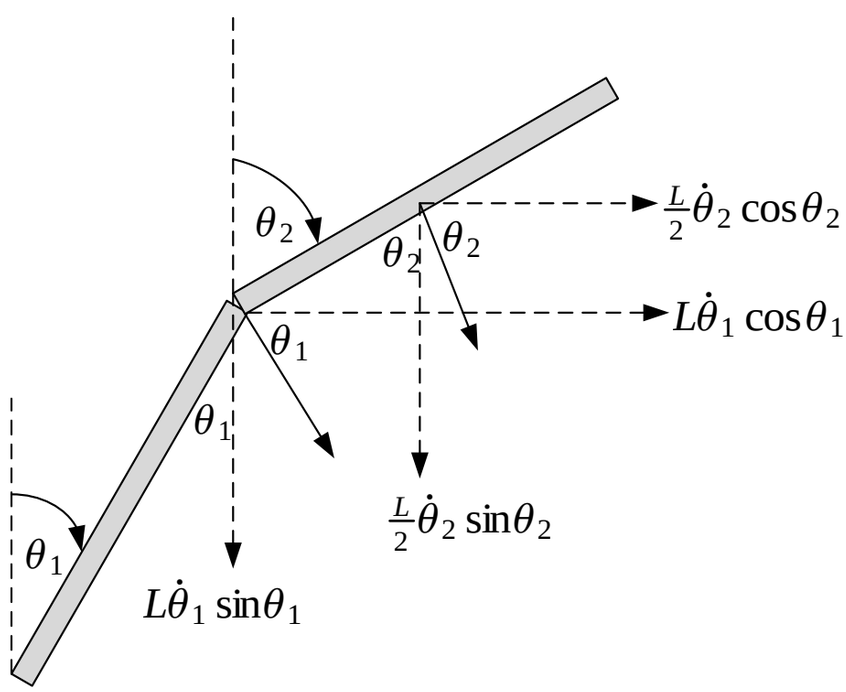
2.) **Problem 4.7** as given in the text. You may use the following equations of motion:

$$mL^2\begin{bmatrix} 4/3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + mgL\begin{bmatrix} 2\beta - 3/2 & -\beta \\ -\beta & \beta - 1/2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Using power balance method, we start by finding the kinetic energy.



Since the top bar does not have a point that is fixed in inertial space as the lower bar does, then we take its moment of inertia around its center of mass, and add a translational kinetic energy due to the motion of its center of mass in space. For the lower bar, since it has a point that is fixed in space, then we take the moment of inertia around that point, and we do not need to account for translational kinetic energy for the lower bar. To find the speed of the center of mass of the top bar, we can either use its coordinates system x, y differentiate these w.r.t time, or we can use the angular motion of the base of the second bar and add it to the speed of the center of mass of the second bar relative to the base. This is what will be done next:



Therefore, the speed components of the center of mass of the top bar is

$$
\begin{aligned}
v_x &= \frac{L}{2}\dot{\theta}_2 \cos\theta_2 + L\dot{\theta}_1 \cos\theta_1 \\
v_y &= -\frac{L}{2}\dot{\theta}_2 \sin\theta_2 - L\dot{\theta}_1 \sin\theta_1
\end{aligned}
$$

So the velocity of the center of mass is

$$
v_{c.g.} = \sqrt{v_x^2 + v_y^2}
$$

Now that we have the translation velocity of the top bar, and we know its moment of inertia around its c.g. then we have all the terms needed to obtain the kinetic energy.

$$
T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 + \frac{1}{2}mv_{c.g.}^2
$$

Again, the important thing is to note that I_1 is taken around the base of lower rod while I_2 is taken around the center of mass of the top rod. Hence

$$
\begin{aligned}
T &= \frac{1}{2}\frac{mL^2}{3}\dot{\theta}_1^2 + \frac{1}{2}\frac{mL^2}{12}\dot{\theta}_2^2 + \frac{1}{2}m\left(v_x^2 + v_y^2\right) \\
&= \frac{1}{2}\frac{mL^2}{3}\dot{\theta}_1^2 + \frac{1}{2}\frac{mL^2}{12}\dot{\theta}_2^2 + \frac{1}{2}m\left(\left(\frac{L}{2}\dot{\theta}_2 \cos\theta_2 + L\dot{\theta}_1 \cos\theta_1\right)^2 + \left(-\frac{L}{2}\dot{\theta}_2 \sin\theta_2 - L\dot{\theta}_1 \sin\theta_1\right)^2\right) \\
&= \frac{1}{2}\frac{mL^2}{3}\dot{\theta}_1^2 + \frac{1}{2}\frac{mL^2}{12}\dot{\theta}_2^2 + \frac{1}{2}m \\
&\left(\left(\frac{L^2}{4}\dot{\theta}_2^2 \cos^2\theta_2 + L^2\dot{\theta}_1^2 \cos^2\theta_1 + L^2\dot{\theta}_2\dot{\theta}_1 \cos\theta_2 \cos\theta_1\right) + \left(\frac{L^2}{4}\dot{\theta}_2^2 \sin^2\theta_2 + L^2\dot{\theta}_1^2 \sin^2\theta_1 + L^2\dot{\theta}_2\dot{\theta}_1 \sin\theta_2 \sin\theta\right.\right.
\end{aligned}
$$

Simplifying the last term, and using $\cos^2\theta_1 + \sin^2 \sin^2\theta_1 = 1$ we obtain

$$
T = \frac{1}{2}\frac{mL^2}{3}\dot{\theta}_1^2 + \frac{1}{2}\frac{mL^2}{12}\dot{\theta}_2^2 + \frac{1}{2}m\left(\frac{L^2}{4}\dot{\theta}_2^2 + L^2\dot{\theta}_1^2 + L^2\dot{\theta}_2\dot{\theta}_1(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1)\right)
$$

To compare with the quadratic form, we collect all terms as follows

$$
T = \dot{\theta}_1^2\left(\frac{1}{2}\frac{mL^2}{3} + \frac{1}{2}L^2m\right) + \dot{\theta}_2^2\left(\frac{1}{2}\frac{mL^2}{12} + \frac{1}{8}mL^2\right) + \dot{\theta}_2\dot{\theta}_1\left(\frac{1}{2}mL^2(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1)\right)
$$

Using $\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 = \cos(\theta_2 - \theta_1)$ the above becomes

$$
T = \dot{\theta}_1^2\left(\frac{4}{6}mL^2\right) + \dot{\theta}_2^2\left(\frac{1}{6}mL^2\right) + \dot{\theta}_2\dot{\theta}_1\left(\frac{1}{2}mL^2 \cos(\theta_2 - \theta_1)\right)
$$

We now compare the above to

$$
T = \frac{1}{2}M_{11}\dot{\theta}_1^2 + \frac{1}{2}M_{22}\dot{\theta}_2^2 + M_{12}\dot{\theta}_2\dot{\theta}_1
$$

Therefore

$$
\begin{aligned}
M_{11} &= \frac{4}{3}mL^2 \\
M_{22} &= \frac{1}{3}mL^2 \\
M_{12} = M_{21} &= \frac{1}{2}mL^2 \cos(\theta_2 - \theta_1)
\end{aligned}
$$

I am not sure how to get the same answer given for the mass matrix. Even if I assume that $\theta_2 - \theta_1$ is very small, hence $M_{12} = \frac{1}{2}mL^2$ then the mass matrix is

$$[M] = mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$$

Now we find the V_{spring} the potential energy due to springs.

$$\begin{aligned} V_{spring} &= \frac{1}{2}\beta mgL\theta_1^2 + \frac{1}{2}\beta mgL(\theta_2 - \theta_1)^2 \\ &= \frac{1}{2}\beta mgL\theta_1^2 + \frac{1}{2}\beta mgL\left(\theta_2^2 + \theta_1^2 - 2\theta_2\theta_1\right) \\ &= \theta_1^2\left(\beta mgL\right) + \theta_2^2\left(\frac{1}{2}\beta mgL\right) + \theta_2\theta_1\left(-\beta mgL\right) \end{aligned}$$

Comparing to quadratic form

$$V_{spring} = \frac{1}{2}K_{11}\theta_1^2 + \frac{1}{2}K_{11}\theta_1^2 + K_{12}\theta_1\theta_2$$

Then

$$\begin{aligned} K_{11} &= 2\beta mgL \\ K_{22} &= \beta mgL \\ K_{12} &= K_{21} = -\beta mgL \end{aligned}$$

Hence the stiffness matrix due to springs only is

$$[K] = mgL\begin{pmatrix} 2\beta & -\beta \\ -\beta & \beta \end{pmatrix}$$

We know need to find the gravity contribution to stiffness. We start by finding the $V_{gravity}$. We take the datum as the horizontal line at the bottom the lower bar.

$$V_{gravity} = mg\frac{L}{2}\cos\theta_1 + mg\left(L\cos\theta_1 + \frac{L}{2}\cos\theta_2\right)$$

Hence

$$V_{11} = \frac{\partial^2 V_g}{\partial^2\theta_1} = -mg\frac{L}{2}\cos\theta_1 - mg(L\cos\theta_1)$$

evaluate at $\theta_1 = 0$ gives

$$\begin{aligned} V_{11} &= -mg\frac{L}{2} - mgL \\ &= -\frac{3}{2}mgL \end{aligned}$$

and

$$V_{22} = \frac{\partial^2 V_g}{\partial^2\theta_2} = -mg\left(\frac{L}{2}\cos\theta_2\right)$$

evaluate at $\theta_2 = 0$ gives

$$V_{22} = -mg\frac{L}{2}$$

and

$$V_{12} = \frac{\partial^2 V_g}{\partial\theta_1\partial\theta_2} = 0$$

Hence the stiffness matrix due to gravity is

$$[K] = mgL\begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Combine the above with the stiffness matrix due to springs we obtain

$$\begin{aligned}[K] &= mgL\begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} + mgL\begin{pmatrix} 2\beta & -\beta \\ -\beta & \beta \end{pmatrix} \\ &= mgL\begin{pmatrix} 2\beta - \frac{3}{2} & -\beta \\ -\beta & \beta - \frac{1}{2} \end{pmatrix}\end{aligned}$$

There is no P_{disp} and no P_{in} hence the equations of motion are

$$mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + mgL\begin{pmatrix} 2\beta - \frac{3}{2} & -\beta \\ -\beta & \beta - \frac{1}{2} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\boxed{\beta = 4}$

$$mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + mgL\begin{pmatrix} \frac{13}{2} & -4 \\ -4 & \frac{7}{2} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To find modes of free vibration, we write

$$\Big[[k] - \omega^2[M]\Big]\{\Phi\} = \{0\}$$

Solving for eigenvalues

$$\det\left[mgL\begin{pmatrix} \frac{13}{2} & -4 \\ -4 & \frac{7}{2} \end{pmatrix} - \omega^2 mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

$$\det\left[\begin{pmatrix} \frac{13}{2} & -4 \\ -4 & \frac{7}{2} \end{pmatrix} - \omega^2\frac{L}{g}\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

Let $\omega^2\frac{L}{g} = \eta$, hence

$$\det\left[\begin{pmatrix} \frac{13}{2} & -4 \\ -4 & \frac{7}{2} \end{pmatrix} - \eta\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} \frac{13}{2} - \frac{4}{3}\eta & -4 - \frac{1}{2}\eta \\ -4 - \frac{1}{2}\eta & \frac{7}{2} - \frac{1}{3}\eta \end{bmatrix} = 0$$

$$\frac{7}{36}\eta^2 - \frac{65}{6}\eta + \frac{27}{4} = 0$$

Hence $\eta = 55.084, \eta = 0.63023$

When $\boxed{\eta = 55.084}$

$$\Big[[k] - \eta[M]\Big]\{\Phi\}_1 = \{0\}$$

$$\begin{bmatrix} \frac{13}{2} - \frac{4}{3}\eta & -4 - \frac{1}{2}\eta \\ -4 - \frac{1}{2}\eta & \frac{7}{2} - \frac{1}{3}\eta \end{bmatrix} \begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -66.945 & -31.542 \\ -31.542 & -14.861 \end{bmatrix} \begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} -66.945 & -31.542 \\ -31.542 & -14.861 \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$-66.945 - 31.542\Phi_{21} = 0$$

$$\Phi_{21} = -\frac{66.945}{31.542} = -2.1224$$

Hence the first mode associated with $\eta = 55.084$ is

$$\begin{Bmatrix} 1 \\ -2.1224 \end{Bmatrix}$$

When $\boxed{\eta = 0.63023}$

$$\Big[[k] - \omega_2^2[M]\Big]\{\Phi\}_2 = \{0\}$$

$$\begin{bmatrix} \frac{13}{2} - \frac{4}{3}\eta & -4 - \frac{1}{2}\eta \\ -4 - \frac{1}{2}\eta & \frac{7}{2} - \frac{1}{3}\eta \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 5.6597 & -4.3151 \\ -4.3151 & 3.2899 \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} 5.6597 & -4.3151 \\ -4.3151 & 3.2899 \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$5.6597 - 4.3151\Phi_{22} = 0$$

$$\Phi_{22} = \frac{-5.6597}{-4.3151} = 1.3116$$

Hence the second mode associated with $\eta = 0.630$ is

$$\begin{Bmatrix} 1 \\ 1.3116 \end{Bmatrix}$$

Summary, $\omega^2 \frac{L}{g} = \eta$ hence $\omega = \sqrt{\eta}\sqrt{\frac{g}{L}}$

ω_n (rad/sec)	mode shape
$7.422\sqrt{\frac{g}{L}}$	$\begin{Bmatrix} 1 \\ -2.1224 \end{Bmatrix}$
$0.794\sqrt{\frac{g}{L}}$	$\begin{Bmatrix} 1 \\ 1.3116 \end{Bmatrix}$

For $\boxed{\beta = 2}$

$$mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + mgL\begin{pmatrix} 4-\frac{3}{2} & -2 \\ -2 & 2-\frac{1}{2} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To find modes of free vibration, we write

$$\Big[[k] - \omega^2[M]\Big]\{\Phi\} = \{0\}$$

Solving for eigenvalues

$$\det\left[mgL\begin{pmatrix} \frac{5}{2} & -2 \\ -2 & \frac{3}{2} \end{pmatrix} - \omega^2 mL^2\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

$$\det\left[\begin{pmatrix} \frac{5}{2} & -2 \\ -2 & \frac{3}{2} \end{pmatrix} - \omega^2\frac{L}{g}\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

Let $\omega^2 \frac{L}{g} = \eta$, hence

$$\det\left[\begin{pmatrix} \frac{5}{2} & -2 \\ -2 & \frac{3}{2} \end{pmatrix} - \eta\begin{pmatrix} \frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} \frac{5}{2} - \frac{4}{3}\eta & -2 - \frac{1}{2}\eta \\ -2 - \frac{1}{2}\eta & \frac{3}{2} - \frac{1}{3}\eta \end{bmatrix} = 0$$

$$\frac{7}{36}\eta^2 - \frac{29}{6}\eta - \frac{1}{4} = 0$$

Hence $\eta = 24.909, \eta = -5.162 \times 10^{-2}$

Since $\omega^2 \frac{L}{g} = \eta$ hence when $\eta = -5.162 \times 10^{-2}$ then $\omega = \sqrt{\eta}\sqrt{\frac{g}{L}}$ which means there will a complex number for ω which is not possible as the frequency must be positive. This means such a $\boxed{\text{system is not stable}}$.It is not possible to obtain the shape functions when ω is complex.

2.10.4 Problem 3

problem 4.11 in text: the mass and stiffness matrices of a system are $[M] = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$kg, $[K] = \begin{bmatrix} 300 & 1 \\ 1 & 200 \end{bmatrix}$kN/m, determine the system natural frequencies and normal vibration modes. (hint, normal modes means mass normalized modes).

Answer:

To find modes of free vibration, we write

$$\Big[[k] - \omega^2[M]\Big]\{v\} = \{0\}$$

Solving for eigenvalues

$$\det\left[\begin{bmatrix} 300\times 10^3 & 1000 \\ 1000 & 200\times 10^3 \end{bmatrix} - \omega^2\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\right] = 0$$

$$\det\begin{bmatrix} 300\times 10^3 - 4\omega^2 & 1000 - \omega^2 \\ 1000 - \omega^2 & 200\times 10^3 - 3\omega^2 \end{bmatrix} = 0$$

$$11\omega^4 - 1698000\omega^2 + 59999000000 = 0$$

Hence the positive roots are $\omega = 234.02, \omega = 315.59$

When $\omega_1 = 234.02$ rad/sec then

$$\Big[[k] - \omega_1^2[M]\Big]\{v\}_1 = \{0\}$$

$$\begin{bmatrix} 300\times 10^3 - 4\omega_1^2 & 1000 - \omega_1^2 \\ 1000 - \omega_1^2 & 200\times 10^3 - 3\omega_1^2 \end{bmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 300\times 10^3 - 4(234.02)^2 & 1000 - (234.02)^2 \\ 1000 - (234.02)^2 & 200\times 10^3 - 3(234.02)^2 \end{bmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 80939. & -53765. \\ -53765. & 35704. \end{bmatrix}\begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let v_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} 80939. & -53765. \\ -53765. & 35704. \end{bmatrix}\begin{Bmatrix} 1 \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$80939. - 53765v_{21} = 0$$

$$v_{21} = -\frac{80939}{53765} = 1.5054$$

Hence

$$\{v\}_1 = \begin{Bmatrix} 1 \\ 1.5054 \end{Bmatrix}$$

When $\omega_2 = 315.59$ rad/sec then

$$\Big[[k] - \omega_2^2[M]\Big]\{v\}_2 = \{0\}$$

$$\begin{bmatrix} 300\times10^3-4\omega_2^2 & 1000-\omega_2^2 \\ 1000-\omega_2^2 & 200\times10^3-3\omega_2^2 \end{bmatrix}\begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 300\times10^3-4(315.59)^2 & 1000-(315.59)^2 \\ 1000-(315.59)^2 & 200\times10^3-3(315.59)^2 \end{bmatrix}\begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -98388. & -98597. \\ -98597. & -98791. \end{bmatrix}\begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let v_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} -98388. & -98597. \\ -98597. & -98791. \end{bmatrix}\begin{Bmatrix} 1 \\ v_{22} \end{Bmatrix}=\begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$-98388-98597v_{22}=0$$

$$v_{22}=-\frac{98388}{98597}=-0.998$$

Hence

$$\{v\}_2=\begin{Bmatrix} 1 \\ -0.998 \end{Bmatrix}$$

To obtain the mass normalized shape functions:

$$\begin{aligned}
\mu_1 &= \{v\}_1^T[M]\{v\}_1 \\
&= \begin{Bmatrix} 1 \\ 1.5054 \end{Bmatrix}^T\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.5054 \end{Bmatrix} \\
&= \begin{bmatrix} 5.5054 & 5.5162 \end{bmatrix}\begin{Bmatrix} 1 \\ 1.5054 \end{Bmatrix} \\
&= 13.809
\end{aligned}$$

And

$$\begin{aligned}
\mu_2 &= \{v\}_2^T[M]\{v\}_2 \\
&= \begin{Bmatrix} 1 \\ -0.999 \end{Bmatrix}^T\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.998 \end{Bmatrix} \\
&= \begin{bmatrix} 3.002 & -1.994 \end{bmatrix}\begin{Bmatrix} 1 \\ -0.998 \end{Bmatrix} \\
&= 4.992
\end{aligned}$$

Hence

$$\{\Phi\}_1=\frac{\{v\}_1}{\sqrt{\mu_1}}=\frac{\begin{Bmatrix} 1 \\ 1.505 \end{Bmatrix}}{\sqrt{13.809}}=\begin{Bmatrix} 0.269 \\ 0.405 \end{Bmatrix}$$

and

$$\{\Phi\}_2=\frac{\{v\}_2}{\sqrt{\mu_2}}=\frac{\begin{Bmatrix} 1 \\ -0.999 \end{Bmatrix}}{\sqrt{4.992}}=\begin{Bmatrix} 0.446 \\ -0.447 \end{Bmatrix}$$

Summary

ω_n (rad/sec)	original mode shape	normal mode shapes
234.02	$\begin{Bmatrix} 1 \\ 1.5054 \end{Bmatrix}$	$\begin{Bmatrix} 0.269 \\ 0.405 \end{Bmatrix}$
315.59	$\begin{Bmatrix} 1 \\ 0.999 \end{Bmatrix}$	$\begin{Bmatrix} 0.44759 \\ -0.44665 \end{Bmatrix}$

Hence

$$[\Phi] = \begin{bmatrix} 0.2691 & 0.448 \\ 0.40511 & -0.447 \end{bmatrix}$$

To verify

$$\begin{aligned}
[\Phi]^T[M][\Phi] &= \begin{bmatrix} 0.269 & 0.448 \\ 0.405 & -0.447 \end{bmatrix}^T \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.269 & 0.446 \\ 0.405 & -0.447 \end{bmatrix} \\
&= \begin{bmatrix} 0.269 & 0.405 \\ 0.448 & -0.447 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.269 & 0.448 \\ 0.405 & -0.447 \end{bmatrix} \\
&= \begin{bmatrix} 1.0 & 8.840 \times 10^{-5} \\ 8.840 \times 10^{-5} & 1.0 \end{bmatrix}
\end{aligned}$$

Which is approximately $\begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$ as expected. calculations were not done with high enough accuracy, so that is why the off-diagonal numerical values were not an exact zeros.

To verify with the $[K]$ matrix

$$\begin{aligned}
[\Phi]^T[K][\Phi] &= \begin{bmatrix} 0.269 & 0.448 \\ 0.405 & -0.447 \end{bmatrix}^T \begin{bmatrix} 300 \times 10^3 & 1000 \\ 1000 & 200 \times 10^3 \end{bmatrix} \begin{bmatrix} 0.269 & 0.448 \\ 0.405 & -0.447 \end{bmatrix} \\
&= \begin{bmatrix} 54765. & 6.594 \\ 6.594 & 99600.0 \end{bmatrix}
\end{aligned}$$

Note $\omega_1^2 = 234.02^2 = 54765.$ and $\omega_2^2 = 315.59^2 = 99597$ and these are the values on the diagonal as expected. The values off the diagonal should be an exact zero, since the $[K]$ matrix should be decoupled. Due to low precision in the above calculations, the values did not come out to be zero.

Verify using Matlab. Note that Matlab eig() returns the shape function that are mass normalized

```
EDU>> M=[4 1;1 3];
EDU>> K=[300*10^3 1000;1000 200*10^3];
EDU>> [eig,lam]=eig(K,M)

eig =

   -0.2691   -0.4476
   -0.4051    0.4467
```

```
lam =

   1.0e+04 *

    5.4764         0
         0    9.9600

EDU>> eig'*M*eig

    1.0000         0
         0    1.0000

EDU>> eig'*K*eig

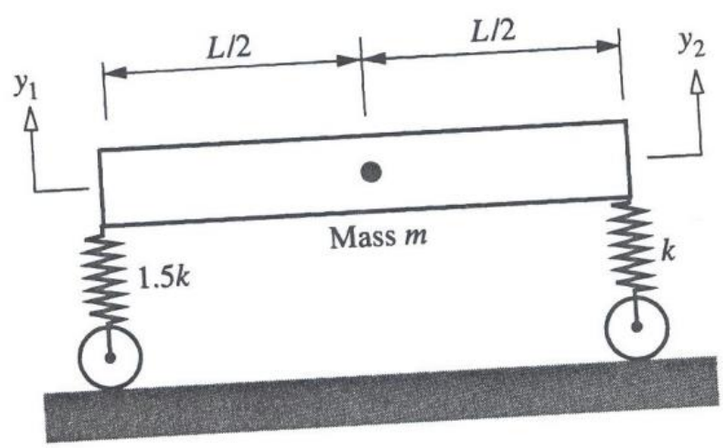
   1.0e+04 *

    5.4764         0
         0    9.9600
```

2.10.5 Problem 4

problem 4.29 in text.

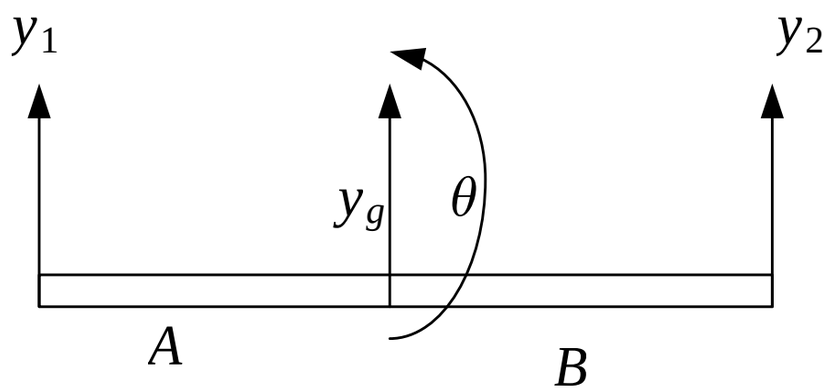
4.29 The diagram models an automobile and its suspension as a rigid block on springs. The mass of the bar is m, and the radius of gyration relative to the center of mass G is $\kappa_G = 0.4L$. Generalized coordinates are the vertical displacements y_1 and y_2 of the ends relative to the static equilibrium



EXERCISE 4.29

position. Consider a situation where the vehicle is released from rest with $y_1 = mg/k$, $y_2 = 0$. Determine the ensuing free vibration as a function of the nondimensional time $t = (k/m)^{1/2}t$.

The mapping between the generalized coordinates y_1, y_2 and y_g, θ is given by



$$
\begin{aligned}
y_g(A+B) &= y_1 A + y_2 B \\
\theta(A+B) &= y_2 - y_1
\end{aligned}
$$

In our care, $A = B = \frac{L}{2}$, hence the above becomes

$$
\begin{aligned}
y_g &= \frac{y_1 \frac{L}{2} + y_2 \frac{L}{2}}{L} = \frac{y_1 + y_2}{2} \\
\theta &= \frac{y_2 - y_1}{L}
\end{aligned}
$$

Hence taking derivative

$$
\begin{aligned}
\dot{y}_g &= \frac{\dot{y}_1 + \dot{y}_2}{2} \\
\dot{\theta} &= \frac{\dot{y}_2 - \dot{y}_1}{L}
\end{aligned}
$$

Using the power balance method, we start by finding the kinetic energy T

$$
\begin{aligned}
T &= \frac{1}{2} m \dot{y}_g^2 + \frac{1}{2} I_{cg} \dot{\theta}^2 \\
&= \frac{1}{2} m \left(\frac{\dot{y}_1 + \dot{y}_2}{2} \right)^2 + \frac{1}{2} \left(m r_G^2 \right) \left(\frac{\dot{y}_2 - \dot{y}_1}{L} \right)^2
\end{aligned}
$$

where r_G is the radius of gyration $0.4L$, hence

$$
\begin{aligned}
T &= \frac{1}{8} m \left(\dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1 \dot{y}_2 \right) + \frac{1}{2} \left(m \left(\frac{4}{10} L \right)^2 \right) \frac{1}{L^2} \left(\dot{y}_2^2 + \dot{y}_1^2 - 2\dot{y}_1 \dot{y}_2 \right) \\
&= \frac{1}{8} m \left(\dot{y}_1^2 + \dot{y}_2^2 + 2\dot{y}_1 \dot{y}_2 \right) + \frac{8}{100} m \left(\dot{y}_2^2 + \dot{y}_1^2 - 2\dot{y}_1 \dot{y}_2 \right) \\
&= \dot{y}_1^2 \left(\frac{1}{8} m + \frac{8}{100} m \right) + \dot{y}_2^2 \left(\frac{1}{8} m + \frac{8}{100} m \right) + \dot{y}_1 \dot{y}_2 \left(\frac{2}{8} m - \frac{16}{100} m \right) \\
&= \frac{41}{200} m \dot{y}_1^2 + \frac{41}{200} m \dot{y}_2^2 + \frac{9}{100} m \dot{y}_1 \dot{y}_2
\end{aligned}
$$

Comparing the above to quadratic form $T = \frac{1}{2} M_{11} \dot{y}_1^2 + \frac{1}{2} M_{22} \dot{y}_2^2 + M_{12} \dot{y}_1 \dot{y}_2$ then

$$
\begin{aligned}
M_{11} &= \frac{41}{100} m = 0.41m \\
M_{22} &= \frac{41}{100} m = 0.41m \\
M_{12} &= 0.09m
\end{aligned}
$$

Hence the mass matrix is

$$[M] = m\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}$$

$$\begin{aligned} V_{spring} &= \frac{1}{2}k_{left}y_1^2 + \frac{1}{2}k_{right}y_1^2 \\ &= \frac{1}{2}\left(\frac{3}{2}k\right)y_1^2 + \frac{1}{2}ky_1^2 \end{aligned}$$

Comparing to quadratic form $\frac{1}{2}K_{11}y_1^2 + \frac{1}{2}K_{22}y_2^2 + K_{12}y_1y_2$ then

$$\left[k_{spring}\right] = k\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_{gravity} = mgy_g = mg\frac{y_1 + y_2}{2}$$

Since this will be evaluated at $y_1 = y_2 = 0$ then we see right away that there is no contribution to potential energy to the stiffness matrix. Hence the EOM are

$$m\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}\begin{bmatrix} \ddot{y}_1(t) \\ \ddot{y}_2(t) \end{bmatrix} + k\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To convert to t' space, given by $t' = \sqrt{\frac{k}{m}}\,t$ as required, then we see that $\frac{dy}{dt} = \frac{dy}{dt'}\frac{dt'}{dt} = \frac{dy}{dt'}\sqrt{\frac{k}{m}}$, and $\frac{d^2y}{dt^2} = \frac{d^2y}{dt'^2}\frac{k}{m}$

Hence the ODE becomes

$$m\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}\begin{bmatrix} \ddot{y}_1(t') \\ \ddot{y}_2(t') \end{bmatrix}\frac{k}{m} + k\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$k\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}\begin{bmatrix} \ddot{y}_1(t') \\ \ddot{y}_2(t') \end{bmatrix} + k\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Since $k \neq 0$ we can divide by it, hence

$$\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}\begin{bmatrix} \ddot{y}_1(t') \\ \ddot{y}_2(t') \end{bmatrix} + \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_1(t') \\ y_2(t') \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To find modes of free vibration, we write

$$\Big[[K] - \omega^2[M]\Big]\{v\} = \{0\}$$

Solving for eigenvalues

$$\det\left[\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} - \omega^2\begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix}\right] = 0$$

$$\det\begin{bmatrix} 1.5 - 0.41\omega^2 & -\omega^2 0.09 \\ -\omega^2 0.09 & 1 - \omega^2 0.41 \end{bmatrix} = 0$$

$$0.16\omega^4 - 1.025\omega^2 + 1.5 = 0$$

Hence the positive roots are $\omega_1 = 2.0357, \omega_2 = 1.5041$

When $\omega_1 = 2.0357$ then

$$\left[[k] - \omega_1^2[M]\right]\{v\}_1 = \{0\}$$

$$\begin{bmatrix} 1.5 - 0.41(2.0357)^2 & -(2.0357)^2 0.09 \\ -(2.0357)^2 0.09 & 1 - (2.0357)^2 0.41 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -0.199 & -0.373 \\ -0.373 & -0.699 \end{bmatrix} \begin{Bmatrix} v_{11} \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let v_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} -0.199 & -0.373 \\ -0.373 & -0.699 \end{bmatrix} \begin{Bmatrix} 1 \\ v_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$-0.199 - 0.373 v_{21} = 0$$

$$v_{21} = -\frac{0.199}{0.373} = -0.534$$

Hence

$$\{v\}_1 = \begin{Bmatrix} 1 \\ -0.534 \end{Bmatrix}$$

When $\omega_2 = 1.5041$ then

$$\left[[k] - \omega_2^2[M]\right]\{v\}_2 = \{0\}$$

$$\begin{bmatrix} 1.5 - 0.41(1.504)^2 & -(1.504)^2 0.09 \\ -(1.504)^2 0.09 & 1 - (1.504)^2 0.41 \end{bmatrix} \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0.572 & -0.204 \\ -0.204 & 0.072 \end{bmatrix} \begin{Bmatrix} v_{12} \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let v_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} 0.572 & -0.204 \\ -0.204 & 0.072 \end{bmatrix} \begin{Bmatrix} 1 \\ v_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$0.572 - 0.204 v_{22} = 0$$

$$v_{22} = \frac{0.572}{0.204} = 2.812$$

Hence

$$\{v\}_2 = \begin{Bmatrix} 1 \\ 2.812 \end{Bmatrix}$$

To obtain the mass normalized shape functions:

$$\begin{aligned} \mu_1 &= \{v\}_1^T [M] \{v\}_1 \\ &= \begin{Bmatrix} 1 \\ -0.534 \end{Bmatrix}^T \begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.534 \end{Bmatrix} \\ &= 0.43073 \end{aligned}$$

And

$$\begin{aligned}\mu_2 &= \{v\}_2^T[M]\{v\}_2\\ &= \begin{Bmatrix}1\\2.812\end{Bmatrix}^T\begin{bmatrix}0.41 & 0.09\\0.09 & 0.41\end{bmatrix}\begin{Bmatrix}1\\2.812\end{Bmatrix}\\ &= 4.1569\end{aligned}$$

Hence

$$\{\Phi\}_1 = \frac{\{v\}_1}{\sqrt{0.43073}} = \frac{\begin{Bmatrix}1\\-0.53374\end{Bmatrix}}{\sqrt{0.43073}} = \begin{Bmatrix}1.5237\\-0.81326\end{Bmatrix}$$

and

$$\{\Phi\}_2 = \frac{\{v\}_2}{\sqrt{\mu_2}} = \frac{\begin{Bmatrix}1\\2.812\end{Bmatrix}}{\sqrt{4.1569}} = \begin{Bmatrix}0.491\\1.379\end{Bmatrix}$$

Summary

ω (rad/sec)	original mode shape	normal mode shapes
2.0357	$\begin{Bmatrix}1\\-0.534\end{Bmatrix}$	$\begin{Bmatrix}1.524\\-0.813\end{Bmatrix}$
1.5041	$\begin{Bmatrix}1\\2.812\end{Bmatrix}$	$\begin{Bmatrix}0.491\\1.379\end{Bmatrix}$

Hence

$$[\Phi] = \begin{bmatrix}1.524 & 0.491\\-0.813 & 1.379\end{bmatrix}$$

To verify

$$\begin{aligned}[\Phi]^T[M][\Phi] &= \begin{bmatrix}1.524 & 0.491\\-0.813 & 1.379\end{bmatrix}^T\begin{bmatrix}0.41 & 0.09\\0.09 & 0.41\end{bmatrix}\begin{bmatrix}1.524 & 0.491\\-0.813 & 1.379\end{bmatrix}\\ &= \begin{bmatrix}1.0 & -1.9688\times10^{-4}\\-1.9688\times10^{-4} & 1.0\end{bmatrix}\end{aligned}$$

Which is approximately $\begin{bmatrix}1.0 & 0\\0 & 1.0\end{bmatrix}$ as expected. calculations were not done with high enough accuracy, so that is why the off-diagonal numerical values were not an exact zeros.

To verify with the $[K]$ matrix

$$\begin{aligned}[\Phi]^T[K][\Phi] &= \begin{bmatrix}1.5237 & 0.49047\\-0.81326 & 1.3790\end{bmatrix}^T\begin{bmatrix}1.5 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1.5237 & 0.49047\\-0.81326 & 1.3790\end{bmatrix}\\ &= \begin{bmatrix}4.1439 & -4.9183\times10^{-4}\\-4.9183\times10^{-4} & 2.2625\end{bmatrix}\end{aligned}$$

Verify using Matlab

```
EDU>> K=[1.5 0;0 1]; M=[0.41  0.09;0.09 0.41];
EDU>> [eig,lam]=eig(K,M)

eig =
   -0.4905   -1.5238
   -1.3789    0.8130

lam =
    2.2624         0
         0    4.1439
```

Now we can solve the problem. Using $\{x\} = [\Phi]\left\{\eta\right\}$, where

$$\begin{aligned}\left\{\eta\right\} &= [\Phi]^{-1}\{x\}\\ &= [\Phi]^T[M]\{x\}\end{aligned}$$

Hence, initial conditions in the $\left\{\eta\right\}$ space is

$$\begin{aligned}\left\{\eta\right\}_0 &= [\Phi]^T[M]\{x\}_0\\ &= [\Phi]^T[M]\begin{Bmatrix} y_1(0)\\ y_2(0)\end{Bmatrix}\\ &= \begin{bmatrix} 1.524 & 0.491\\ -0.813 & 1.379\end{bmatrix}^T\begin{bmatrix} 0.41 & 0.09\\ 0.09 & 0.41\end{bmatrix}\begin{Bmatrix} \frac{mg}{k}\\ 0\end{Bmatrix}\\ &= \begin{Bmatrix} 0.552\frac{g}{k}m\\ 0.325\frac{g}{k}m\end{Bmatrix}\end{aligned}$$

and

$$\begin{aligned}\left\{\eta'\right\}_0 &= [\Phi]^T[M]\{x'\}_0\\ &= [\Phi]^T[M]\begin{Bmatrix} y_1'(0)\\ y_2'(0)\end{Bmatrix}\\ &= [\Phi]^T[M]\begin{Bmatrix} 0\\ 0\end{Bmatrix}\\ &= \begin{Bmatrix} 0\\ 0\end{Bmatrix}\end{aligned}$$

So, we need to solve

$$\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix} \ddot{\eta}_1\\ \ddot{\eta}_2\end{Bmatrix} + \begin{bmatrix} 4.1439 & 0\\ 0 & 2.2625\end{bmatrix}\begin{Bmatrix} \eta_1\\ \eta_2\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\end{Bmatrix}$$

with the initial conditions

$$\begin{Bmatrix} \eta_1(0)\\ \eta_2(0)\end{Bmatrix} = \begin{Bmatrix} 0.552\frac{g}{k}m\\ 0.325\frac{g}{k}m\end{Bmatrix}$$

$$\begin{Bmatrix} \eta_1'(0)\\ \eta_2'(0)\end{Bmatrix} = \begin{Bmatrix} 0\\ 0\end{Bmatrix}$$

The solution is given by

$$\eta_1(t) = A\cos\sqrt{4.144}\,t + B\sin\sqrt{4.144}\,t$$

When $t = 0$, $\eta_1(0) = 0.55152\frac{g}{k}m = A$. Taking derivative gives $\eta_1'(t) = -A\sin\sqrt{4.144}\,t + B\cos\sqrt{4.144}\,t$, hence when $t = 0$ we have $0 = B$, therefore

$$\eta_1(t) = 0.552\frac{g}{k}m\cos\sqrt{4.144}\,t$$

Now we solve for $\eta_2(t)$,The solution is given by

$$\eta_2(t) = A\cos\sqrt{2.263}\,t + B\sin\sqrt{2.263}\,t$$

When $t = 0$, $\eta_2(0) = 0.3252\frac{g}{k}m = A$. and $0 = B$, therefore

$$\eta_2(t) = 0.325\frac{g}{k}m\cos\sqrt{2.263}\,t$$

Now we obtain the solution in the y space

$$\begin{aligned}
\left\{y\right\} &= [\Phi]\left\{\eta\right\} \\
\begin{Bmatrix} y_1(t') \\ y_2(t') \end{Bmatrix} &= \begin{bmatrix} 1.524 & 0.491 \\ -0.813 & 1.379 \end{bmatrix} \begin{Bmatrix} 0.552\frac{g}{k}m\cos\sqrt{4.1439}\,t \\ 0.325\frac{g}{k}m\cos\sqrt{2.2625}\,t \end{Bmatrix} \\
&= \begin{Bmatrix} 0.840\frac{g}{k}m\cos(2.036t') + 0.1595\frac{g}{k}m\cos(1.504t') \\ 0.448\frac{g}{k}m\cos(1.504t') - 0.449\frac{g}{k}m\cos(2.036t') \end{Bmatrix}
\end{aligned}$$

We are supposed to obtain the answer

$$\begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} 0.16\cos(1.5t') + 0.84\cos(2t') \\ 0.45\cos(1.5t') - 0.45\cos(2t') \end{Bmatrix}$$

The answers agree. The scalar $\frac{g}{k}m$ for some reason is not shown in the key solution.

2.10.6 Key solution for HW 9

Homework #9
EMA 545, Spring 2013

For the following problems, you may have the equations of motion for some of these systems in your past homework assignments or in the solutions to those that were posted online, so you may use those if you wish.

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

1.) **Problem 4.3** in Ginsberg. Sketch the deformation of the system when it moves in each of the modes. (Notice that you can pull out factors such as *k* and *m* so that only numbers remain in the mass and stiffness matrices. Then it is possible to check your answers using Matlab.)

2.) **Problem 4.7** as given in the text. You may use the following equations of motion:

$$mL^2\begin{bmatrix} 4/3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + mgL\begin{bmatrix} 2\beta - 3/2 & -\beta \\ -\beta & \beta - 1/2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

3.) **Problem 4.11** as given in the text. Hint: normal modes = mass normalized modes

4.) **Problem 4.29** as given in the text. Also, plot the motion of the automobile as a function of time. Is the response a pure-sinusoid? Why or why not? Note: The answer provided by the book is incorrect. The correct answer is:

$y_1(t) = 0.16\cos(1.5t') + 0.84\cos(2.0t')$
$y_2(t) = 0.45\cos(1.5t') - 0.45\cos(2.0t')$

4.3 4/6/2011 (1)

- Solve by hand for ω_r's and $\{\phi_r\}$'s
- Sketch deformation shape in each mode.

From HW#2 (Prob 1.30)

$$\begin{bmatrix} \frac{1}{3}m_1L^2 & 0 \\ 0 & \left(\frac{1}{12}+\frac{1}{16}\right)m_2L^2 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} \left(\frac{9}{16}K_1L^2+K_2L^2\right) & \left(\frac{9}{16}K_1L^2+\frac{1}{2}K_2L^2\right) \\ \text{sym} & \left(\frac{9}{16}K_1L^2+\frac{1}{4}K_2L^2\right)\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ \frac{FL}{2}\end{Bmatrix}$$

θ_1 θ_2

let $m_1 = m$, $m_2 = 2m$, $K_1 = K$, $K_2 = K/2$

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & \frac{7}{24}mL^2 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} \left(\frac{9}{16}+\frac{1}{2}\right) & \left(\frac{9}{16}+\frac{1}{4}\right) \\ \text{sym} & \left(\frac{9}{16}+\frac{1}{8}\right)\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}\cdot KL^2 = \begin{Bmatrix}0\\ \frac{FL}{2}\end{Bmatrix}$$

$K_{11} = (51/48)KL^2$ $K_{12} = (39/48)KL^2$ $K_{22} = (33/48)KL^2$

let M + K be matrices containing only the numbers,

EVP: $\left([K]KL^2 - \omega^2\cdot mL^2[M]\right)\{\phi\} = 0$

let $\lambda = \omega^2\cdot\frac{mL^2}{KL^2} = \omega^2\frac{m}{K}$

multiply by 48 to clear fractions

$$\left(\begin{bmatrix}51 & 39\\ 39 & 33\end{bmatrix} - \lambda\begin{bmatrix}16 & 0\\ 0 & 14\end{bmatrix}\right)\{\phi\} = 0$$

$$\left(\begin{bmatrix}51-16\lambda & 39\\ 39 & 33-14\lambda\end{bmatrix}\right)\{\phi\} = 0$$

char eq: $(51-16\lambda)(33-14\lambda) - 39^2 = 0$

$$51\cdot 33 - (51\cdot 14 + 16\cdot 33) + 16\cdot 14\lambda^2 - 39^2 = 0$$

polynomial solver on calculator or in Matlab:
roots([16·14, -(51·14+16·33), 51·33-39^2])

$\lambda_1 = 0.1337$ $\lambda_2 = 5.411$

1st eq
$51-16\lambda\,\phi_1 + 39\phi_2 = 0 \longrightarrow \phi_2 = \frac{16\lambda - 51}{39}\phi_1$

value in 2nd row value in 1st row

$\{\phi_1\} = \begin{Bmatrix}1\\ -1.2529\end{Bmatrix}$ $\phi_2 = \begin{Bmatrix}1\\ 0.9122\end{Bmatrix}$

$\omega_1 = 0.3656\sqrt{\frac{K}{m}}$ $\omega_2 = 2.3262\sqrt{\frac{K}{m}}$

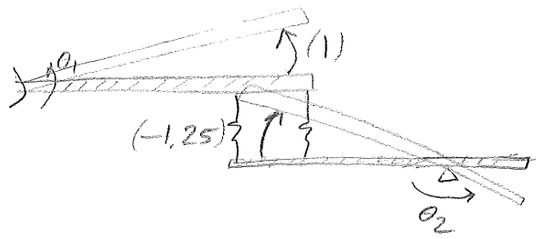
Matlab gives the exact same answer

4.3 - continued (2)

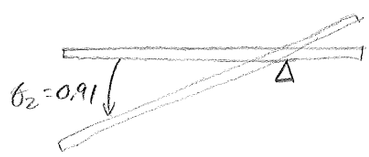
Sketch

Mode 1: $\{\theta\} = \mathrm{Re}(B\{\phi_1\}e^{i\omega_1 t})$, so

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ -1.25\end{Bmatrix}$$



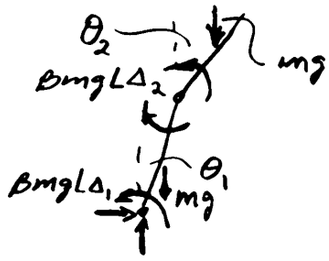
Mode 2: $\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ 0.91\end{Bmatrix}$

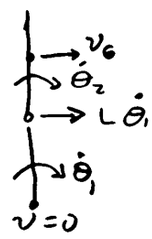


→ Second mode involves more strain in springs, and hence a higher frequency.

→ The effective mass of the two bars is similar since $m_2 = 2m$ (even though its pivot is closer to the center of mass.) $m_{11} = (1/3\, mL^2)$; $m_{22} = 0.29\, mL^2$ Because this is the case, both bars rotate by similar ammounts in each mode.

Exercise 4.7





$$\bar v_G = L\dot\theta_1\,\bar i + \tfrac{L}{2}\dot\theta_2\,\bar i$$

$$T = \tfrac{1}{2}\left(\tfrac{1}{3}mL^2\right)\dot\theta_1^2 + \tfrac{1}{2}m\left(L\dot\theta_1 + \tfrac{L}{2}\dot\theta_2\right)^2 + \tfrac{1}{2}\left(\tfrac{1}{12}mL^2\right)\dot\theta_2^2 = \tfrac{1}{2}mL^2\left(\tfrac{4}{3}\dot\theta_1^2 + \tfrac{1}{3}\dot\theta_2^2 + \dot\theta_1\dot\theta_2\right)$$

$$M_{11} = \tfrac{4}{3}mL^2,\quad M_{22} = \tfrac{1}{3}mL^2,\quad M_{12} = M_{21} = \tfrac{1}{2}mL^2$$

$$V_{sp} = \tfrac{1}{2}(\beta mgL)\theta_1^2 + \tfrac{1}{2}(\beta mgL)(\theta_2-\theta_1)^2 = \tfrac{1}{2}\beta mgL\left[2\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2\right]$$

$$(K_{11})_{sp} = 2\beta mgL,\quad (K_{22})_{sp} = \beta mgL,\quad (K_{12})_{sp} = (K_{21})_{sp} = -\beta mgL$$

$$V_{gr} = mg\tfrac{L}{2}\cos(\theta_1) + mg\left[L\cos(\theta_1) + \tfrac{L}{2}\cos(\theta_2)\right]$$

$$(K_{11})_{gr} = \left.\frac{\partial^2 V_{gr}}{\partial\theta_1^2}\right|_{\theta_1=\theta_2=0} = -\tfrac{3}{2}mgL$$

$$(K_{22})_{gr} = \left.\frac{\partial^2 V_{gr}}{\partial\theta_2^2}\right|_{\theta_1=\theta_2=0} = -\tfrac{1}{2}mgL$$

$$(K_{12})_{gr} = \left.\frac{\partial^2 V}{\partial\theta_1\partial\theta_2}\right|_{\theta_1=\theta_2=0} = 0$$

$$\left[[K] - \omega^2[M]\right]\{\phi\} = \left[mgL\begin{bmatrix}(2\beta-\frac{3}{2}) & -\beta\\ -\beta & (\beta-\frac{1}{2})\end{bmatrix} - mL^2\omega^2\begin{bmatrix}\frac{4}{3} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\right]\{\phi\} = \{0\}$$

Let $\lambda = \dfrac{mL^2\omega^2}{mgL} \Rightarrow \omega^2 = \dfrac{g}{L}\lambda$, Then

$$\begin{bmatrix}\left((2\beta-\frac{3}{2}) - \frac{4}{3}\lambda\right) & -(\beta+\frac{1}{2}\lambda)\\ -(\beta+\frac{1}{2}\lambda) & \left((\beta-\frac{1}{2}) - \frac{1}{3}\lambda\right)\end{bmatrix}\{\phi\} = \{0\}$$

When $\beta = 4$;
$$\begin{bmatrix}\left(\frac{13}{2} - \frac{4}{3}\lambda\right) & -(4+\frac{1}{2}\lambda)\\ -(4+\frac{1}{2}\lambda) & \left(\frac{7}{2} - \frac{1}{3}\lambda\right)\end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Char eq $\quad \frac{7}{36}\lambda^2 - \frac{65}{6}\lambda + \frac{27}{4} = 0 \Rightarrow \lambda = 0.6302,\ 55.08$

$$\begin{bmatrix}\frac{13}{2} - \frac{4}{3}\lambda_j & -(4+\frac{1}{2}\lambda_j)\\ \times & \times\end{bmatrix}\begin{Bmatrix}1\\ \phi_{2j}\end{Bmatrix} = \begin{Bmatrix}0\\ \times\end{Bmatrix}$$

$$\phi_{2j} = \frac{13/2 - (4/3)\lambda_j}{4 + \lambda_j/2}$$

$$\lambda_1 = 0.6302 \Rightarrow \omega_1 = 0.794\left(\frac{g}{L}\right)^{1/2}, \{\phi_1\} = \begin{Bmatrix} 1 \\ 1.312 \end{Bmatrix} \Leftarrow$$

$$\lambda_2 = 55.08 \Rightarrow \omega_2 = 7.422\left(\frac{g}{L}\right)^{1/2}; \{\phi_2\} = \begin{Bmatrix} 1 \\ -2.122 \end{Bmatrix}$$

When $\beta = 2$ $\begin{bmatrix} \frac{5}{2} - \frac{4}{3}\lambda & -(2+\frac{\lambda}{2}) \\ -(2+\frac{\lambda}{2}) & \frac{3}{2} - \frac{1}{3}\lambda \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Char eq: $\frac{7}{36}\lambda^2 - \frac{29}{6}\lambda - \frac{1}{4} = 0 \Rightarrow \lambda_1 = -0.053, 24.91 \Leftarrow$

Negative $\lambda_1 \Rightarrow$ imaginary $\omega_1 \Rightarrow$ divergence instability

Exercise 4.11

$$\left|[K]-\omega^2[M]\right| = \begin{vmatrix} (3(10^5)-4\omega^2) & -\omega^2 \\ -\omega^2 & (2(10^5)-3\omega^2) \end{vmatrix}$$

$$= 11\omega^4 - 17(10^5)\omega^2 + 6(10^{10}) = 0$$

$$\omega^2 = 5.455(10^4),\ 10(10^4)$$

$$\begin{bmatrix} (3(10^5)-4\omega_j^2) & -\omega_j^2 \\ \times & \times \end{bmatrix} \begin{Bmatrix} 1 \\ \phi_{2j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$\phi_{2j} = \frac{3(10^5)-4\omega_j^2}{\omega_j^2}$$

$$\omega_1 = 233.5 \text{ rad/s} \Rightarrow \{\phi_1\} = \begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix} \quad \Leftarrow$$

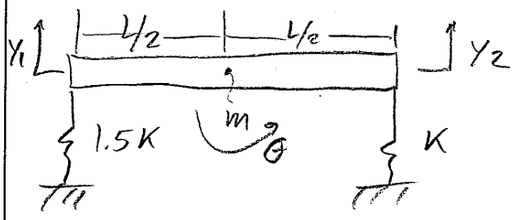
$$\omega_2 = 316.2 \text{ rad/s} \Rightarrow \{\phi_2\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\mu_j = \{\phi_j\}^T[M]\{\phi_j\} \Rightarrow \mu_1 = 13.75,\ \mu_2 = 5$$

$$\{\Phi_1\} = \frac{1}{\sqrt{13.75}} \begin{Bmatrix} 1 \\ 1.5 \end{Bmatrix} = \begin{Bmatrix} 0.2697 \\ 0.4045 \end{Bmatrix} \quad \Leftarrow$$

$$\{\Phi_2\} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.4472 \\ -0.4472 \end{Bmatrix}$$

4.29 Solution



$K_g = 0.4L \rightarrow I_g = mK_g^2$

$y_1(0) = \frac{mg}{K} \quad y_2(0) = 0$

find $\{y(t)\}$ for $t' = \sqrt{K/m}\, t$

$$T = \frac{1}{2} m v_g \cdot v_g + \frac{1}{2} I_g \dot{\theta}^2$$

Use velocity equations to relate $\dot{y}_1, \dot{y}_2$ to $\dot{y}_g, \dot{\theta}$

$$\dot{y}_2 \hat{j} = \dot{y}_1 \hat{j} + L\dot{\theta}\hat{j} \longrightarrow L\dot{\theta} = \dot{y}_2 - \dot{y}_1$$

$$\dot{y}_g \hat{j} = \dot{y}_1 \hat{j} + \frac{L}{2}\dot{\theta} \qquad \dot{y}_g = \dot{y}_1 + \frac{L}{2}\left(\frac{\dot{y}_2 - \dot{y}_1}{L}\right) = \frac{\dot{y}_1 + \dot{y}_2}{2}$$

$$T = \frac{1}{2} m \left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2 + \frac{1}{2} m (0.4\cancel{L})^2 \cdot \left(\frac{\dot{y}_2 - \dot{y}_1}{\cancel{L}}\right)^2$$

$$T = \frac{1}{2} m \left[\frac{1}{4}(\dot{y}_1^2 + 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2) + 0.4^2(\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2)\right]$$

$$= \frac{1}{2} m \left[(0.25 + 0.4^2)\dot{y}_1^2 + 2(0.25 - 0.4^2)\dot{y}_1\dot{y}_2 + (0.25 + 0.4^2)\dot{y}_2^2\right]$$

$$M = \begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix} m \quad K = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} K$$

$\left(V_{spr} = \frac{1}{2}K\Delta_1^2 + \frac{1}{2}K \cdot 1.5\Delta_2^2\right)$

$V_{grav} \Rightarrow$ no contribution to K matrix

$$[M]\ddot{y} + [K]y = 0 \qquad \frac{dy}{dt} = \frac{\partial y}{\partial t'}\frac{\partial t'}{\partial t} = y' \cdot \sqrt{\frac{K}{m}}$$

$$\longrightarrow [M]\cancel{m} \cdot \frac{\cancel{K}}{\cancel{m}} y'' + \cancel{K}[K]y = 0$$ – K, m cancel with $t = t'$

solve EVP in Matlab

$$\{y\} = [\Phi]\{\eta\} \longrightarrow \{\eta(0)\} = [\Phi]^T[M]\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \cdot \frac{mg}{K}$$

$$\eta_1(t) = c_1\cos(\omega_1 t) + 0 \quad \eta_2(t) = c_2\cos(\omega_2 t) + 0$$

$$\{y(t)\} = [\Phi]\begin{Bmatrix} c_1\cos(\omega_1 t) \\ c_2\cos(\omega_2 t) \end{Bmatrix}$$ (See Matlab results)

$$\{y(t)\} = \begin{Bmatrix} 0.1595 \\ 0.4485 \end{Bmatrix}\cos(1.504\, t') + \begin{Bmatrix} 0.8405 \\ -0.4485 \end{Bmatrix}\cos(2.036\, t')$$

Solution: Ch. 4, Problem 4.29

See handwritten notes for derivation.
Matlab Code:

```
% Define Mass and Stiffness matrices:
M = [0.5+0.4^2, 0.5-0.4^2;
    0.5-0.4^2, 0.5+0.4^2];
K = [1.5, 0; 0, 1];
% q_0 = [1; 1];
% q_0 = [1; 0];
q_0 = [1; 0];
q_dot_0 = [0; 0];
% Solve the eigenvalue problem:
[Phi,Lam] = eig(K,M);
% Matlab solves an eigenvalue problem [A]{x} = lam*[B]{x}, where lam is
a
% scalar if we input eig(A,B).  the matrix Lam output by Matlab is a
% diagonal matrix of eigenvalues lam, so by comparing with our
eigenvalue
% problem, we see that each nat. freq wn = sqrt(lam), or the vector of
% natural frequencies is wns = diag(Lam).^(1/2)
Phi
wns = diag(Lam).^(1/2)
% Check that the eigenvectors are mass normalized.  If this is not an
% identity, then we need to normalize
Phi.'*M*Phi
% Note that Matlab sometimes gives the first eigenvector as the
negative of
% what we found.  Either is a valid mode for the system.

% Now the initial conditions give:
eta_0 = Phi.'*M*q_0
eta_dot_0 = Phi.'*M*q_dot_0

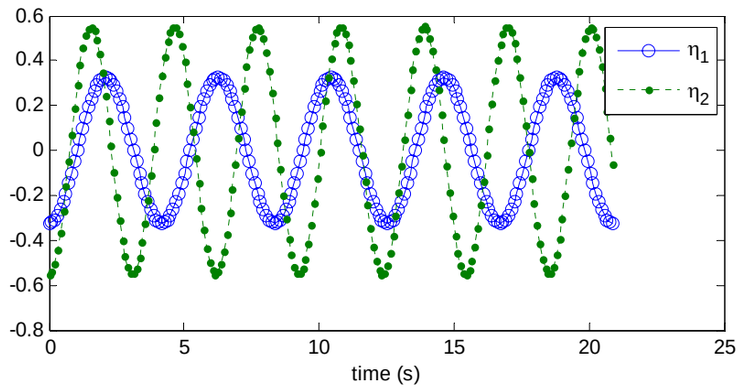
% and the constants in the solutions eta(t) = a1*cos(wns(1)*t)+etc...
a1 = eta_0(1); a2 = eta_dot_0(1)/wns(1);
b1 = eta_0(2); b2 = eta_dot_0(2)/wns(2);

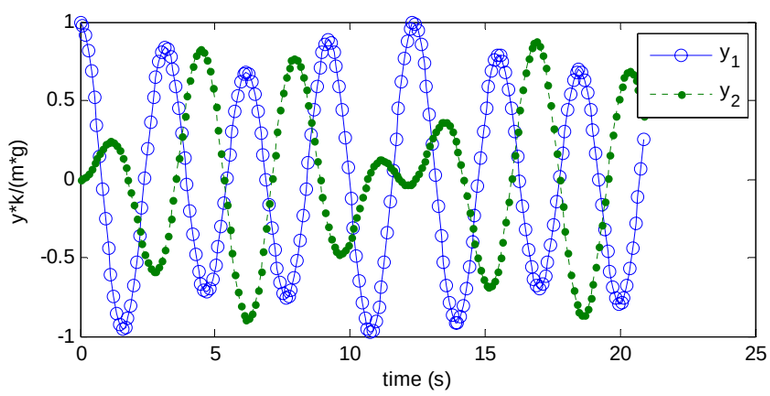
% Define a time vector with 5 cycles of the lowest frequency:
ts = [0:1:200]/200*5*(2*pi/wns(1));
eta_t = [a1*cos(wns(1)*ts)+a2*sin(wns(1)*ts);
    b1*cos(wns(2)*ts)+b2*sin(wns(2)*ts)];
% each column of the matrix above gives {eta(t)} at some instant t.
Since
% the eta values are in columns, we obtain {q} by multiplying by [Phi]
q_t = Phi*eta_t;

figure(1);
subplot(2,1,1)
plot(ts,eta_t(1,:),'-o',ts,eta_t(2,:),':.'); legend('\eta_1','\eta_2');
xlabel('time (s)');
subplot(2,1,2)
plot(ts,q_t(1,:),'-o',ts,q_t(2,:),':.'); legend('y_1','y_2');
xlabel('time (s)');  ylabel('y*k/(m*g)');
```

Results:

```
>> M
M =
         0.41         0.09
         0.09         0.41
>> K
K =
          1.5            0
            0            1
Phi =
      -0.4905      -1.5238
      -1.3789      0.81302
wns =
       1.5041
       2.0357
ans =
            1  5.5511e-017
 -5.5511e-017            1
eta_0 =
     -0.32521
     -0.55158
eta_dot_0 =
     0
     0
```






```
Response plotted over 5 cycles.  Notice that the initial
conditions are satisfied.  Both modes are excited and
oscillate at different frequencies.  The superposition of
both modes causes the response in y1, y2 coordinates to
look quite complicated.
```

2.11 HW10

Local contents

2.11.1 problem description

Homework #10
EMA 545, Spring 2013

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

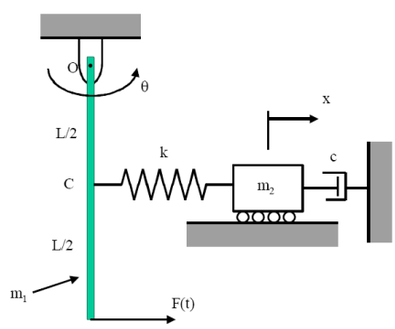
1.) Exercise 4.8 from Ginsberg. (Note: the spring constants are defined such that the frequencies given are the natural frequencies that each spring-mass system would have if it were attached to a rigid base. Notice that there is not a simple relationship between those frequencies and the natural frequencies of the system as a whole.)

2.) Exercise 4.30 as given in the text. Repeat the analysis for $k=2mg/L$ and graph that response as well. (Questions to consider: What do you notice about the natural frequencies of this system in each case? How does that affect the way the response looks? Why?)

3.) Exercise 4.43 from Ginsberg. How does the time required to reach steady state compare with $\tau_r=1/(\zeta_r\omega_r)$ for each mode, r=1,2,3?

4.) Exercise 4.47 from Ginsberg.

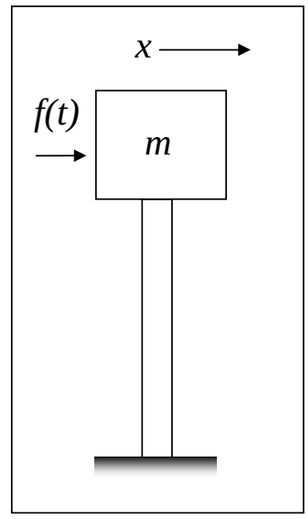
5.) (Covering material from Chapter 5)
A uniform rod of length L and mass m_1 is attached to a cart having mass m_2 by means of a spring k. A viscous damper c resists the motion of the cart.



a.) Let F(t)=Re[Fexp(iωt)], x(t) = Re[Xexp(iωt)] and θ(t)=Re[Yexp(iωt)]. Find analytical expressions for the complex transfer functions X/F and Y/F.
b.) Find the magnitude and phase of the response of x and θ when the system is forced at its natural frequencies $\omega=\omega_1$ and $\omega=\omega_2$. Compare these values to the eigenvectors for modes 1 and 2. Use the following numerical values: $m_1=m_2=1$ kg, k=3 N/m, L=1 m, g=9.81 m/s^2, and c=0.1 N-s/m.

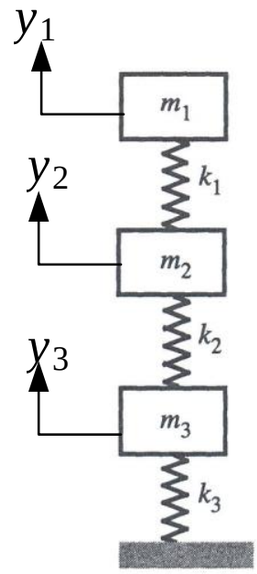
c.) Plot the transfer functions Y/F and X/F over a range of frequencies encompassing both modes of vibration. Use the plot to determine at what frequency m_2 acts as a vibration absorber for the rod. How does that frequency compare with the natural frequency that the system would have if the rod were held fixed: $\omega_{cart} = (k/m_2)^{1/2}$?

6.) Consider **Exercise 3.45 and 3.46** in the text (you solved this in problem #3 in HW#6). Use the steady-state displacement that you computed using FFT techniques for $\tau = 3\pi/\omega_n$ to compute the maximum stress in the spring. Assume that the spring is a cantilever beam (in bending) modeled after one of the pillars supporting the ERB, which have length L=40m, rectangular cross section with height h, equal to the width b=h=0.6m, and is constructed from a material with modulus E=30 GPa and ultimate tensile strength σ=40 MPa. (The mass of the beam is assumed to be included in *m*, so its density is not needed.) Let the mass *m* be such that the natural frequency of the mass-spring system is ω_n=0.2Hz. What is the amplitude of the force, P, such that the beam fails due to the dynamic load? Compare that to the static load required to cause the beam to fail (also in bending).



2.11.2 problem 1

4.8 The system shown in the sketch represents a scale model used to study the vibration of a three-story building. The masses for the model are $m_1 = 100$, $m_2 = 200$, and $m_3 = 300$ kg. The springs are selected such that $k_j = \omega_j^2 m_j$, where $\omega_j = 40, 50$, and 60 rad/s. Determine the characteristic equation, natural frequencies, and mode shapes of this system.



Generalized coordinates are y_3, y_2, y_1. Kinetic energy is $T = \frac{1}{2}m_3\left(y_3'\right)^2 + \frac{1}{2}m_2\left(y_2'\right)^2 + \frac{1}{2}m_1\left(y_1'\right)^2$. Potential energy due to springs is $V_{spring} = \frac{1}{2}k_3y_3^2 + \frac{1}{2}k_2\left(y_2 - y_3\right)^2 + \frac{1}{2}k_1\left(y_1 - y_2\right)^2$. Therefore

$$\begin{aligned} V_{spring} &= \frac{1}{2}k_3y_3^2 + \frac{1}{2}k_2\left(y_2^2 + y_3^2 - 2y_2y_3\right) + \frac{1}{2}k_1\left(y_1^2 + y_2^2 - 2y_1y_2\right) \\ &= y_3^2\left(\frac{1}{2}k_3 + \frac{1}{2}k_2\right) + y_2^2\left(\frac{1}{2}k_2 + \frac{1}{2}k_1\right) + y_1^2\left(\frac{1}{2}k_1\right) + y_1y_2(-k_1) + y_1y_3(0) + y_2y_3(-k_2) \end{aligned}$$

The EOM is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_2 + k_1 & -k_2 \\ 0 & -k_2 & k_3 + k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Following values are for mass (units in kg) $m_1 = 100, m_2 = 200, m_3 = 300$. Following values are for spring constants (units in N/m) $k_1 = 40^2(100), k_2 = 50^2(200), k_3 = 60^2(300)$. EOM becomes

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} + \begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Characteristic equation is

$$\det\left([K] - \omega^2[M]\right) = 0$$

$$\det\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - \omega^2\begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 160000 - 100\omega^2 & -160000 & 0 \\ -160000 & 660000 - 200\omega^2 & -500000 \\ 0 & -500000 & 1580000 - 300\omega^2 \end{bmatrix} = 0$$

$$-6 \times 10^6\omega^6 + 6.1 \times 10^{10}\omega^4 - 1.54 \times 10^{14}\omega^2 + 8.64 \times 10^{16} = 0$$

Positive roots of the above polynomial are the natural frequencies (units in rad/sec).

$$\begin{aligned}\omega_1 &= 28.1\\ \omega_2 &= 52.6\\ \omega_3 &= 81.3\end{aligned}$$

To obtain mode shapes, the eigenvector associated with each eigenvalue is found. Starting with $\omega_1 = 28.1$

$$\left(\begin{bmatrix}160000 & -160000 & 0\\ -160000 & 660000 & -500000\\ 0 & -500000 & 1580000\end{bmatrix} - 28.1^2\begin{bmatrix}100 & 0 & 0\\ 0 & 200 & 0\\ 0 & 0 & 300\end{bmatrix}\right)\begin{bmatrix}1\\ \varphi_{21}\\ \varphi_{31}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence

$$\begin{bmatrix}8.1\times10^4 & -160000 & 0\\ -160000 & 5.02\times10^5 & -500000\\ 0 & -500000 & 1.34\times10^6\end{bmatrix}\begin{bmatrix}1\\ \varphi_{21}\\ \varphi_{31}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}8.1\times10^4 - 1.6\times10^5\varphi_{21}\\ 5.02\times10^5\varphi_{21} - 5.0\times10^5\varphi_{31} - 1.6\times10^5\\ 1.34\times10^6\varphi_{31} - 5\times10^5\varphi_{21}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving gives $\varphi_{21} = 0.506$ and $\varphi_{31} = 0.188$. First eigenvector is

$$\varphi_1 = \begin{bmatrix}1\\0.506\\0.188\end{bmatrix}$$

For $\omega_2 = 52.6$,

$$\left(\begin{bmatrix}160000 & -160000 & 0\\ -160000 & 660000 & -500000\\ 0 & -500000 & 1580000\end{bmatrix} - 52.6^2\begin{bmatrix}100 & 0 & 0\\ 0 & 200 & 0\\ 0 & 0 & 300\end{bmatrix}\right)\begin{bmatrix}1\\ \varphi_{22}\\ \varphi_{32}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Hence

$$\begin{bmatrix}-1.17\times10^5 & -1.6\times10^5 & 0\\ -1.6\times10^5 & 1.07\times10^5 & -5.0\times10^5\\ 0 & -5.0\times10^5 & 7.50\times10^5\end{bmatrix}\begin{bmatrix}1\\ \varphi_{22}\\ \varphi_{32}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1.6\times10^5\varphi_{22} - 1.17\times10^5\\ 1.07\times10^5\varphi_{22} - 5.0\times10^5\varphi_{32} - 1.6\times10^5\\ 7.5\times10^5\varphi_{32} - 5.0\times10^5\varphi_{22}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Solving gives $\varphi_{22} = -0.731$ and $\varphi_{32} = -0.476$ .Second eigenvector is

$$\varphi_2 = \begin{bmatrix}1\\-0.731\\-0.476\end{bmatrix}$$

For $\omega_3 = 81.3$

$$\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - 81.3^2 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right)\begin{bmatrix} 1 \\ \varphi_{23} \\ \varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$\begin{bmatrix} -5.01\times 10^5 & -1.6\times 10^5 & 0 \\ -1.6\times 10^5 & -6.62\times 10^5 & -5.0\times 10^5 \\ 0 & -5.0\times 10^5 & -4.03\times 10^5 \end{bmatrix}\begin{bmatrix} 1 \\ \varphi_{23} \\ \varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1.6\times 10^5\varphi_{23} - 5.01\times 10^5 \\ -6.62\times 10^5\varphi_{23} - 5.0\times 10^5\varphi_{33} - 1.6\times 10^5 \\ -5.0\times 10^5\varphi_{23} - 4.03\times 10^5\varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{23} = -3.13$ and $\varphi_{32} = \varphi_{33} = 3.82$. Third eigenvector is

$$\varphi_3 = \begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix}$$

Eigenvectors are mass normalized. Mass normalization factors μ_i are found for each eigenvector

$$\begin{aligned} \mu_1 &= \varphi_1^T[M]\varphi_1 \\ &= \begin{bmatrix} 1 \\ 0.506 \\ 0.188 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\begin{bmatrix} 1 \\ 0.506 \\ 0.188 \end{bmatrix} = 162. \end{aligned}$$

and

$$\begin{aligned} \mu_2 &= \varphi_2^T[M]\varphi_2 \\ &= \begin{bmatrix} 1 \\ -0.731 \\ -0.476 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\begin{bmatrix} 1 \\ -0.731 \\ -0.476 \end{bmatrix} = 275. \end{aligned}$$

and

$$\begin{aligned} \mu_3 &= \varphi_3^T[M]\varphi_3 \\ &= \begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix} = 6.44\times 10^3 \end{aligned}$$

Normalized eigenvectors are

$$\Phi_1 = \frac{\varphi_1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{162}}\begin{bmatrix}1\\0.506\\0.188\end{bmatrix} = \begin{bmatrix}7.86\times10^{-2}\\3.98\times10^{-2}\\1.48\times10^{-2}\end{bmatrix}$$

$$\Phi_2 = \frac{\varphi_2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{275.}}\begin{bmatrix}1\\-0.731\\-0.476\end{bmatrix} = \begin{bmatrix}6.03\times10^{-2}\\-4.41\times10^{-2}\\-2.87\times10^{-2}\end{bmatrix}$$

$$\Phi_3 = \frac{\varphi_3}{\sqrt{\mu_3}} = \frac{1}{\sqrt{6.44\times10^{3}}}\begin{bmatrix}1\\-3.13\\3.82\end{bmatrix} = \begin{bmatrix}1.25\times10^{-2}\\-0.039\\4.76\times10^{-2}\end{bmatrix}$$

Verification of the above result follows

```
EDU>> k=[160000 -160000 0;-160000 660000 -500000;0 -500000 1580000];
EDU>> M=[100 0 0;0 200 0;0 0 300];
EDU>> [eigV,lam]=eig(k,M)

eigV =
    0.0786    0.0606    0.0124
    0.0398   -0.0437   -0.0389
    0.0148   -0.0289    0.0477

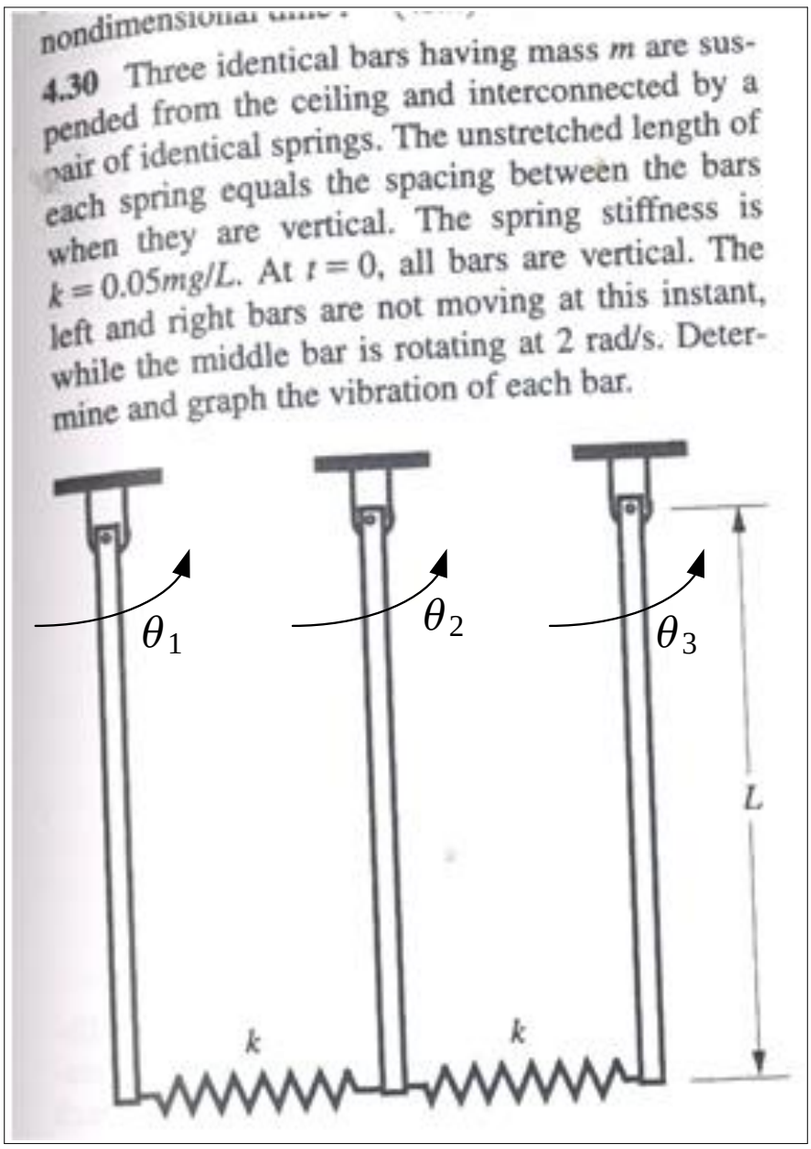
lam =

   1.0e+03 *
    0.7897         0         0
         0    2.7528         0
         0         0    6.6242
EDU>> sqrt(diag(lam))

ans =

   28.1013
   52.4674
   81.3889
```

2.11.3 Problem 2



nondimensional time.

4.30 Three identical bars having mass m are suspended from the ceiling and interconnected by a pair of identical springs. The unstretched length of each spring equals the spacing between the bars when they are vertical. The spring stiffness is $k = 0.05mg/L$. At $t = 0$, all bars are vertical. The left and right bars are not moving at this instant, while the middle bar is rotating at 2 rad/s. Determine and graph the vibration of each bar.

Initial conditions are $\theta_i(0) = 0$ for $i = 1, 2, 3$ and $\theta_1'(0) = \theta_3'(0) = 0$ but $\theta_2'(0) = 2$ rad/sec. The generalized coordinates are shown above. kinetic energy is

$$T = \frac{1}{2}I\left(\theta_1'\right)^2 + \frac{1}{2}I\left(\theta_2'\right)^2 + \frac{1}{2}I\left(\theta_3'\right)^2$$

where $I = \frac{1}{3}mL^2$. Mas matrix becomes

$$[M] = \frac{1}{3}mL^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta_1''\\ \theta_2''\\ \theta_3''\end{bmatrix}$$

Spring potential energy is

$$\begin{aligned}
V_{spring} &= \frac{1}{2}k(L\theta_2 - L\theta_1)^2 + \frac{1}{2}k(L\theta_3 - L\theta_2)^2\\
&= \frac{1}{2}kL^2\left(\theta_2^2 + \theta_1^2 - 2\theta_1\theta_2\right) + \frac{1}{2}kL^2\left(\theta_3^2 + \theta_2^2 - 2\theta_2\theta_3\right)\\
&= \theta_1^2\left(\frac{1}{2}kL^2\right) + \theta_2^2\left(\frac{1}{2}kL^2 + \frac{1}{2}kL^2\right) + \theta_3^2\left(\frac{1}{2}kL^2\right) + \theta_1\theta_2\left(-kL^2\right) + \theta_1\theta_3(0) + \theta_2\theta_3\left(-kL^2\right)
\end{aligned}$$

Hence stiffness matrix due to spring is

$$[K]_{spring} = kL^2\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

Assume the zero PE for gravity is taken as the top of the bar. Stiffness due to gravity is

$$V_g = -mg\frac{L}{2}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$$

$V_{11} = \frac{\partial^2 V_g}{\partial\theta_1^2} = mg\frac{L}{2}(\cos\theta_1)$. Evaluate this at static position $\theta_1 = 0$,hence $V_{11} = m\frac{L}{2}$. Similarly, $V_{22} = V_{33} = m\frac{L}{2}$. All other terms are zero.

Hence stiffness matrix due to gravity is

$$[K]_g = mg\frac{L}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Therefore, complete stiffness matrix is

$$kL^2\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} + mg\frac{L}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

There are no generalized forces. Hence EOM is

$$\frac{1}{3}mL^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta_1''\\ \theta_2''\\ \theta_3''\end{bmatrix} + \left(kL^2\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} + mg\frac{L}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

2.11.3.1 Part (a) $k = 0.05\frac{mg}{L}$

For case $k = 0.05\frac{mg}{L}$, Hence for $\sigma = 0.05$ then $k = \sigma\frac{mg}{L}$. EOM becomes

$$\frac{1}{3}mL^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta_1''\\ \theta_2''\\ \theta_3''\end{bmatrix} + \left(\sigma mgL\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix} + mg\frac{L}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\frac{1}{3}mL^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta_1''\\ \theta_2''\\ \theta_3''\end{bmatrix} + mgL\begin{bmatrix}\frac{1}{2}+\sigma & -\sigma & 0\\ -\sigma & \frac{1}{2}+2\sigma & -\sigma\\ 0 & -\sigma & \frac{1}{2}+\sigma\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\theta_1''\\ \theta_2''\\ \theta_3''\end{bmatrix} + \frac{3g}{L}\begin{bmatrix}\frac{1}{2}+\sigma & -\sigma & 0\\ -\sigma & \frac{1}{2}+2\sigma & -\sigma\\ 0 & -\sigma & \frac{1}{2}+\sigma\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Let $L = 1, g = 10$. The above becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + \begin{bmatrix} 15+30\sigma & -30\sigma & 0 \\ -30\sigma & 15+60\sigma & -30\sigma \\ 0 & -30\sigma & 15+30\sigma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Natural frequencies of the system are found by solving the eigenvalue problem.

$$\det\left(\begin{bmatrix} 15+30\sigma & -30\sigma & 0 \\ -30\sigma & 15+60\sigma & -30\sigma \\ 0 & -30\sigma & 15+30\sigma \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Substituting $\sigma = 0.05$ gives

$$\det\left(\begin{bmatrix} 16.5 & -1.5 & 0 \\ -1.5 & 18.0 & -1.5 \\ 0 & -1.5 & 16.5 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} 16.5-\omega^2 & -1.5 & 0 \\ -1.5 & 18-\omega^2 & -1.5 \\ 0 & -1.5 & 16.5-\omega^2 \end{bmatrix} = 0$$

$$-\omega^6 + 51\omega^4 - 861.75\omega^2 + 4826.3 = 0$$

Positive roots of this polynomial are $\omega = 3.87, \omega = 4.062, \omega = 4.416$.

Associated eigenvectors are found by solving for φ_i in $\big([K] - \omega^2[M]\big)\varphi_i = 0$ for each eigenvalue ω_i.

For $\omega_1 = 3.87$

$$\begin{bmatrix} 16.5-\omega_1^2 & -1.5 & 0 \\ -1.5 & 18-\omega_1^2 & -1.5 \\ 0 & -1.5 & 16.5-\omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16.5-3.87^2 & -1.5 & 0 \\ -1.5 & 18-3.87^2 & -1.5 \\ 0 & -1.5 & 16.5-3.87^2 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.523 & -1.5 & 0 \\ -1.5 & 3.023 & -1.5 \\ 0 & -1.5 & 1.523 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.523 - 1.5\varphi_{21} \\ 3.023\,\varphi_{21} - 1.5\varphi_{31} - 1.5 \\ 1.523\,\varphi_{31} - 1.5\varphi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{21} = 1.015\,3$ and $\varphi_{31} = 1.046\,2$. First eigenvector is

$$\varphi_1 = \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.0153 \\ 1.0462 \end{bmatrix}$$

Similarly, second and the third eigenvectors are found.

Eigenvectors are mass normalized. First the mass normalization factors μ_i are found for each eigenvector

$$\begin{aligned}\mu_1 &= \varphi_1^T[M]\varphi_1\\ &= \begin{bmatrix}1\\1.0153\\1.0462\end{bmatrix}^T\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1.0153\\1.0462\end{bmatrix} = 3.1254\end{aligned}$$

Normalized eigenvector is

$$\Phi_1 = \frac{\varphi_1}{\sqrt{3.1254}} = \frac{1}{\sqrt{3.792}}\begin{bmatrix}1\\1.0153\\1.0462\end{bmatrix} = \begin{bmatrix}0.51353\\0.52139\\0.53726\end{bmatrix}$$

Verification of the above result (Matlab result is more accurate due to more accurate method used)

```
EDU>>  k=[0.55 -0.05 0;-0.05 0.6 -0.05;0 -0.05 0.55];
EDU>> M=eye(3);
EDU>> [eigV,lam]=eig(k,M)

eigV =

   -0.5774   -0.7071    0.4082
   -0.5774   -0.0000   -0.8165
   -0.5774    0.7071    0.4082

EDU>> sqrt(diag(lam))

ans =

    0.7071
    0.7416
    0.8062
```

Transformation matrix (based on Matlab more accurate result) is

$$\Phi = [\Phi_1\Phi_2\Phi_3] = \begin{bmatrix}-0.577 & -0.7073 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.7069 & 0.4082\end{bmatrix}$$

Mapping from physical coordinates θ to modal coordinates η is

$$= [\Phi]$$

Bold face is used to indicate a column vector. EOM's are written in modal coordinates resulting in

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\eta_1''\\\eta_2''\\\eta_3''\end{bmatrix} + \begin{bmatrix}\omega_1^2&0&0\\0&\omega_2^2&0\\0&0&\omega_2^2\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\\\eta_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\eta_1''\\\eta_2''\\\eta_3''\end{bmatrix} + \begin{bmatrix}0.7071^2&0&0\\0&0.7416^2&0\\0&0&0.8062^2\end{bmatrix}\begin{bmatrix}\eta_1\\\eta_2\\\eta_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Initial conditions are transformed to modal coordinates using $(0) = [\Phi]^T[M](0)$ and $'(0) = [\Phi]^T[M]'(0)$, since $(0) = 0$ then $(0) = 0$, however $'(0)$ is not all zero, hence

$$\begin{bmatrix}\eta_1'(0)\\ \eta_2'(0)\\ \eta_3'(0)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7073 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.7069 & 0.4082\end{bmatrix}^T \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\\ 0\end{bmatrix}$$
$$= \begin{bmatrix}-1.154\\ 0\\ -1.633\end{bmatrix}$$

Initial conditions in modal coordinates are found. The solution can be found. The solution to $\eta'' + \omega^2\eta = 0$ with initial conditions $\eta(0)$ and $\eta'(0)$ is $\eta(t) = \eta(0)\cos\omega t + \frac{\eta'(0)}{\omega}\sin\omega t$. Therefore modal solutions are

$$\begin{aligned}\eta_1(t) &= \frac{-1.154}{0.7071}\sin(0.7071t) = -1.632\sin(0.707\,t)\\ \eta_2(t) &= 0\\ \eta_3(t) &= \frac{-1.633}{0.8062}\sin(0.8062t) = -2.026\sin(0.8062t)\end{aligned}$$

Solution in the normal coordinates is

$$\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\\ \theta_3(t)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7073 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.706\,9 & 0.4082\end{bmatrix}\begin{bmatrix}-1.632\sin(0.707\,t)\\ 0\\ -2.026\sin(0.8062t)\end{bmatrix}$$
$$= \begin{bmatrix}0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\\ 0.94166\sin(0.707\,t) + 1.6542\sin(0.8062t)\\ 0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\end{bmatrix}$$

2.11.3.2 Part (b) $k = 2\frac{mg}{L}$

Using part (a), but with $\sigma = 2$ results in

$$\det\left(\begin{bmatrix}15+30\sigma & -30\sigma & 0\\ -30\sigma & 15+60\sigma & -30\sigma\\ 0 & -30\sigma & 15+30\sigma\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}15+30(2) & -30(2) & 0\\ -30(2) & 15+60(2) & -30(2)\\ 0 & -30(2) & 15+30(2)\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix}75.0-\omega^2 & -60.0 & 0\\ -60.0 & 135.0-\omega^2 & -60.0\\ 0 & -60.0 & 75.0-\omega^2\end{bmatrix} = 0$$

Similar steps as repeated as part (a) above. The final result are shown below using Matlab

```
EDU>> k=[75 -60 0;-60 135 -60;0 -60 75]
EDU>> M=eye(3);
[eigV,lam]=eig(k,M)

eigV =

   -0.5774   -0.7071    0.4082
   -0.5774    0.0000   -0.8165
   -0.5774    0.7071    0.4082
EDU>> sqrt(diag(lam))

    3.8730
    8.6603
   13.9642
```

Transformation matrix is $\Phi = [\Phi_1 \Phi_2 \Phi_3] = \begin{bmatrix} -0.577 & -0.7071 & 0.4082 \\ -0.577 & 0 & -0.816\,5 \\ -0.577 & 0.7071 & 0.4082 \end{bmatrix}$. Mapping from θ to modal coordinates η is

$$= [\Phi]$$

Bold face is used to indicate a column vector. EOM's are written in modal coordinates resulting in

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} 3.8730^2 & 0 & 0 \\ 0 & 8.6603^2 & 0 \\ 0 & 0 & 13.9642^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Initial conditions are transformed to modal coordinates using $(0) = [\Phi]^T[M]\mathbf{x}(0)$ and $'(0) = [\Phi]^T[M]\theta'(0)$, since $\theta(0) = 0$ then $(0) = 0$, however $\theta'(0)$ is not all zero. Similar to part (a), initial conditions are found

$$\begin{bmatrix} \eta_1'(0) \\ \eta_2'(0) \\ \eta_3'(0) \end{bmatrix} = \begin{bmatrix} -1.154 \\ 0 \\ -1.633 \end{bmatrix}$$

The solution to $\eta'' + \lambda^2\eta = 0$ with initial conditions $\eta(0)$ and $\eta'(0)$ is given by $\eta(t) = \eta(0)\cos\lambda t + \frac{\eta'(0)}{\lambda}\sin\lambda t$. The solutions are

$$\begin{aligned} \eta_1(t) &= \frac{-1.154}{3.8730}\sin(3.873t) = -0.297\,96\sin(3.873\,t) \\ \eta_2(t) &= 0 \\ \eta_3(t) &= \frac{-1.633}{13.9642}\sin(13.9642) = -0.116\,94\sin(13.9642) \end{aligned}$$

Solution in the physical coordinates is

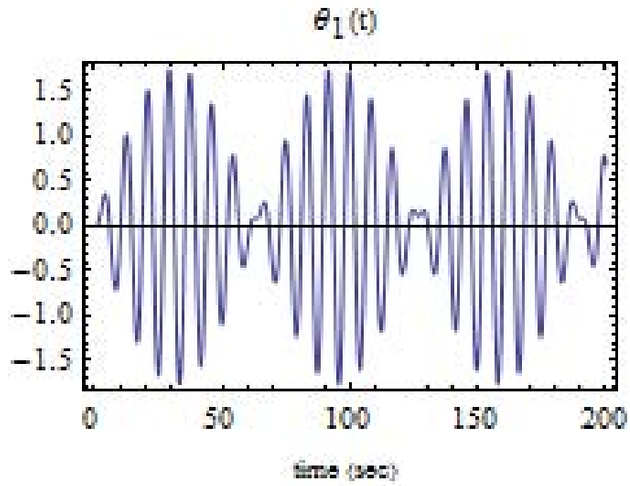
$$\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\\ \theta_3(t)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7071 & 0.4082\\ -0.577 & 0 & -0.816\,5\\ -0.577 & 0.7071 & 0.4082\end{bmatrix}\begin{bmatrix}-0.297\,96\sin(3.8730\,t)\\ 0\\ -0.116\,94\sin(13.9642t)\end{bmatrix}$$
$$= \begin{bmatrix}0.171\,92\sin(3.873t) - 4.773\,5\times10^{-2}\sin(13.964t)\\ 9.548\,2\times10^{-2}\sin(13.964t) + 0.171\,92\sin(3.873t)\\ 0.171\,92\sin(3.873t) - 4.773\,5\times10^{-2}\sin(13.964t)\end{bmatrix}$$

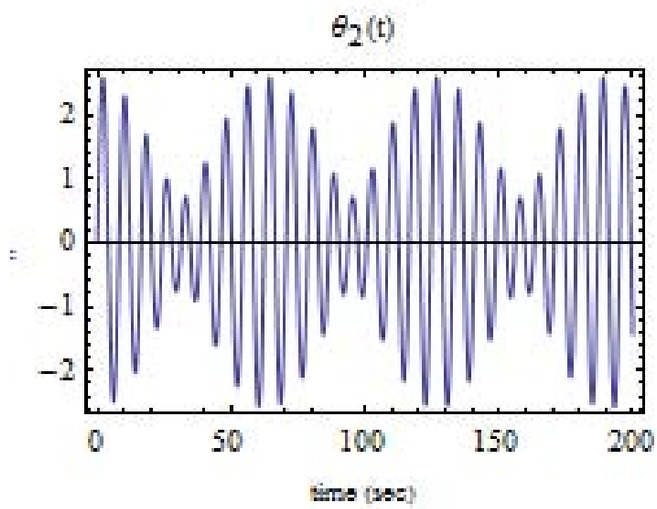
Summary table

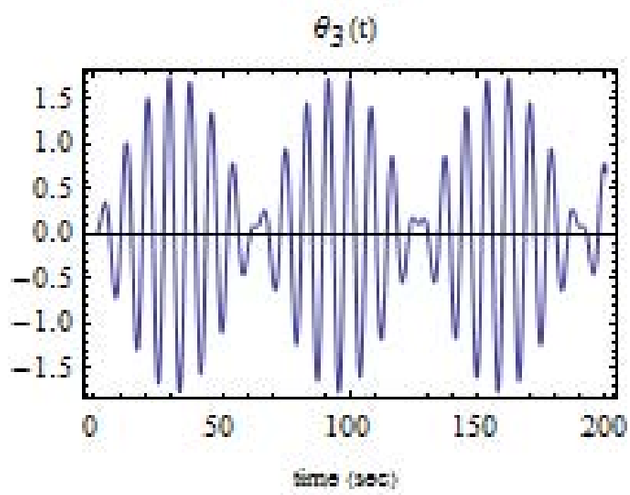
k	frequencies	$[\Phi]$	solutions in θ
$0.05\frac{mg}{L}$	$\begin{Bmatrix}0.7071\\ 0.7416\\ 0.8062\end{Bmatrix}$	$\begin{bmatrix}-0.5774 & -0.7071 & 0.4082\\ -0.5774 & 0 & -0.8165\\ -0.5774 & 0.7071 & 0.4082\end{bmatrix}$	$\begin{bmatrix}0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\\ 0.94166\sin(0.707\,t) + 1.6542\sin(0.8062t)\\ 0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\end{bmatrix}$
$2\frac{mg}{L}$	$\begin{Bmatrix}3.8730\\ 8.6603\\ 13.9642\end{Bmatrix}$	$\begin{bmatrix}-0.5774 & -0.7071 & 0.4082\\ -0.5774 & 0 & -0.8165\\ -0.5774 & 0.7071 & 0.4082\end{bmatrix}$	$\begin{bmatrix}0.17192\sin(3.873t) - 4.773\,5\times10^{-2}\sin(13.964t)\\ 9.5482\times10^{-2}\sin(13.964t) + 0.171\,92\sin(3.873t)\\ 0.17192\sin(3.873t) - 4.773\,5\times10^{-2}\sin(13.964t)\end{bmatrix}$

Even though the normalized natural frequencies are different, the shape functions are the same.

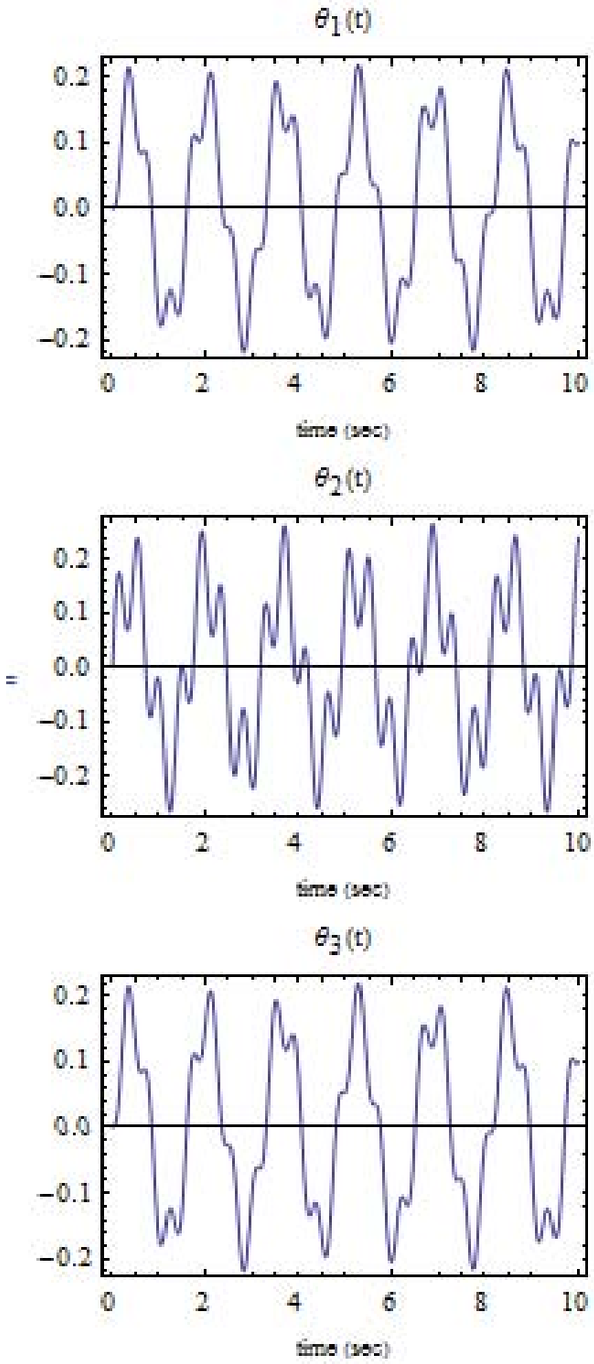
Plots of the solutions of $\theta_i(t)$ for both cases are made. For the case of $k = 0.05\frac{mg}{L}$







For $k = 2\frac{mg}{L}$



In addition, a small program is written to animate both the full solution and the modal solutions. The program to animate the full solution is at `http://12000.org/my_courses/univ_wisconson_madison/spring_2013/EMA_545_Mechanical_Vibrations/HWs/HW10/HW10p2.m.txt` while the program that animate the modal solution is number 112 at bottom of this page `http://12000.org/my_notes/my_matlab_functions/index.htm`

2.11.4 Problem 3

4.43 The following properties are known for a certain three-degree-of-freedom system:

$$[M] = \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \text{kg},$$

$$[K] = \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \text{kN/m},$$

$$[C] = \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} \text{N-s/m},$$

$$\{Q\} = \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix} \text{N}$$

The system was initially at rest at its static equilibrium position. Use the light damping approximation to determine the response. Graph each generalized coordinate as a function of time. From that result, estimate the time required to attain the steady-state condition.

EOM is

$$\begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{Bmatrix} + \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix} + 10^3 \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix}$$

Initial conditions are $\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

Solve the eigenvalue problem to determine the natural frequencies of the system

$$\det\left([K] - \omega^2[M]\right) = 0$$

$$\det\left(\begin{bmatrix} 300\times10^3 & 0 & -200\times10^3 \\ 0 & 500\times10^3 & 300\times10^3 \\ -200\times10^3 & 300\times10^3 & 700\times10^3 \end{bmatrix} - \omega^2 \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix}\right) = 0$$

$$-4.0\times10^8\omega^6 + 1.044\times10^{12}\omega^4 - 4.72\times10^{14}\omega^2 + 5.8\times10^{16} = 0$$

Positive roots are $\{\omega = 15.052, \omega = 17.562, \omega = 45.552\}$. For each natural frequency the corresponding eigenvector is found. A program is now used to compute these values.

```
EDU>> k = [300 0 -200;0 500 300;-200 300 700]*10^3;
M = [600 400 200;400 1200 0;200 0 800];
C = [500 300 -400;300 900 600;-400 600 1300];
[PHI,lam] = eig(k,M);
PHI
lam   = sqrt(diag(lam))
CC    = PHI'*C*PHI;
zeta1 = CC(1,1)/(2*lam(1))
zeta2 = CC(2,2)/(2*lam(2))
zeta3 = CC(3,3)/(2*lam(3))

PHI =
   -0.0216    0.0232   -0.0373
    0.0203    0.0168    0.0201
   -0.0220    0.0023    0.0302

lam =
   15.0519
   17.5624
   45.5522


zeta1 =
    0.0018

zeta2 =
    0.0219

zeta3 =
    0.0376
```

$[\Phi] = \begin{bmatrix} -0.0216 & 0.0232 & -0.0373 \\ 0.0203 & 0.0168 & 0.0201 \\ -0.0220 & 0.0023 & 0.0302 \end{bmatrix}$. In modal coordinates EOM is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + [\Phi]^T \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} 5.419\times10^{-2} & 5.331\times10^{-2} & -0.416 \\ 5.331\times10^{-2} & 0.768 & -3.52\times10^{-4} \\ -0.4156 & -3.52\times10^{-4} & 3.428 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix}$$

In the above $2\zeta_1\omega_1 = 0.0542$, $2\zeta_2\omega_2 = 0.7676$ and $2\zeta_3\omega_3 = 3.4247$. Hence $\zeta_1 = \frac{5.4193\times10^{-2}}{2(15.0519)} = 0.0018$ and $\zeta_2 = \frac{0.76755}{2(17.5624)} = 0.0219$ and $\zeta_3 = \frac{3.4247}{2(45.5522)} = 0.0376$

Final EOM in modal coordinates is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\eta_1''\\\eta_2''\\\eta_3''\end{Bmatrix}+\begin{bmatrix}0.0542&0&0\\0&0.768&0\\0&0&3.425\end{bmatrix}\begin{Bmatrix}\eta_1'\\\eta_2'\\\eta_3'\end{Bmatrix}+\begin{bmatrix}226.56&0&0\\0&308.44&0\\0&0&2075\end{bmatrix}\begin{Bmatrix}\eta_1\\\eta_2\\\eta_3\end{Bmatrix}=\begin{Bmatrix}-4.32\cos(16.0t)\\4.64\cos(16.0t)\\-7.46\cos(16.0t)\end{Bmatrix}$$

EOM's to solve are

$$\begin{aligned}\eta_1''+2\zeta_1\omega_1\eta_1'+\omega_1^2\eta_1&=-4.32\cos(16.0t)\\\eta_2''+2\zeta_2\omega_2\eta_2'+\omega_2^2\eta_2&=4.64\cos(16.0t)\\\eta_3''+2\zeta_3\omega_3\eta_3'+\omega_3^2\eta_3&=-7.46\cos(16.0t)\end{aligned}$$

Initial conditions are zero. The solution in modal coordinates is given in appendix B for underdamped case. Complete solution for the case of underdamped is given in appendix B as

$$\eta(t)=\frac{F_0}{\beta^2+4\zeta^2\omega^2\varpi^2}\left\{\beta\cos(\varpi t)+2\zeta\omega\varpi\sin(\varpi t)-e^{-\zeta\omega t}\left[\beta\cos(\omega_d t)+\frac{\zeta\omega\beta}{\omega_d}\sin(\omega_d t)\right]\right\}h(t)$$

$\beta=\left(\omega^2-\varpi^2\right)$, $\omega_d=\omega\sqrt{1-\zeta^2}$.

The solutions in modal coordinates are now found. Recall that $\omega_1=15.0519,\omega_2=17.5624,\omega_3=45.5522$ and $\zeta_1=0.0018,\zeta_2=0.0219$ and $\zeta_3=0.0376$

Next step is to transform the solution to the physical coordinates using $q_j=\sum_{m=1}^{3}\Phi(j,m)\eta(m)$, or

$$\mathbf{q}=[\Phi]$$

In component form

$$\begin{aligned}q_1(t)&=\Phi(1,1)\eta_1(t)+\Phi(1,2)\eta_2(t)+\Phi(1,3)\eta_3(t)\\q_2(t)&=\Phi(2,1)\eta_1(t)+\Phi(2,2)\eta_2(t)+\Phi(2,3)\eta_3(t)\\q_3(t)&=\Phi(3,1)\eta_1(t)+\Phi(3,2)\eta_2(t)+\Phi(3,3)\eta_3(t)\end{aligned}$$

Program was written to complete the computation and make plots. Here is the result showing plots of each of the above $q_i(t)$ vs. time

```
function nma_HW10_problem_3_EMA_545()
%solve for q(t) using modal analysis, by Nasser M. Abbasi
close all;

syms t;
N = 3;
k = [300 0 -200;0 500 300;-200 300 700]*10^3;
M = [600 400 200;400 1200 0;200 0 800];
C = [500 300 -400;300 900 600;-400 600 1300];
wF = 16;
F = [200*cos(wF*t); 0; 0];

[PHI,lam] = eig(k,M);
lam   = sqrt(diag(lam));
CC    = PHI'*C*PHI

F = PHI.'*F;
eta = sym(zeros(N, 1));
time_constant = zeros(3,1);

for i=1:N
```

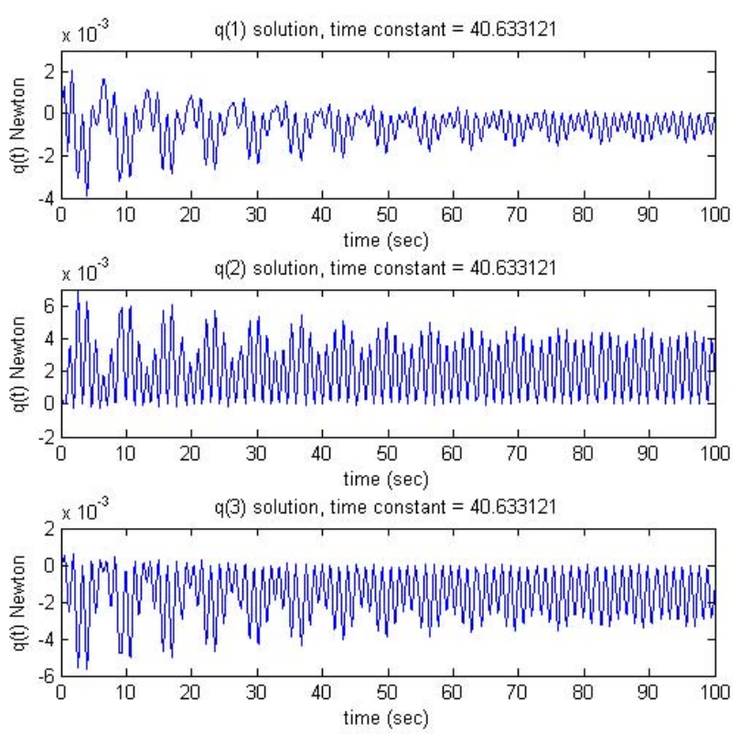
```
    w    = lam(i);
    b    = w^2-wF^2;
    zeta = CC(i,i)/(2*w);
    wd   = w*sqrt(1-zeta^2);
    eta(i) = F(i)/(b^2+4*zeta^2*w^2*wF^2) * ...
        ( b*cos(wF*t)+2*zeta*w*wF*sin(wF*t)- ...
        exp(-zeta*w*t)* (  b*cos(wd*t)+ zeta*w*b/wd * sin(wd*t)  ) ...
        );
    time_constant(i) = 1/(zeta*w);
end

q=PHI*eta;
time_constant
time_constant = sum(time_constant);

% plot the generalized solutions
lims= [-0.004 0.003;
      -0.002  0.007;
      -0.006  0.002
      ];

for i=1:N
    subplot(3,1,i);
    ezplot(q(i),[0,100]);
    ylim(lims(i,:));
    title(sprintf('q(%d) solution, time constant = %f',i,time_constant));
    xlabel('time (sec)');
    ylabel('q(t) Newton');
end

end
```



From above, the time to reach steady state is about 90 seconds based on looking at $q_1(t)$ since that takes the longest time to each steady state out of the three coordinates.

The time constant for each $\eta_i(t)$ solution was calculated giving $\tau_1 = \frac{1}{\zeta_1 \omega_1} = 37.4471$ and

$\tau_2 = 2.602$ and $\tau_3 = 0.58$. The first time constant $\tau_1 = 37.4$ seconds dominated the result in the response in the physical coordinates.

This means the dominant time constant found in modal analysis is one to use to estimate how long it will take for the response in physical coordinates to reach steady state. Each modal solution contributes to each physical solution. The one with the longest time constant affects more than any other mode how long the physical solution takes to reach steady state.

2.11.5 Problem 4

4.47 The mass matrix, natural frequencies, and unnormalized vibration modes for a two-degree-of-freedom system are

$$[M] = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \text{kg}$$

$$\omega_1 = 15.68 \text{ rad/s}, \qquad \omega_2 = 40.78 \text{ rad/s}$$

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}, \qquad \{\phi_2\} = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

The modal damping ratios are estimated to be $\zeta_1 = \zeta_2 = 0.08$. The system is subjected to a harmonic excitation for which the generalized forces are $Q_1 = 50 \sin(20t)$, $Q_2 = 100 \cos(20t)$ N. Determine the steady-state response of the generalized coordinates. Express the result in the form $q_n = \text{Re}[Y_n \exp(i20t)]$ and give the values of Y_1 and Y_2.

$$[M] = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \text{kg}, \omega_1 = 15.68 \text{ rad/sec}, \omega_2 = 40.78 \text{ rad/sec.}\ {}_1 = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}, {}_2 = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

$$\mu_1 = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}^T \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix} = 4.2678$$

$$\mu_2 = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}^T \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = 7.7318$$

Normalized eigenvectors are

$${}_1 = \frac{1}{\sqrt{\mu_1}} {}_1 = \frac{1}{\sqrt{4.2678}} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = \begin{Bmatrix} 0.48406 \\ -0.17717 \end{Bmatrix}$$

$${}_2 = \frac{1}{\sqrt{\mu_2}} {}_2 = \frac{1}{\sqrt{7.7318}} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = \begin{Bmatrix} 0.35963 \\ -0.13163 \end{Bmatrix}$$

Hence

$$[\Phi] = [_{12}] = \begin{bmatrix} 0.484\,06 & 0.359\,63 \\ -0.177\,17 & -0.131\,63 \end{bmatrix}$$

EOM in modal coordinates is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 2(0.08)(15.68) & 0 \\ 0 & 2(0.08)(40.78) \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 15.68^2 & 0 \\ 0 & 40.78^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 50\sin(20t) \\ 100\cos(20t) \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 2.509 & 0 \\ 0 & 6.524\,8 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 245.86 & 0 \\ 0 & 1663 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 24.203\sin(20.0t) - 17.717\cos \\ 17.982\sin(20.0t) - 13.163\cos \end{bmatrix}$$

The two EOMs to solve are

$$\eta_1''(t) + 2.509\eta_1'(t) + 245.86\eta_1(t) = 24.203\sin(20t) - 17.717\cos(20t) = \mathrm{Re}\left\{\frac{24.203}{i}e^{i20t}\right\} + \mathrm{Re}\left\{-17.717e^{i20t}\right\}$$

$$\eta_2''(t) + 6.525\eta_2'(t) + 1663\eta_2(t) = 17.982\sin(20t) - 13.163\cos(20t) = \mathrm{Re}\left\{\frac{17.982}{i}e^{i20t}\right\} + \mathrm{Re}\left\{-13.163e^{i20t}\right\}$$

Hence

$$\eta_1''(t) + 2.509\eta_1'(t) + 245.86\eta_1(t) = 24.203\sin(20t) - 17.717\cos(20t) = \mathrm{Re}\left\{(-24.203i - 17.717)e^{i20t}\right\}$$

$$\eta_2''(t) + 6.525\eta_2'(t) + 1663\eta_2(t) = 17.982\sin(20t) - 13.163\cos(20t) = \mathrm{Re}\left\{(-17.982i - 13.163)e^{i20t}\right\}$$

In matrix form

$$[I]'' + [C]' + [K] = \mathrm{Re}\left\{\mathbf{F}e^{i\varpi t}\right\}$$

Where $\varpi = 20$ rad/sec. $\mathbf{F}$ is the complex amplitude of the input

$$\mathbf{F} = \begin{Bmatrix} -24.203i - 17.717 \\ -17.982i - 13.163 \end{Bmatrix}$$

Using method of transfer functions (since steady state response is needed), response is

$$= \mathrm{Re}\left\{\mathbf{X}e^{i20t}\right\}$$

Where

$$X_j = \frac{F_j}{-\varpi^2 + 2i\zeta_j\omega_j\varpi + \omega_j^2}$$

Steady state solutions in modal coordinates is

$$\begin{aligned}
\eta_1(t) &= \mathrm{Re}\left\{\frac{-24.203i - 17.717}{-\varpi^2 + 2.5088i\varpi + 245.86}e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left\{\frac{-24.203i - 17.717}{-400 + 50.176i + 245.86}e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left\{\left(5.77\times10^{-2} + 0.176i\right)e^{i\varpi t}\right\} \\
\eta_2(t) &= \mathrm{Re}\left\{\frac{-17.982i - 13.163}{-\varpi^2 + 6.525i\varpi + 1663}e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left\{\frac{-17.982i - 13.163}{-400 + 130.5i + 1663}e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left\{\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)e^{i\varpi t}\right\}
\end{aligned}$$

Solutions are transformed back to normal coordinates

$$\mathbf{q} = [\Phi]$$

Hence

$$\begin{aligned} q_j(t) &= \sum_n^2 \Phi(j,n)\eta(n) \\ &= \sum_n^2 \Phi(j,n)\,\mathrm{Re}\left\{X(n)e^{i\omega t}\right\} \\ &= \mathrm{Re}\sum_n^2 \Phi(j,n)X(n)e^{i\omega t} \end{aligned}$$

Since$[\Phi] = \begin{bmatrix} 0.484\,06 & 0.359\,63 \\ -0.177\,17 & -0.131\,63 \end{bmatrix}$ then

$$q_1(t) = \mathrm{Re}\left(\left\{0.484\,06\left(5.77\times10^{-2} + 0.176i\right) + 0.359\,63\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)\right\}e^{i20t}\right)$$
$$q_2(t) = \mathrm{Re}\left(\left\{-0.177\,17\left(5.77\times10^{-2} + 0.176i\right) - 0.131\,63\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)\right\}e^{i20t}\right)$$

or

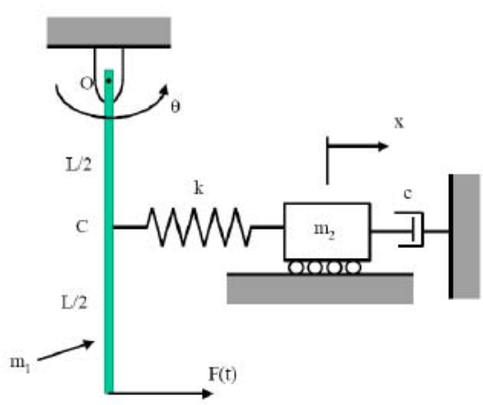
$$\begin{aligned} q_1(t) &= \mathrm{Re}\left(\left\{2.369\times10^{-2} + 8.051\times10^{-2}i\right\}e^{i20t}\right) \\ q_2(t) &= \mathrm{Re}\left(\left\{-8.672\times10^{-3} - 2.947\times10^{-2}i\right\}e^{i20t}\right) \end{aligned}$$

Therefore

$$\begin{aligned} Y_1 &= 2.369\times10^{-2} + 8.051\times10^{-2}i \\ Y_2 &= -8.672\times10^{-3} - 2.947\times10^{-2}i \end{aligned}$$

sectionProblem 5

5.) (Covering material from Chapter 5)
A uniform rod of length L and mass m_1 is attached to a cart having mass m_2 by means of a spring k. A viscous damper c resists the motion of the cart.



a.) Let F(t)=Re[Fexp(iωt)], x(t) = Re[Xexp(iωt)] and θ(t)=Re[Yexp(iωt)]. Find analytical expressions for the complex transfer functions X/F and Y/F.
b.) Find the magnitude and phase of the response of x and θ when the system is forced at its natural frequencies $\omega=\omega_1$ and $\omega=\omega_2$. Compare these values to the eigenvectors for modes 1 and 2. Use the following numerical values: $m_1=m_2=1$ kg, k=3 N/m, L=1 m, g=9.81 m/s^2, and c=0.1 N-s/m.

c.) Plot the transfer functions Y/F and X/F over a range of frequencies encompassing both modes of vibration. Use the plot to determine at what frequency m_2 acts as a vibration absorber for the rod. How does that frequency compare with the natural frequency that the system would have if the rod were held fixed: $\omega_{cart} = (k/m_2)^{1/2}$?

2.11.5.1 Part(a)

First step is to determine EOM. The kinetic energy T is

$$T = \frac{1}{2}I(\theta')^2 + \frac{1}{2}m_2(x')^2$$

$I = \frac{1}{3}m_1L^2$.Assuming small angle, stiff spring approximation and zero gravity datum at the level where pendulum is hinged, spring potential energy V is

$$\begin{aligned} V &= \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2 \\ &= \frac{1}{2}k\left(x^2 + \frac{L^2}{4}\theta^2 - xL\theta\right) \\ &= \theta^2\left(\frac{L^2}{8}k\right) + x^2\left(\frac{1}{2}k\right) + x\theta\left(-\frac{kL}{2}\right) \end{aligned}$$

Stiffness matrix due to spring is

$$K_{spring} = \begin{bmatrix} \frac{L^2}{4}k & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}$$

Potential energy due to gravity is $V_g = -mg\frac{L}{2}\cos\theta$. Hence $V_{g_{11}} = \frac{\partial^2 V_g}{\partial^2\theta} = \left(mg\frac{L}{2}\cos\theta\right)_{\theta=0} = mg\frac{L}{2}$. All other terms are zero. The stiffness matrix due to gravity is

$$K_{spring} = \begin{bmatrix} mg\frac{L}{2} & 0 \\ 0 & 0 \end{bmatrix}$$

Combined stiffness matrix is

$$K = \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}$$

EOM is

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} Q_\theta \\ Q_x \end{Bmatrix}$$

Generalized forces are now found. $Q_\theta = FL$ since F is only external forces acting on the first d.o.f. θ and the work done by this force is $FL\delta\theta$ for small virtual angle. For Q_x work is done only by damper and acts to remove energy, hence negative in sign. $Q_x = -cx'$. The above becomes

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} FL \\ -cx' \end{Bmatrix}$$

Rearranging

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{Bmatrix} \theta' \\ x' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + m_2g\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} FL \\ 0 \end{Bmatrix}$$

Each EOM is

$$I\theta'' + \left(\frac{L^2}{4}k + m_1 g\frac{L}{2}\right)\theta - \frac{kL}{2}x = FL$$
$$m_2 x'' + cx' - \frac{kL}{2}\theta + kx = 0$$

Units checking: First EOM. each term must have units of torque. $\frac{L^2}{4}k\theta$ have units of torque OK. $mg\frac{L}{2}\theta$ have units of torque OK. kLx have units of torque OK.

second EOM Each term must have units of force. cx' have units of force OK. $kL\theta$ have units of force, OK. kx have units of force, OK.

Transfer function is now found Let $x = \text{Re}\left\{Xe^{i\omega t}\right\}, \theta = \text{Re}\left\{Ye^{i\omega t}\right\}, F = \text{Re}\left\{\hat{F}e^{i\omega t}\right\}$.Substitute in the above EOM

$$\text{Re}\left\{\left[\left(-I\omega^2 Y\right) + \left(\frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}X\right]e^{i\omega t}\right\} = \text{Re}\left\{\hat{F}Le^{i\omega t}\right\}$$
$$\text{Re}\left\{\left[-m_2\omega^2 X + ic\omega X - \frac{kL}{2}Y + kX\right]e^{i\omega t}\right\} = 0$$

Simplify

$$\left(-I\omega^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}X = \hat{F}L \tag{2.107}$$
$$\left(-m_2\omega^2 + ic\omega + k\right)X = \frac{kL}{2}Y \tag{2.108}$$

The above two equations are solved to obtain the required transfer functions X/F and Y/F . To obtain Y/F, the second equation solved for X in terms of Y

$$X = \frac{\frac{kL}{2}}{-m_2\omega^2 + ic\omega + k}Y$$

X in first equation is replaced by the giving

$$\left(-I\omega^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}\frac{\frac{kL}{2}}{-m_2\omega^2 + ic\omega + k}Y = \hat{F}L$$
$$\left(-I\omega^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2} - \frac{\frac{k^2L^2}{4}}{-m_2\omega^2 + ic\omega + k}\right)Y = \hat{F}L$$

Hence

$$Y = \frac{1}{\left(-\frac{1}{3}m_1 L\omega^2 + \frac{L}{4}k + \frac{m_2 g}{2} - \frac{k^2L/4}{-m_2\omega^2+ic\omega+k}\right)}\hat{F}$$

To obtain the transfer function X/F, the second equation is solved for Y in terms of X

$$Y = \frac{\left(-m_2\omega^2 + ic\omega + k\right)}{kL/2}X$$

This is substituted in the first equation giving

$$\left(-I\omega^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)\frac{\left(-m_2\omega^2 + ic\omega + k\right)}{kL/2}X - \frac{kL}{2}X = \hat{F}L$$
$$\left[\frac{\left(-\frac{1}{3}m_1 L\omega^2 + \frac{L}{4}k + \frac{m_2 g}{2}\right)\left(-m_2\omega^2 + ic\omega + k\right)}{k/2} - \frac{kL}{2}\right]X = \hat{F}L$$

Hence

$$X = \frac{kL}{\left(-\frac{1}{3}m_1 L\varpi^2 + \frac{L}{4}k + \frac{m_2 g}{2}\right)\left(-m_2\varpi^2 + ic\varpi + k\right) - k^2 L}\hat{F}$$

This complete part(a). These are the analytical expressions for the transfer functions.

2.11.5.2 Part(b)

Let $m_1 = m_2 = 1$ kg, $k = 3$ N/m, $L = 1$ m, $g = 9.81$ m/s^2, $c = 0.1$ N-s/m.

A program was written to plot the magnitude and phase spectrums of $x(t)$ and $\theta(t)$ using the above numerical values. This was done for a range of forcing frequencies to cover both natural frequencies and beyond. Natural frequencies are found by solving the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$

$$\omega_1 = 1.1308 \text{ rad/sec}$$
$$\omega_2 = 4.3228 \text{ rad/sec}$$

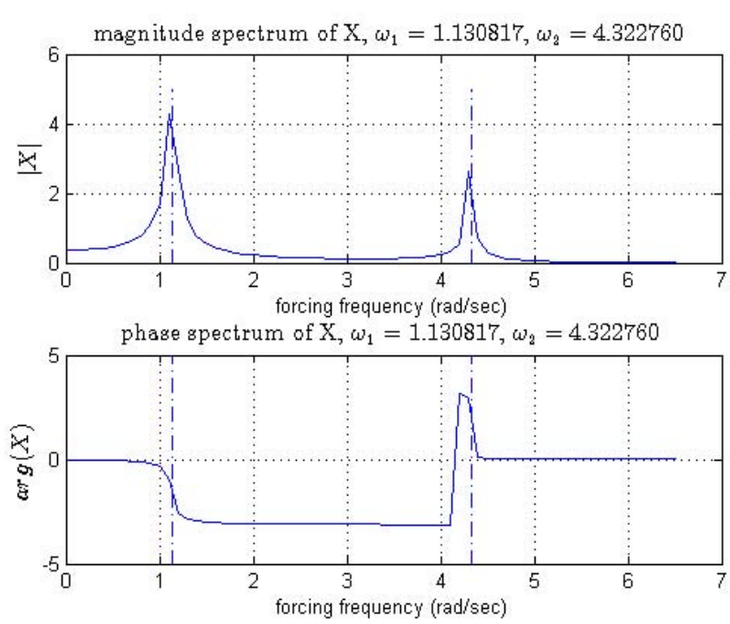
The magnitude and phase of each transfer function are evaluated when $\varpi = \omega_1$ and when $\varpi = \omega_2$.$F = 1$ was assumed since its numerical value was not given. Result is shown below. From these plots, magnitude and phase values are determined at the natural frequencies.

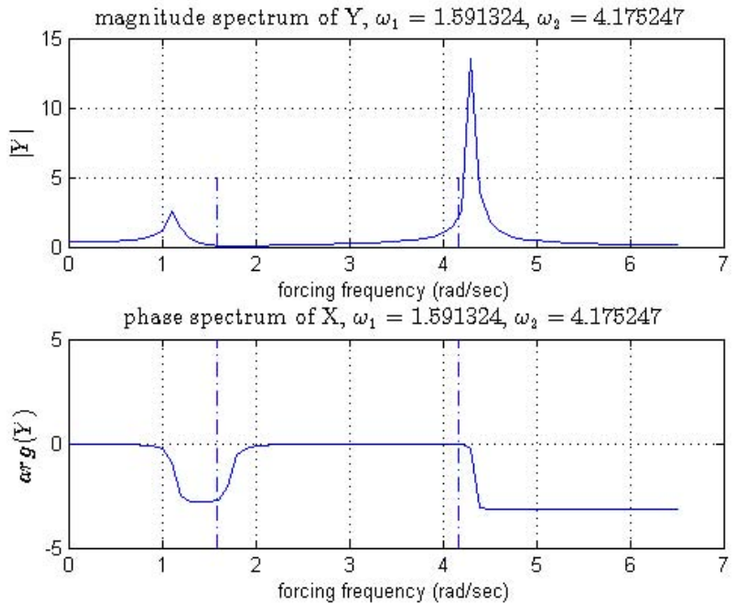
$$x(t) = \text{Re}\left\{Xe^{i\varpi t}\right\}$$
$$\theta(t) = \text{Re}\left\{Ye^{i\varpi t}\right\}$$

Table of results

response	magnitude at ω_1	phase at ω_1	magnitude at ω_2	phase at ω_2
$x(t)$	4.25	-83^0	2.62	131.7^0
$\theta(t)$	2.55	-80^0	11.5	-50^0
ratio	4.25/2.55 = 1.666 7		11.5/2.62 = 4. 389 3	

Plots used to obtain these results





The function used to generate the plots

```
function nma_HW10_problem_5_EMA_545_spectrum()
%plots the spectrums of problem 5, HW10, by Nasser M. Abbasi
close all;

c  = 0.1;
g  = 9.81;
L  = 1;
k  = 3;
m1 = 1;
m2 = 1;
F  = 1;

M  = [1/3*m1*L^2 0;0 m2];
K  = [L^2/4*k+m2*g*L/2  -k*L/2;-k*L/2 k];
C  = [0 0;0 c];

[PHI,w] = eig(K,M);
lam     = sqrt(diag(w))

I = sqrt(-1);
X = @(wf) ((k*L)./((-1/3*m1*L*wf.^2+L/4*k+m2*g/2).*(-m2*wf.^2+I*c*wf+k)- (k^2*L)))*F;
Y = @(wf) (1./((-1/3*m1*L*wf.^2+L/4*k+m2*g/2-( k^2*L./(-m2*wf.^2+I*c*wf+k)))))*F;

N = 2;

for i=1:N
    figure(i);
    wf  = 0:0.1:6.5;

    if i==1
       name_='X';
       tf_ = X(wf);
    else
       name_='Y';
       tf_ = Y(wf);
    end

    subplot(2,1,1);
```

```
    plot(wf,abs(tf_));
    hold on;
    line([lam(1) lam(1)],[0 5],'LineStyle','-.');
    line([lam(2) lam(2)],[0 5],'LineStyle','-.');
    title(sprintf('magnitude spectrum of %c, $\\omega_1=%f$, $\\omega_2=%f$',name_,lam(1
    xlabel('forcing frequency (rad/sec)');
    ylabel(sprintf('$|%c|$',name_),'interpreter','latex','FontSize',12);
    grid;

    subplot(2,1,2);
    plot(wf,angle(tf_));
    line([lam(1) lam(1)],[-5 5],'LineStyle','-.');
    line([lam(2) lam(2)],[-5 5],'LineStyle','-.');
    title(sprintf('phase spectrum of X, $\\omega_1=%f$, $\\omega_2=%f$',lam(1),lam(2)),'
    xlabel('forcing frequency (rad/sec)');
    ylabel(sprintf('$arg(%c)$',name_),'interpreter','latex','FontSize',12);
    grid;
end

end
```

Eigenvectors Φ_1 and Φ_1 are now found, using modal analysis, which de-couples the EOM. The ratio of one component of the same eigenvector to its other component is found and compared with the result found above. The eigenvectors found are

$$\Phi_1 = \begin{Bmatrix} -0.5446 \\ -0.9493 \end{Bmatrix}$$
$$\Phi_2 = \begin{Bmatrix} -1.6442 \\ 0.3145 \end{Bmatrix}$$

The ratios are 0.9493/0.5446 = 1.7431 and 1.6442/0.3145 = 5.2280. Compare these to the ratios found

response	magnitude at ω_1	phase at ω_1	magnitude at ω_2	phase at ω_2
$x(t)$	4.25	-83^0	2.62	131.7^0
$\theta(t)$	2.55	-80^0	11.5	-50^0
ratio	4.25/2.55 = 1.6667		11.5/2.62 = 4.3893	

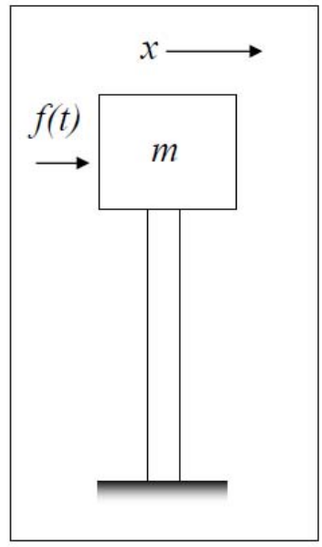
These ratios are close to each others. Ratio Φ_{1j}/Φ_{2j} shows how much one dof (1) will change relative to dof (2) in mode j

2.11.5.3 Part(c)

Transfer functions are plotted in part(a). From magnitude spectrum of Y it is seen that $|Y| = 0$ when ϖ between 1.5 and 2.0 rad/sec and also when $\varpi > 6$ rad/sec. $\omega_{cart} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{3}{1}} = 1.7321$ rad/sec. This agrees with range found in plots. When $\sqrt{\frac{k}{m_2}} = 1.73$, top mass acts as vibration absorber, and rod will not oscillate when $F(t)$ is at this specific frequency.

2.11.6 Problem 6

6.) Consider **Exercise 3.45 and 3.46** in the text (you solved this in problem #3 in HW#6). Use the steady-state displacement that you computed using FFT techniques for $\tau = 3\pi/\omega_n$ to compute the maximum stress in the spring. Assume that the spring is a cantilever beam (in bending) modeled after one of the pillars supporting the ERB, which have length L=40m, rectangular cross section with height h, equal to the width b=h=0.6m, and is constructed from a material with modulus E=30 GPa and ultimate tensile strength σ=40 MPa. (The mass of the beam is assumed to be included in *m*, so its density is not needed.) Let the mass *m* be such that the natural frequency of the mass-spring system is ω_n=0.2Hz. What is the amplitude of the force, P, such that the beam fails due to the dynamic load? Compare that to the static load required to cause the beam to fail (also in bending).



From HW6, problem 3

$$f(t) = \begin{array}{ll} \frac{P}{\tau}t & 0 < t < \tau \\ 0 & \tau < t < 2\tau \end{array}$$

Let $y_{ss}(t)$ be the solution from problem 3 found using FFT technique. Let the full solution for deflection of the above pillar be

$$\chi(y,t) = y(t)\psi(y)$$

$y(t)$ is the time dependent (dynamic) part of the solution. This solution is $y_{ss}(t)$ found in problem 3. $\psi(y)$ is solution due to static loading. Also called the shape function. For cantilever beam with static force P at its end, deflection curve due to static loading P at end is

$$\psi(x) = \frac{P}{6EI}\left(3Lx^2 - x^3\right)$$

Internal bending moment $M(x,t) = EI\frac{d^2\chi(x,t)}{dx^2}$ and direct stress $\sigma = \frac{M(x,t)c}{I}$ where c is the section modulus. Assume $c = \frac{h}{2}$. For yield, let $\sigma = 40MPa$, then

$$\begin{aligned} M(x,t) &= \frac{\sigma I}{c} \\ EI\frac{d^2\chi(x,t)}{dx^2} &= \frac{\sigma I}{c} \end{aligned}$$

$I = \frac{1}{12}bh^3$.

$$\begin{aligned} \frac{d^2\chi(x,t)}{dx^2} &= y_{ss}(t)\frac{d^2}{dx^2}\frac{P}{6EI}\left(3Lx^2 - x^3\right) \\ &= y_{ss}(t)\frac{PL}{EI} \end{aligned}$$

Solve for P at yield

$$\begin{aligned} y_{ss}(t)\frac{PL}{EI} &= \frac{\sigma_{yield}I}{c} \\ P &= \frac{\sigma_{yield}I}{y_{ss}(t)\frac{h}{2}L}EI \end{aligned}$$

$y_{ss}(t)$ from problem 3 has maximum value of 1.8 at $t = 10$ sec. Given numerical values in the problem and using this maximum value of $y_{ss}(t)$ then P can be found from above.

I am not sure this is the correct approach to solve this.We did not have any practice or examples on solving this type of vibration problem before. Need more time to study this subject.

2.11.7 Key solution for HW 10

Homework #10
EMA 545, Spring 2013

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

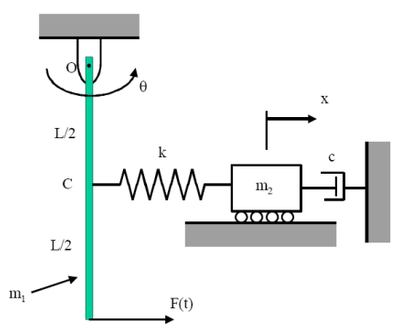
1.) Exercise 4.8 from Ginsberg. (Note: the spring constants are defined such that the frequencies given are the natural frequencies that each spring-mass system would have if it were attached to a rigid base. Notice that there is not a simple relationship between those frequencies and the natural frequencies of the system as a whole.)

2.) Exercise 4.30 as given in the text. Repeat the analysis for $k=2mg/L$ and graph that response as well. (Questions to consider: What do you notice about the natural frequencies of this system in each case? How does that affect the way the response looks? Why?)

3.) Exercise 4.43 from Ginsberg. How does the time required to reach steady state compare with $\tau_r=1/(\zeta_r\omega_r)$ for each mode, r=1,2,3?

4.) Exercise 4.47 from Ginsberg.

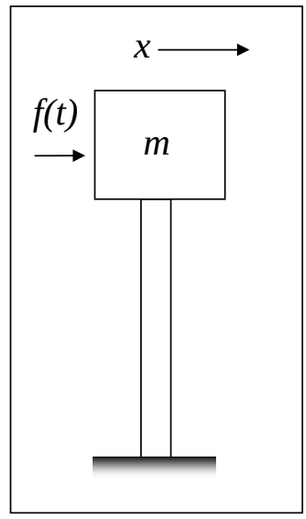
5.) (Covering material from Chapter 5)
A uniform rod of length L and mass m_1 is attached to a cart having mass m_2 by means of a spring k. A viscous damper c resists the motion of the cart.



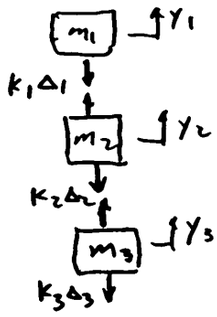
a.) Let F(t)=Re[Fexp(iωt)], x(t) = Re[Xexp(iωt)] and θ(t)=Re[Yexp(iωt)]. Find analytical expressions for the complex transfer functions X/F and Y/F.
b.) Find the magnitude and phase of the response of x and θ when the system is forced at its natural frequencies $\omega=\omega_1$ and $\omega=\omega_2$. Compare these values to the eigenvectors for modes 1 and 2. Use the following numerical values: $m_1=m_2=1$ kg, k=3 N/m, L=1 m, g=9.81 m/s^2, and c=0.1 N-s/m.

c.) Plot the transfer functions Y/F and X/F over a range of frequencies encompassing both modes of vibration. Use the plot to determine at what frequency m_2 acts as a vibration absorber for the rod. How does that frequency compare with the natural frequency that the system would have if the rod were held fixed: $\omega_{cart} = (k/m_2)^{1/2}$?

6.) Consider **Exercise 3.45 and 3.46** in the text (you solved this in problem #3 in HW#6). Use the steady-state displacement that you computed using FFT techniques for $\tau = 3\pi/\omega_n$ to compute the maximum stress in the spring. Assume that the spring is a cantilever beam (in bending) modeled after one of the pillars supporting the ERB, which have length L=40m, rectangular cross section with height h, equal to the width b=h=0.6m, and is constructed from a material with modulus E=30 GPa and ultimate tensile strength σ=40 MPa. (The mass of the beam is assumed to be included in *m*, so its density is not needed.) Let the mass *m* be such that the natural frequency of the mass-spring system is ω_n=0.2Hz. What is the amplitude of the force, P, such that the beam fails due to the dynamic load? Compare that to the static load required to cause the beam to fail (also in bending).



Exercise 4.8



$m_1 = 100,\ m_2 = 200,\ m_3 = 300$ kg

$k_j = \omega_j^2 m_j,\ \omega_1 = 40,\ \omega_2 = 50,\ \omega_3 = 60$ rad/s

$q_1 = y_1,\ q_2 = y_2,\ q_3 = y_3$

$$[M] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2+k_3 \end{bmatrix} = \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 6.6 & -5 \\ 0 & -5 & 15.0 \end{bmatrix} \times 10^5$$

Divide all equations by 100:

$$[[K]-\omega^2[M]]\{\phi\} = \begin{bmatrix} 1600-\omega^2 & -1600 & 0 \\ -1600 & 6600-2\omega^2 & -5000 \\ 0 & -5000 & 15800-3\omega^2 \end{bmatrix}\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

Char eq: $(1600-\omega^2)(6600-2\omega^2)(15800-3\omega^2)$

$$-(1600)^2(15800-3\omega^2)-(5000)^2(1600-\omega^2)=0$$

$$6\omega^6 - 61(10^3)\omega^4 + 153.84(10^6)\omega^2 - 86.4(10^9) = 0 \quad \Leftarrow$$

$\omega^2 = 789.7,\ 2752.8,\ 6624.2$

Set $\phi_{1j} = 1$ & use first two eqs

$$\begin{Bmatrix} 1600-\omega_j^2 \\ -1600 \end{Bmatrix} + \begin{bmatrix} -1600 & 0 \\ 6600-2\omega_j^2 & -5000 \end{bmatrix}\begin{Bmatrix}\phi_{2j}\\ \phi_{3j}\end{Bmatrix} = \begin{Bmatrix}0\\0\end{Bmatrix}$$

$$\phi_{2j} = \frac{1600-\omega_j^2}{1600} \quad \& \quad \phi_{3j} = \frac{-1600+(6600-2\omega_j^2)\phi_{2j}}{5000}$$

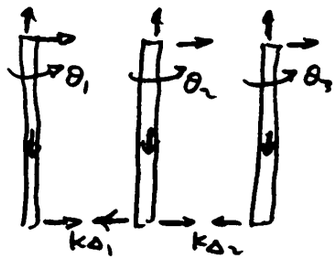
Thus

$\omega_1 = 28.10$ rad/s, $\{\phi_1\} = \begin{Bmatrix}1\\0.5065\\0.1885\end{Bmatrix}$

$\omega_2 = 52.47$ rad/s, $\{\phi_2\} = \begin{Bmatrix}1\\-0.7205\\-0.4777\end{Bmatrix}$ $\Leftarrow$

$\omega_3 = 81.39$ rad/s $\{\phi_3\} = \begin{Bmatrix}1\\-3.140\\3.855\end{Bmatrix}$

Exercise 4.30



$q_1 = \theta_1,\ q_2 = \theta_2,\ q_3 = \theta_3$

$T = \frac{1}{2} I_0 (\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2)$

$I_0 = \frac{1}{3} m L^2$

$V = V_{sp} + V_{gr}$

$V_{sp} = \frac{1}{2} k \Delta_1^2 + \frac{1}{2} k \Delta_2^2$

$\Delta_1 = L(\theta_2 - \theta_1),\ \Delta_2 = L(\theta_3 - \theta_2)$

$V_{sp} = \frac{1}{2} k L^2 [(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2]$

$= \frac{1}{2} k L^2 [\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3]$

$(K_{11})_{sp} = kL^2,\ (K_{22})_{sp} = 2kL^2,\ (K_{33})_{sp} = kL^2$

$(K_{12})_{sp} = -kL^2,\ (K_{23})_{sp} = -kL^2,\ (K_{13})_{sp} = 0$

$V_{gr} = -mg\frac{L}{2}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$

$(K_{11})_{gr} = \left(\frac{\partial^2 V_{sp}}{\partial \theta_1^2}\right)_{\theta_j = 0} = mg\frac{L}{2}$, similarly $(K_{22})_{gr} = (K_{33})_{gr} = mg\frac{L}{2}$

$(K_{12})_{gr} = \left(\frac{\partial^2 V}{\partial\theta_1 \partial\theta_2}\right)_0 = 0 = (K_{23})_{gr} = (K_{13})_{gr}$

Set $k = 0.05 mgL$

$$[K] = [K_{sp}] + [K_{gr}] = mgL \begin{bmatrix} 0.55 & -0.05 & 0 \\ -0.05 & 0.60 & -0.05 \\ 0 & -0.05 & 0.55 \end{bmatrix}$$

Evaluate ω_j & $[\Phi]$ Scaling factor for ω_j is $\left(\frac{g}{L}\right)^{1/2}$

Set $\{q\} = [\Phi]\{\eta\} \Rightarrow \ddot\eta_j + \omega_j^2 \eta_j = 0$

Initial conditions $\{q\} = \{0\}$ & $\{\dot q\} = \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$ rad/s @ $t=0$

Thus $\{\eta\} = 0$ & $\{\dot\eta\} = [\Phi]^T[M]\begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$ at $t=0$

Complementary solution $\eta_j = A_j \cos(\omega_j t) + B_j \sin(\omega_j t)$

Satisfy initial condition $\Rightarrow A_j = 0,\ B_j = \dot\eta_j(0)/\omega_j$

Evaluate η_j at discrete t_n covering four fundamental periods: $[\eta] = [\{\eta(t_1)\}\ \{\eta(t_2)\} \ldots]$

Then $[q] = [\Phi][\eta]$: Rotations as a function of $\left(\frac{g}{L}\right)^{1/2} t$

HW 4.30 Solution
M.S. Allen
Spring 2013

Using the equations of motion and modal responses derived on the previous page, the following Matlab code can then be used to find the transient response:

```
M = eye(3)/3; %*mL^2
 K = [1 -1 0; -1 2 -1; 0 -1 1]*0.05 + eye(3)*0.5; % Kspr + Kgrav

 [phi,lam] = eig(K,M);
 wns = sqrt(diag(lam));

 % Sort & Normalize Eigenvectors to unity modal mass and Check
Orthogonality
[lam_sort,lam_indx] = sort(diag(lam));
wns = sqrt(lam_sort) % *sqrt(k/m)

phi_sort = (phi(:,lam_indx));
mu = phi_sort.'*M*phi_sort;
PHI = real(phi_sort*sqrt(inv(mu)))

check_orth = norm(PHI.'*M*PHI-eye(size(phi)))

n_0 = [0; 0; 0];
nd_0 = PHI'*M*[0; 2; 0]./wns % *m/k
t = [0:0.5:80];
q = PHI*[nd_0(1)*sin(wns(1)*t);
        nd_0(2)*sin(wns(2)*t);
        nd_0(3)*sin(wns(3)*t)];

figure(1)
plot(t,q(1,:), t,q(2,:), t,q(3,:), '.'); grid on;
xlabel('time t*(k/m)^0^.^5'); ylabel('Displacement (m)*k/m');
legend('\theta_1', '\theta_2', '\theta_3');
title('Response to Initial Velocity in \theta_2')

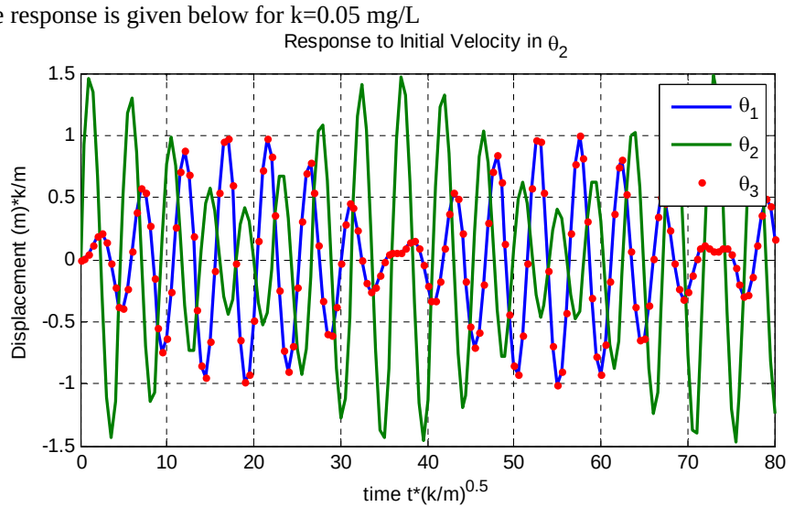
% To animate the solution
%{
figure(2)
for ii = 1:1:length(t);
    sf = 50;
    plot([-0.5 0.5].', [0 0].','o:',[-0.7 0.7].', [0 0].','k');
    line([-0.5 -0.5+10*sin(q(1,ii)/sf)],[0 -
10*cos(q(1,ii)/sf)],'LineWidth', 4); grid on;
    line([0 0+10*sin(q(2,ii)/sf)],[0 -10*cos(q(2,ii)/sf)],'LineWidth',
4); grid on;
    line([0.5 0.5+10*sin(q(3,ii)/sf)],[0 -
10*cos(q(3,ii)/sf)],'LineWidth', 4); grid on;
    xlabel('X-position (*L)'); ylabel('Displacement (m)*k/m');
    title(['Time (m/k)^0.5 = ' num2str(t(ii))])
    axis([-0.7 0.7 -12 2]);
    mov1(ii) = getframe(2);
end
```

```
movie(mov1,2,20)
%}
```

The natural frequencies and mode shapes are (only the first and third modes are excited):

```
wns =
       1.2247
       1.2845
       1.3964
PHI =
            -1      -1.2247      0.70711
            -1  -7.4506e-09      -1.4142
            -1       1.2247      0.70711
```

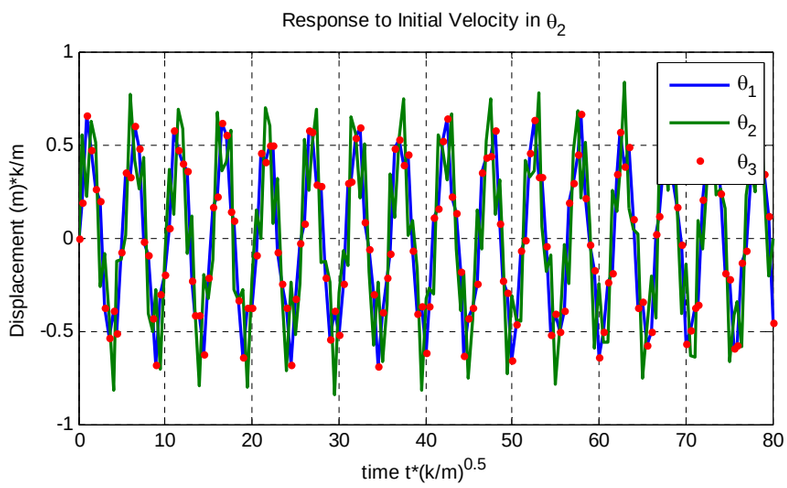
The response is given below for k=0.05 mg/L



The response shows a beating phenomenon, since each bar is influenced by modes 1 and 3 and the modes' frequencies are close (1.22 and 1.39 rad/s).

On the other hand, for k=2*mg/L, the natural frequencies differ by a factor of more than three and the response does not look as simple:

```
wns =
       1.2247
       2.7386
       4.4159
PHI =
            -1      -1.2247      0.70711
            -1  -1.2905e-08      -1.4142
            -1       1.2247      0.70711
```


Response to Initial Velocity in θ2
Displacement (m)*k/m
time t*(k/m)^0.5
θ1
θ2
θ3
1
0.5
0
-0.5
-1
0
10
20
30
40
50
60
70
80

Exercise 4.43

$$M := \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \qquad K := 1000 \cdot \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \qquad C := \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix}$$

Eigensolution

$$\lambda := \text{genvals}(K, M) \qquad \phi := \text{genvecs}(K, M)$$

$$\phi := \text{submatrix}\left(\text{rsort}\left(\text{stack}\left(\lambda^T, \phi\right), 1\right), 2, \text{rows}(\lambda) + 1, 1, \text{rows}(\lambda)\right)$$

$$\lambda := \text{sort}(\lambda) \qquad \omega := \sqrt{\lambda}$$

$$\lambda^T = \begin{bmatrix} 226.558845 & 308.438284 & 2.075003 \cdot 10^3 \end{bmatrix}$$

$$\omega^T = (15.051872 \quad 17.562411 \quad 45.552199)$$

$$\phi = \begin{bmatrix} -0.586168 & -0.806945 & 0.717 \\ 0.549759 & -0.585395 & -0.386503 \\ -0.595123 & -0.078432 & -0.580109 \end{bmatrix} \qquad j := 1 .. \text{rows}(\lambda)$$

$$\Phi^{<j>} := \text{if}\left[\phi_{1,j} \geq 0, \frac{\phi^{<j>}}{\sqrt{\left|\left(\left(\phi^{<j>}\right)^T \cdot M \cdot \phi^{<j>}\right)_{1,1}\right|}}, -\frac{\phi^{<j>}}{\sqrt{\left|\left(\left(\phi^{<j>}\right)^T \cdot M \cdot \phi^{<j>}\right)_{1,1}\right|}}\right]$$

$$\Phi = \begin{bmatrix} 0.021637 & 0.023197 & 0.037332 \\ -0.020293 & 0.016828 & -0.020124 \\ 0.021968 & 2.254691 \cdot 10^{-3} & -0.030205 \end{bmatrix} \qquad \Phi^T \cdot M \cdot \Phi - \text{identity}(\text{rows}(\lambda)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Light damping approximation

$$C' := \Phi^T \cdot C \cdot \Phi$$

$$C' = \begin{bmatrix} 0.053409 & -0.052984 & -0.413171 \\ -0.052984 & 0.768463 & 2.49957 \cdot 10^{-3} \\ -0.413171 & 2.49957 \cdot 10^{-3} & 3.428128 \end{bmatrix}$$

$$\zeta_j := \frac{C'_{j,j}}{2 \cdot \omega_j} \qquad \zeta^T = \begin{bmatrix} 1.774153 \cdot 10^{-3} & 0.021878 & 0.037629 \end{bmatrix}$$

Unit cosine response from Appendix B

$$c_{trans}(t,\omega,\omega_{nat},\zeta) := \frac{\left[\left(\omega_{nat}^2-\omega^2\right)\cdot\cos(\omega\cdot t)+2\cdot\zeta\cdot\omega_{nat}\cdot\omega\cdot\sin(\omega\cdot t)\ \ldots + -\exp\left(-\zeta\cdot\omega_{nat}\cdot t\right)\cdot\left[\left(\omega_{nat}^2-\omega^2\right)\cdot\left(\cos\left(\omega_{nat}\cdot\sqrt{1-\zeta^2}\cdot t\right)\right)\ldots + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\left(\omega_{nat}^2+\omega^2\right)\cdot\sin\left(\omega_{nat}\cdot\sqrt{1-\zeta^2}\cdot t\right)\right]\right]}{\left[\left(\omega_{nat}^2-\omega^2\right)^2+4\cdot\zeta^2\cdot\omega_{nat}^2\cdot\omega^2\right]}$$

Generalized force coefficents

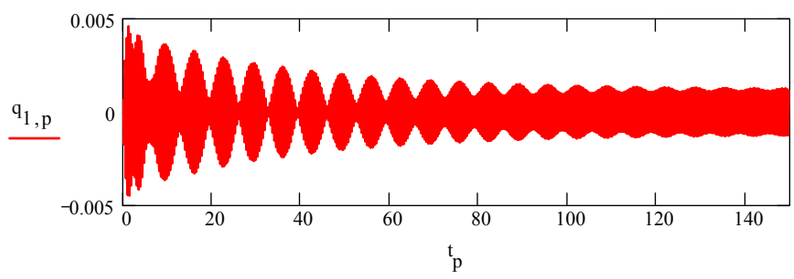
$$F := (200\ \ 0\ \ 0)^T$$

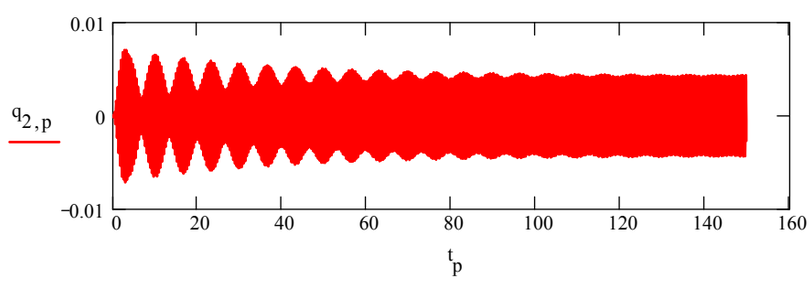
Transient solution for modal coordinates when ω = 16 rad/s:

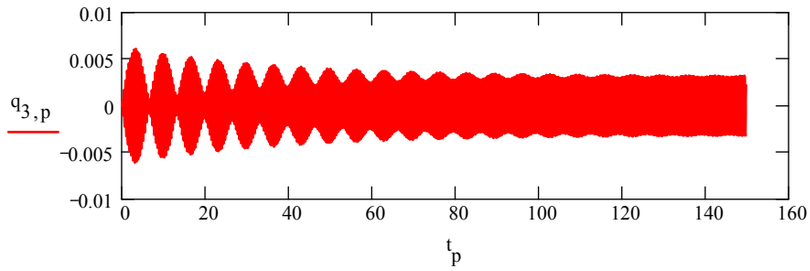
$$T_{max} := \frac{4}{\min\left(\overrightarrow{(\zeta\cdot\omega)}\right)} \qquad \Delta t := \frac{1}{4}\cdot\frac{2\cdot\pi}{\max(\omega)} \qquad P := \mathrm{ceil}\left(\frac{T_{max}}{\Delta t}\right)$$

$$p := 1..P \qquad t_p := (p-1)\cdot\Delta t \qquad P = 4.344\cdot10^3$$

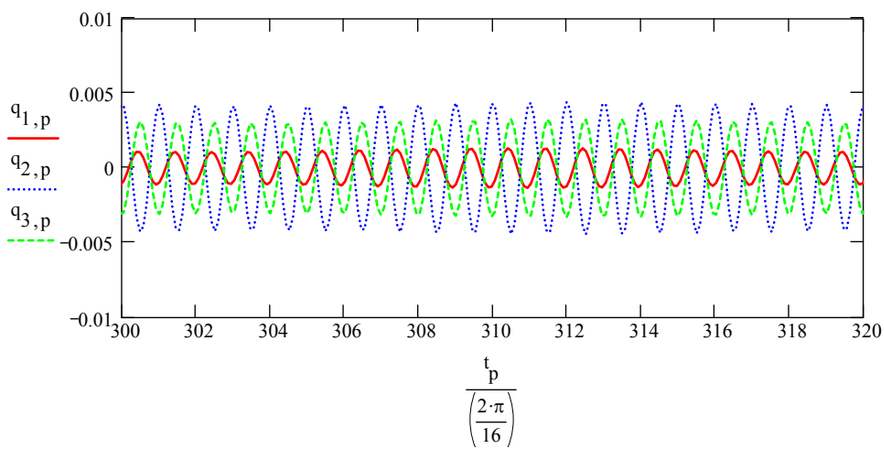
$$\eta_{j,p} := \Phi_{1,j}\cdot F_1\cdot c_{trans}\left(t_p,16,\omega_j,\zeta_j\right) \qquad q := \Phi\cdot\eta$$







Plot responses vs. t nondimensionalized by the forced period to check for steady-state



Response seems to periodic at ω=16 rad/s after 300 forced cycles

$$300 \cdot \frac{2 \cdot \pi}{16} = 117.809725$$

Exercise 4.47

$$M := \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \qquad \omega_1 := 15.38 \qquad \omega_2 := 40.78$$

$$\phi := \begin{bmatrix} 1 & 1 \\ 1.366 & -0.366 \end{bmatrix} \qquad \zeta_1 := 0.08 \qquad \zeta_2 := 0.08$$

$$j := 1..2$$

$$\Phi^{<j>} := \frac{\phi^{<j>}}{\sqrt{\left(\phi^{<j>T} \cdot M \cdot \phi^{<j>}\right)_{1,1}}} \qquad \Phi = \begin{bmatrix} 0.484057 & 0.359633 \\ 0.661222 & -0.131626 \end{bmatrix}$$

$$F := \begin{bmatrix} \frac{50}{i} \\ 100 \end{bmatrix} \qquad \Omega := 20$$

$$X_j := \frac{\left(\Phi^{<j>T} \cdot F\right)_{1,1}}{\left(\omega_j\right)^2 + 2i \cdot \zeta_j \cdot \omega_j \cdot \Omega - \Omega^2}$$

$$Y := \Phi \cdot X \qquad Y = \begin{bmatrix} -0.203556 + 6.975392i \cdot 10^{-3} \\ -0.270729 + 0.017639i \end{bmatrix}$$

Solution P4e01 → column failure 5/3/2013

Given:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m}$$

non-dimensionalize:

$$\left.\begin{aligned} t' &= \omega_n t \\ dt' &= \omega_n dt \end{aligned}\right\} \quad \frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{dx}{dt'}\omega_n$$

EOM becomes:

$$\omega_n^2 x'' + 2\zeta\omega_n^2 x' + \omega_n^2 x = \frac{f(t)}{m} \qquad \omega_n^2 \cdot m = K$$

$$x'' + 2\zeta x' + x = \frac{f(t)}{K}$$

- now the effective nat. freq. is $\omega_n = 1$
- Everywhere we had t before, use $t' = \omega_n t$

$$\longrightarrow \tau = \frac{3\pi}{\omega_n} \longrightarrow \tau' = 3\pi$$

Solve for response to $f(t)$, with $K = 1$, $P = 1$

$\max(x(t)) = 1.88P$ (Matlab) if $K = 2 \rightarrow x(t) = (1.88 \cdot P/K)$ meters etc...

$\max(x(t)) = (1.88\, P/K)$ (amplification factor = 1.88)

(Small since the 3rd harmonic is on resonance, not the first)

For the column given:

$k = \frac{3EI}{L^3}$ $E = 30\,GPa$ $L = 40\,m$ $I = bh^3/12 = (0.6m)^4/12$

$\boxed{K = 182\,KN/m}$ (could use this to find m w/ $f_n = 0.2\,Hz$)

$m = \frac{K}{\omega_n^2} = 115{,}000\,Kg$

for ref, mass of column $= 2300\,kg/m^3 \cdot 0.6^2 \cdot 40 = 33{,}000\,Kg$

stress: (static)

$$\sigma_{max} = \frac{Mc}{I} = \frac{FL(h/2)}{I}$$

$$\sigma_{max} = \frac{K x_{max} L \cdot h/2}{h^4/12} = \frac{K \cdot 1.88LP/K}{h^3/6} = \frac{6 \cdot 1.88\,PL}{h^3}$$

$$P_{max}|_{dyn} = \sigma_{max} \cdot \frac{h^3}{L}\left(\frac{1}{6 \cdot 1.88}\right) = \boxed{19\,KN = P_{max}|_{dyn}}$$

$$P_{max}|_{static} \Rightarrow \sigma_{max} = \frac{PLh/2}{h^4/6} \longrightarrow P_{max}|_{static} = \frac{\sigma h^3}{6L} = 19\,KN \cdot 1.88$$

$$\boxed{P_{max}|_{static} = 36\,KN}$$

✓ (If the beam were cantilevered horizontally, it couldn't hold it's own weight. Probably reasonable)
→ steel reinforcement would help a lot...

2.12 HWA1

Local contents

2.12.1 problem description

Homework #A1
EMA 545, Spring 2013

Instructions:

- If you scored 23 or below on Problem #2 on Exam #1, do Problems 1, 2 and 3.
- If you scored 23 or below on Problem #4 on Exam #1, do Problems 4 and 5
- If neither of those criteria apply to you then you do not need to turn in this assignment.

When working out your solutions to the following problems, you must derive your answers starting from the following. You may not use any equations from the book without first deriving them from these basic principles:
The general solution to an underdamped SDOF system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^{\ 2}x = 0$$

is:

$$x(t) = \mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{\mathrm{i}\omega_d t}\right)$$

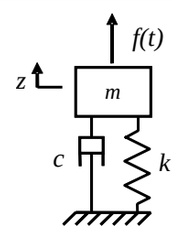
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and A is a complex constant.
You are free to use Appendix B as needed and the fact that the forced response of a system is

$$x(t) = x_{IC} + x_F$$

Where x_{IC} and x_F are found in Appendix B for a variety of forcing functions.

$$f(t) = \mathrm{Re}\left(Fe^{\mathrm{i}\omega t}\right) \qquad \rightarrow \qquad x(t) = \mathrm{Re}\left(Xe^{\mathrm{i}\omega t}\right)$$

The half power points in a transfer function occur at frequencies $\omega \cong \omega_{peak} \pm \zeta\omega_n$



Problem 1: 3.1 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with q(t)).

Problem 2: 3.11 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with x(t)).

Problem 3: (3e2)
The equations of motion for the 2DOF system studied in class are given below. If the applied force is $f(t)$=Fcos(ωt), then the response of both coordinates x_1 and x_2 will also be harmonic. Use this fact to derive the transfer function between the force F and the response X_1.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Problem 4: 2.52 from Ginsberg. Check your answer by comparing it to the solution from ode45 in Matlab.

Problem 5: 2.54 from Ginsberg. Choose values for the parameters and check your answer by plotting the response and comparing it to the solution from ode45 in Matlab.

2.12.2 problem 1

A system has mass $M = 20kg$ and $\omega_n = 100$ rad/sec. It is observed that steady state response is $q = 20\cos(110t - 1.5)\ mm$, where t is in seconds. Determine harmonic excitation causing this response for $\zeta = 0$ and $\zeta = 0.4$

Let the harmonic excitation be

$$F(t) = \text{Re}\left\{\hat{F}e^{i\omega t}\right\}$$

where $\hat{F}$ is its complex amplitude. Also let

$$q = \text{Re}\left\{\hat{Q}e^{i\omega t}\right\}$$

be the steady state response. We are given that $q = 20 \times 10^{-3}\cos(110t - 1.5)$, therefore

$$\begin{aligned} q &= \text{Re}\left\{20 \times 10^{-3}e^{i(100t-1.5)}\right\} \\ &= \text{Re}\left\{20 \times 10^{-3}e^{-1.5i}e^{i100t}\right\} \end{aligned}$$

Therefore

$$\hat{Q} = 20 \times 10^{-3}e^{-1.5i}$$

But the transfer function for second order system is

$$\hat{Q} = \frac{\hat{F}}{k}\frac{1}{\left(1 - r^2\right) + 2i\zeta r}$$

where $r = \frac{\omega}{\omega_n}$, hence we can now solve for $\hat{F}$ from the above.

$$\hat{F} = \hat{Q}\left(k\left(\left(1 - r^2\right) + 2i\zeta r\right)\right)$$

But $k = M\omega_n^2$ hence

$$\boxed{\hat{F} = \hat{Q}\left(M\omega_n^2\left(\left(1 - r^2\right) + 2i\zeta r\right)\right)}$$

$\boxed{\text{When } \zeta = 0}$ we find

$$\begin{aligned} \hat{F} &= 20 \times 10^{-3}e^{-1.5i}\left(20 \times 100^2\left(1 - \left(\frac{110}{100}\right)^2\right)\right) \\ &= 20 \times 10^{-3}e^{-1.5i}(-42000.0) \\ &= -42000.0 \times 20 \times 10^{-3}e^{-1.5i} \\ &= -840.0e^{-1.5i} \end{aligned}$$

Hence

$$\begin{aligned} F(t) &= \text{Re}\left\{\hat{F}e^{i\omega t}\right\} \\ &= \text{Re}\left\{-840.0e^{-1.5i}e^{i110t}\right\} \\ &= \text{Re}\left\{-840.0e^{i(100t-1.5)}\right\} \end{aligned}$$

Therefore

$$\boxed{F(t) = -840\cos(100t - 1.5)}$$

$\boxed{\text{When } \zeta = 0.4}$ we find

$$\begin{aligned}
\hat{F} &= \hat{Q}\left(M\omega_n^2\left(\left(1-r^2\right)+2i\zeta r\right)\right)\\
&= 20\times 10^{-3}e^{-1.5i}\left(20\times 100^2\left(\left(1-\left(\frac{110}{100}\right)^2\right)+i2(0.4)\left(\frac{110}{100}\right)\right)\right)\\
&= 20\times 10^{-3}e^{-1.5i}\left(20\times 100^2(-0.21+0.88i)\right)\\
&= 4000e^{-1.5i}(-0.21+0.88i)\\
&= 4000e^{-1.5i}\left(\sqrt{(0.21)^2+(0.88)^2}\,e^{i\tan^{-1}\left(\frac{.88}{-0.21}\right)}\right)
\end{aligned}$$

```
In[4]:= ArcTan[-0.21, 0.88]

Out[4]= 1.80505
```

Hence

$$\begin{aligned}
\hat{F} &= 4000e^{-1.5i}\left(0.90471e^{i1.80505}\right)\\
&= 3618.8e^{-1.5i+1.80505i}\\
&= 3618.8e^{0.30505i}
\end{aligned}$$

Therefore

$$\begin{aligned}
F(t) &= \mathrm{Re}\left\{\hat{F}e^{i\omega t}\right\}\\
&= \mathrm{Re}\left\{3618.8e^{0.30505i}e^{i110t}\right\}\\
&= \mathrm{Re}\left\{3618.8e^{i(100t+0.30505)}\right\}
\end{aligned}$$

Hence

$$\boxed{F(t) = 3618.8\cos(100t+0.30505)}$$

2.12.3 Problem 2

3.11 Measurement of the steady-state response of a one-degree-of-freedom system to a harmonic excitation $F\cos(\omega t)$ indicates that at a frequency of 100 Hz, the response is $x = 4\sin(\omega t)$. It also is observed that 105 Hz is a half-power point.

(a) Determine the phase lag of the response relative to the excitation at 105 Hz.

(b) Determine the amplitude and phase lag of the response at 110 Hz.

Let

$$P(t) = \mathrm{Re}\left\{\hat{F}e^{i\omega t}\right\}$$

where $\hat{F}$ is the complex amplitude of the excitation. Hence by comparing this to $P(t) = F\cos\omega t = \mathrm{Re}\left\{Fe^{i\omega t}\right\}$ we see that $\hat{F} = F$.

When $\omega = 2\pi 100$ then the response was $q = \text{Re}\left\{\hat{Q}e^{i\omega t}\right\} = 4\sin(2\pi 100t)$ hence $q = \text{Re}\left\{\overbrace{4e^{-i\frac{\pi}{2}}}^{\hat{Q}}e^{i\omega t}\right\} = q = \text{Re}\left\{4e^{i\left(\omega t-\frac{\pi}{2}\right)}\right\}$ therefore

$$\hat{Q} = 4e^{-i\frac{\pi}{2}}$$

But, from the transfer function of second order system we know that

$$\hat{Q} = \frac{\hat{F}}{k}\frac{1}{\left(1-r^2\right)+2i\zeta r}$$

Hence

$$\begin{aligned} 4e^{-i\frac{\pi}{2}} &= \frac{\hat{F}}{k}\frac{1}{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)+2i\zeta\left(\frac{2\pi 100}{\omega_n}\right)} \\ &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 100}{\omega_n}\right)\right)^2}}e^{-i\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)} \end{aligned} \tag{2.109}$$

By comparing sides we see that

$$\begin{aligned} \frac{\pi}{2} &= \frac{2\zeta r}{1-r^2} \\ &= \frac{2\zeta\frac{2\pi 100}{\omega_n}}{1-\left(\frac{2\pi 100}{\omega_n}\right)^2} \end{aligned} \tag{2.110}$$

When $\omega = 105Hz$ we are told it is half power point, which means the amplitude there is 0.707 of the maximum amplitude which occurs when $r = 1$. Hence

$$\begin{aligned} 0.707\frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-(1)^2\right)^2+(2\zeta(1))^2}} &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 105}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 105}{\omega_n}\right)\right)^2}} \\ 0.707\frac{1}{2\zeta} &= \frac{1}{\sqrt{\left(1-\left(\frac{2\pi 105}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 105}{\omega_n}\right)\right)^2}} \end{aligned} \tag{2.111}$$

We now have 2 equations 2.110 and 2.111 to solve numerically for ζ and ω_n . Solving and keeping the positive solutions results in

$$\begin{aligned} \zeta &= 0.0309 \\ \omega_n &= 640.8 \text{ rad/sec} \\ &= 101.987 \text{ Hz} \end{aligned}$$

Hence at $\omega = 105$ hz the phase is

$$\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.0309)\frac{2\pi(105)}{640.8}}{1-\left(\frac{2\pi(105)}{640.8}\right)^2}\right) = 133.305^0$$

```
In[35]:= ArcTan[1 - ((2 Pi 105)/640.8)^2, 2 (0.0309) ((2 Pi 105)/640.8)]*180/Pi
Out[35]= 133.305
```

2.12.3.1 Part(b)

When $\omega = 100$ Hz we found from Eq 2.109 that

$$4 = \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2\pi 100}{\omega_n}\right)\right)^2}}$$

But we found that $\omega_n = 640.8$ rad/sec and $\zeta = 0.0309$, hence

$$\begin{aligned}\frac{\hat{F}}{k} &= 4\sqrt{\left(1-\left(\frac{2\pi 100}{640.8}\right)^2\right)^2 + \left(2(0.0309)\left(\frac{2\pi 100}{640.8}\right)\right)^2}\\ &= 0.28733\end{aligned}$$

At $\omega = 110$ Hz

$$\begin{aligned}|\hat{Q}| &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 110}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2\pi 110}{\omega_n}\right)\right)^2}}\\ &= 0.28733\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 110}{640.8}\right)^2\right)^2 + \left(2(0.0309)\left(\frac{2\pi 110}{640.8}\right)\right)^2}}\\ &= \boxed{1.6288}\end{aligned}$$

The phase is

$$\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.0309)\left(\frac{2\pi 110}{640.8}\right)}{1-\left(\frac{2\pi 110}{640.8}\right)^2}\right) = \boxed{157.798^0}$$

```
In[37]:= ArcTan[1 - ((2 Pi 110)/640.8)^2, 2 (0.0309) ((2 Pi 110)/640.8)]*180/Pi
Out[37]= 157.798
```

2.12.4 Problem 3

Problem 3: (3e2)
The equations of motion for the 2DOF system studied in class are given below. If the applied force is $f(t)$=Fcos(ωt), then the response of both coordinates x_1 and x_2 will also be harmonic. Use this fact to derive the transfer function between the force F and the response X_1.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

The two equations are

$$\begin{aligned} mx_1'' + 2kx_1 - kx_2 &= 0\\ mx_2'' - kx_1 + kx_2 &= f(t)\end{aligned}$$

Since the responses are harmonic and the input is harmonic, then we can write

$$\begin{aligned} x_1(t) &= \mathrm{Re}\left\{\hat{X}_1 e^{i\omega t}\right\}\\ x_1(t) &= \mathrm{Re}\left\{\hat{X}_2 e^{i\omega t}\right\}\end{aligned}$$

Therefore the two equations can be written in terms of the complex amplitudes as

$$-m\omega^2\hat{X}_1 + 2k\hat{X}_1 - k\hat{X}_2 = 0 \tag{2.112}$$
$$-m\omega^2\hat{X}_2 - k\hat{X}_1 + k\hat{X}_2 = F \tag{2.113}$$

From Eq 2.117

$$\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 = \hat{X}_2$$

Substitute the above into Eq 2.113 gives

$$\begin{aligned}
-m\omega^2\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 - k\hat{X}_1 + k\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 &= F \\
\left(\frac{\left(-m^2\omega^4 - m\omega^2 2k\right)}{k} + k - m\omega^2\right)\hat{X}_1 &= F \\
\hat{X}_1 &= kF\frac{1}{\left(-m^2\omega^4 - m\omega^2 2k + k^2 - km\omega^2\right)}
\end{aligned}$$

Dividing the numerator and denominator of the RHS by k^2, and using $k^2 = \omega_n^4 m^2$ and using $r = \frac{\omega}{\omega_n}$

$$\begin{aligned}
\hat{X}_1 &= \frac{F}{k}\frac{1}{\left(\frac{-m^2\omega^4}{\omega_n^4 m^2} - \frac{m\omega^2 2}{\omega_n^2 m} + 1 - \frac{m\omega^2}{\omega_n^2 m}\right)} \\
\hat{X}_1 &= \frac{F}{k}\frac{1}{\left(-r^4 - 2r^2 + 1 - r^2\right)}
\end{aligned}$$

Hence the transfer function is

$$\hat{X}_1 = \frac{F}{k}\frac{1}{\left(1 - r^4 - 3r^2\right)}$$

2.12.5 Problem 4

2.52 An undamped system is subjected to a sequence of step excitations, $Q(t) = 100$ N if $0 < t < 3\pi/\omega_{\text{nat}}$, $Q(t) = 200$ N if $t > 3\pi/\omega_{\text{nat}}$. The system mass is 5 kg and $\omega_{\text{nat}} = 50$ rad/s. Determine and graph the response.

Summary of method of solution: There are 2 ways to solve these problem. We will solve it using both methods. The first method is using known standard solution for step input, the solution $y(t)$ is found for the period of $0 < t < \frac{3\pi}{\omega_n}$ using zero initial conditions. Next, the solution $y(t)$ and $y'(t)$ is evaluated again at $t = \frac{3\pi}{\omega_n}$. These values are now used as the initial conditions for the solution for $t > \frac{3\pi}{\omega_n}$. The solution for $t > \frac{3\pi}{\omega_n}$ will have the same form, but the step input now is $200N$ instead of 100N.

The second method as follows: Let $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ or $F(t) = 100h(t) + 100h\left(\tilde{t}\right)$ where $\tilde{t} = t - \frac{3\pi}{\omega_n}$, then assuming the transient solution to $h(t)$ is $s(t)$ then the solution to $F(t)$ is $100s(t) + 100s\left(\tilde{t}\right)$. The second method is simplet than the first method.

Solution using first method:

The system is

$$my''(t) + ky(t) = F(t)$$

When $F(t)$ is a fixed input, such as a step input of magnitude F then the response is given by

$$y(t) = \left(y_0 - \frac{F}{k}\right)\cos\omega_n t + \frac{y_0'}{\omega_n}\sin\omega_n t + \frac{F}{k}$$

Where in the above, y_0 and y_0' are the initial position and initial velocity. For $0 < t < 1.5T_n$ the solution is

$$\begin{aligned} y(t) &= -\frac{F}{k}\cos\omega_n t + \frac{F}{k} \\ &= \frac{F}{k}(1 - \cos\omega_n t) \end{aligned}$$

Let $F = Q_1 = 100\text{N}$, and since $k = m\omega_n^2$ then the above becomes

$$y(t) = \frac{Q_1}{m\omega_n^2}(1 - \cos\omega_n t) \qquad 0 < t < \frac{3\pi}{\omega_n}$$

Now we need first to evaluate $y\left(t = \frac{3\pi}{\omega_n}\right)$ and $y'\left(t = \frac{3\pi}{\omega_n}\right)$. From the above

$$y'(t) = \frac{Q_1}{m\omega_n}\sin\omega_n t \qquad 0 < t < \frac{3\pi}{\omega_n}$$

Hence

$$y\left(t = \frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n^2}\left(1 - \cos\omega_n\frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n^2}(1 - \cos 3\pi) = \frac{Q_1}{m\omega_n^2}(1 - (-1)) = \frac{2Q_1}{m\omega_n^2}$$

and

$$y'\left(t = \frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n}\sin\left(\omega_n\frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n}\sin(3\pi) = 0$$

Now let $\tilde{t} = t - \frac{3\pi}{\omega_n}$Hence the solution for $\tilde{t} > 0$ is

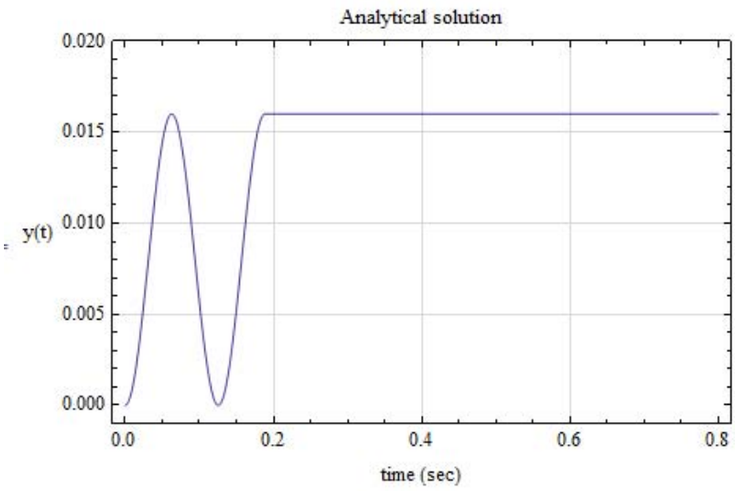
$$\begin{aligned} y(\tilde{t}) &= \left(y(\tilde{t} = 0) - \frac{Q_2}{m\omega_n^2}\right)\cos\omega_n\tilde{t} + \frac{y'(\tilde{t} = 0)}{\omega_n}\sin\omega_n\tilde{t} + \frac{Q_2}{k} \\ &= \left(\frac{2Q_1}{m\omega_n^2} - \frac{Q_2}{m\omega_n^2}\right)\cos\omega_n\tilde{t} + \frac{Q_2}{k} \end{aligned}$$

Therefore, we have obtain the complete solution, which is

time	solution
$0 < t < \frac{3\pi}{\omega_n}$	$\frac{Q_1}{m\omega_n^2}(1 - \cos\omega_n t) = \frac{100}{5(50)^2}(1 - \cos 50t) = 0.008(1 - \cos 50t)$
$\tilde{t} = t - \frac{3\pi}{\omega_n}$	$\left(\frac{2Q_1}{m\omega_n^2} - \frac{Q_2}{k}\right)\cos\omega_n\tilde{t} + \frac{Q_2}{k} = \left(\frac{2(100)}{5(50)^2} - \frac{200}{5(50)^2}\right)\cos\omega_n\tilde{t} + \frac{200}{5(50)^2} = 0.016$

This is a plot of the solution. Then a numerical ODE solver is used to verify the result

```
y[t_] := Piecewise[{{0.008 (1 - Cos[50 t]), 0 ≤ t < 3 π/50}, {0.016, True}}];
Plot[y[t], {t, 0, 0.8}, Exclusions → None,
 PlotRange → {Automatic, {-0.001, 0.02}}, AxesOrigin → {0, 0},
 Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "Analytical solution"}},
 RotateLabel → False]
```



Now a numerical ODE solver was used to verify. Here is the result

```
force[t_?NumericQ] :=
  Piecewise[{{100, 0 ≤ t < 3. π/50}, {200, True}, {0, t < 0}}];
ode = y''[t] + 50^2 y[t] == force[t]/5;
sol = NDSolve[{ode, y[0] == 0, y'[0] == 0}, y, {t, 0, 0.8}];

Plot[Evaluate[y[t] /. sol], {t, 0, 0.8}, Exclusions → None,
 PlotRange → {Automatic, {-0.001, 0.02}}, AxesOrigin → {0, 0},
 Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
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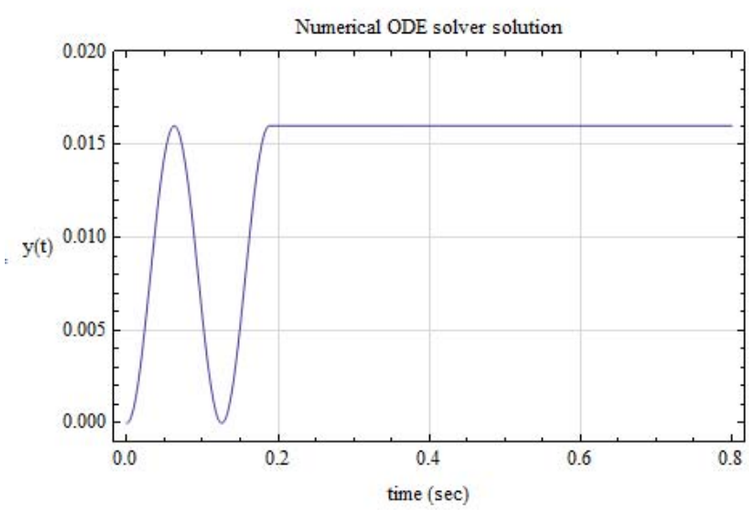
  {{"y(t)", None}, {"time (sec)", "Numerical ODE solver solution"}},
 RotateLabel → False]
```



We can see the solutions match very well.

Solution using second method:

Let $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ then assuming the transient solution to $h(t)$ is $s(t)$ then

the solution to $F(t)$ is $100s(t) + 100s\left(t - \frac{3\pi}{\omega_n}\right)h\left(t - \frac{3\pi}{\omega_n}\right)$. From appendix B, the solution to $h(t)$ is given by

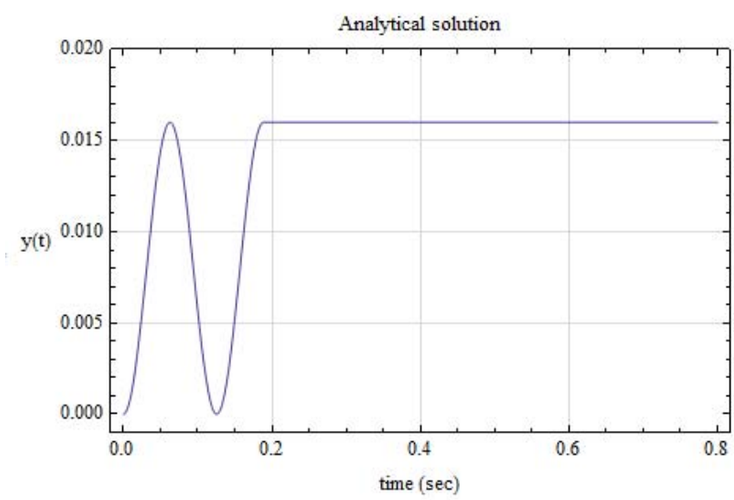
$$s(t) = \frac{1}{m\omega_n^2}(1 - \cos\omega_n t)$$

hence the solution to $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ is

$$\begin{aligned} y(t) &= 100s(t) + 100s\left(t - \frac{3\pi}{\omega_n}\right)h\left(t - \frac{3\pi}{\omega_n}\right) \\ &= \frac{100}{m\omega_n^2}(1 - \cos\omega_n t) + \frac{100}{m\omega_n^2}\left(1 - \cos\omega_n\left(t - \frac{3\pi}{\omega_n}\right)\right)h\left(t - \frac{3\pi}{\omega_n}\right) \end{aligned}$$

To verify, this is a plot of the above solution. We see it is the same as the first analytical solution, and it is the same solution as the one using numerical ODE solver as well.

```
y[t_] := 100/(m wn^2) (1 - Cos[wn t]) +
   100/(m wn^2) (1 - Cos[wn (t - 3 π/50)]) UnitStep[t - 3 π/50]
Plot[Evaluate[y[t] /. {m → 5, wn → 50}], {t, 0, 0.8},
 PlotRange → {Automatic, {-0.001, 0.02}}, Exclusions → None,
 AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "Analytical solution"}},
 RotateLabel → False]
```



2.12.6 Problem 5

2.54 An undamped system is subjected to a constant force for an interval T, after which it begins to roll off exponentially. Specifically, $Q(t) = F_0$ if $0 < t < T$, $Q(t) = F_0 \exp[-\beta(t - T)]$ if $t > T$. Derive an expression for the response.

The input can be written as $F_0h(t) - F_0h(t - T) + F_0e^{-\beta(t-T)}h(t - T)$ or, by letting $\tilde{t} = t - T$, the input becomes

$$F_0h(t) - F_0h(\tilde{t}) + F_0e^{-\beta\tilde{t}}h(\tilde{t})$$

If the response to $h(t)$ is $s(t)$ and the response to $e^{-\beta\tilde{t}}$ is $s_1(\tilde{t})$ then the response to the above becomes

$$F_0 s(t) - F_0 s(\tilde{t}) h(\tilde{t}) + F_0 s_1(\tilde{t}) h(\tilde{t})$$

From appendix B, we see that

$$s(t) = \frac{1}{m\omega_n^2}(1 - \cos\omega_n t)$$

and

$$s_1(\tilde{t}) = \frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta\tilde{t}} - \left(\cos(\omega_n\tilde{t}) + \frac{-\beta}{\omega_n}\sin\omega_n\tilde{t}\right)\right)h(\tilde{t})$$

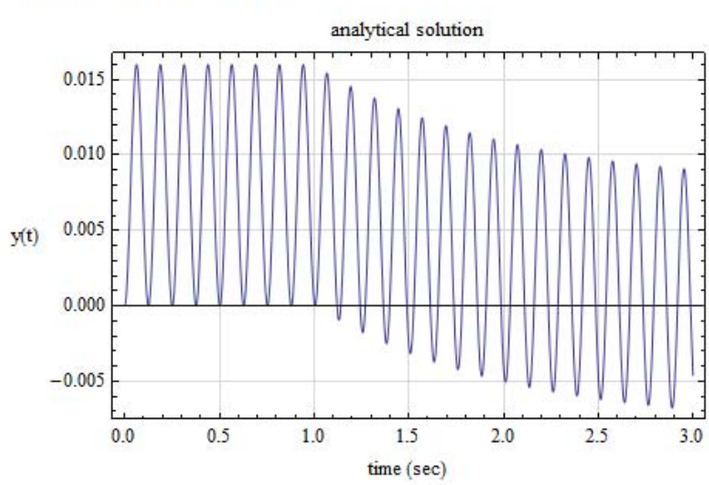
Therefore the the final response is

$$\begin{aligned}
y(t) &= F_0 h(t) - F_0 h(\tilde{t}) + F_0 e^{-\beta\tilde{t}} h(\tilde{t}) \\
&= F_0 \frac{1}{m\omega_n^2}(1 - \cos\omega_n t)h(t) - F_0 \frac{1}{m\omega_n^2}(1 - \cos\omega_n\tilde{t})h(\tilde{t}) + \\
&F_0 \frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta\tilde{t}} - \left(\cos(\omega_n\tilde{t}) + \frac{-\beta}{\omega_n}\sin\omega_n\tilde{t}\right)\right)h(\tilde{t}) \\
&= F_0 \frac{1}{m\omega_n^2}(1 - \cos\omega_n t) - F_0 \frac{1}{m\omega_n^2}(1 - \cos(\omega_n(t-T)))h(t-T) + \\
&F_0 \frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta(t-T)} - \left(\cos(\omega_n(t-T)) + \frac{-\beta}{\omega_n}\sin\omega_n(t-T)\right)\right)h(t-T)
\end{aligned}$$

To plot this, we need to choose values for parameters. Let $F_0 = 100, \omega_n = 50 rad/\sec, m = 5kg, \beta = 1, T = 1$, then a plot of the above is below, followed by solution from numerical ODE solver.

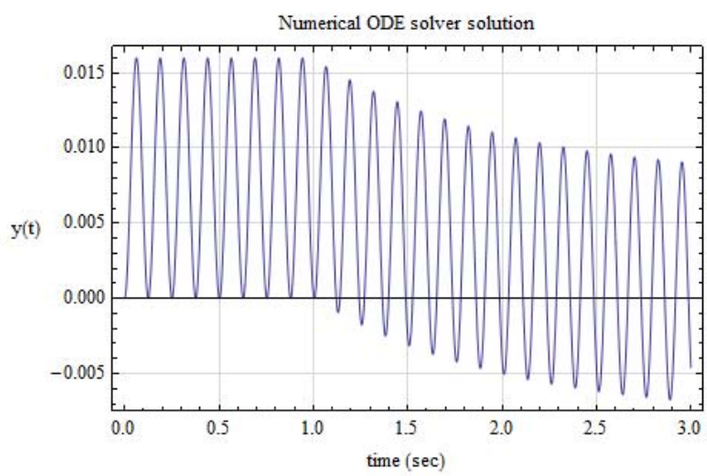
Plot of the analytical solution

```
y[t_] := f0/(m wn^2) (1 - Cos[wn t]) - f0/(m wn^2) (1 - Cos[wn (t - T)]) UnitStep[t - T] +
   f0/(m (wn^2 β^2)) (Exp[-β (t - T)] - (Cos[wn (t - T)] - β/wn Sin[wn (t - T)]))
    UnitStep[t - T]
parms = {T → 1, β → 1, wn → 50, m → 5, f0 → 100};
Plot[Evaluate[y[t] /. parms], {t, 0, 3}, Exclusions → None,
 PlotRange → All, AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "analytical solution"}},
 RotateLabel → False]
```



To verify, this is the result from numerical ODE solver

```
Clear[t, y]
T = 1; β = 1; wn = 50; m = 5; f0 = 100;
force[t_?NumericQ] :=
  Piecewise[{{f0, 0 ≤ t < T}, {f0 Exp[-β (t - T)] UnitStep[t - T], True}}];
ode = y''[t] + wn^2 y[t] == force[t]/m;
sol = NDSolve[{ode, y[0] == 0, y'[0] == 0}, y, {t, 0, 3}];
Plot[Evaluate[y[t] /. sol], {t, 0, 3}, Exclusions → None,
 PlotRange → All, AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel →
  {{"y(t)", None}, {"time (sec)", "Numerical ODE solver solution"}},
 RotateLabel → False]
```



We can see that the solutions agree.

2.12.7 Key solution for HW A1

Homework #A1
EMA 545, Spring 2013

Instructions:

- If you scored 23 or below on Problem #2 on Exam #1, do Problems 1, 2 and 3.
- If you scored 23 or below on Problem #4 on Exam #1, do Problems 4 and 5
- If neither of those criteria apply to you then you do not need to turn in this assignment.

When working out your solutions to the following problems, you must derive your answers starting from the following. You may not use any equations from the book without first deriving them from these basic principles:
The general solution to an underdamped SDOF system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^{\ 2}x = 0$$

is:

$$x(t) = \mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{\mathrm{i}\omega_d t}\right)$$

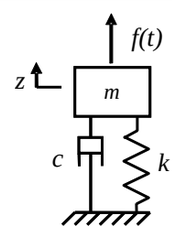
where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and A is a complex constant.
You are free to use Appendix B as needed and the fact that the forced response of a system is

$$x(t) = x_{IC} + x_F$$

Where x_{IC} and x_F are found in Appendix B for a variety of forcing functions.

$$f(t) = \mathrm{Re}\left(Fe^{\mathrm{i}\omega t}\right) \qquad \rightarrow \qquad x(t) = \mathrm{Re}\left(Xe^{\mathrm{i}\omega t}\right)$$

The half power points in a transfer function occur at frequencies $\omega \cong \omega_{peak} \pm \zeta\omega_n$



Problem 1: 3.1 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with q(t)).

Problem 2: 3.11 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with x(t)).

Problem 3: (3e2)
The equations of motion for the 2DOF system studied in class are given below. If the applied force is $f(t)$=Fcos(ωt), then the response of both coordinates x_1 and x_2 will also be harmonic. Use this fact to derive the transfer function between the force F and the response X_1.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Problem 4: 2.52 from Ginsberg. Check your answer by comparing it to the solution from ode45 in Matlab.

Problem 5: 2.54 from Ginsberg. Choose values for the parameters and check your answer by plotting the response and comparing it to the solution from ode45 in Matlab.

Exercise 3.1

$M = 20\,kg,\ \omega_{nat} = 100\ rad/s \Rightarrow k = M\omega_{nat}^2 = 2(10^5)\,N/m$

$q = 0.02\cos(110t - 1.5)$ meter

$= Re\left[0.02\, e^{i(\omega t - 1.5)}\right] \quad \omega = 110\ rad/s$

$Q(t) = M\ddot{q} + C\dot{q} + Kq$

$= Re\left[(-M\omega^2 + Ci\omega + K)(0.02)e^{i(\omega t - 1.5)}\right]$

But $\frac{K}{M} = \omega_{nat}^2$ & $C/M = 2\zeta\omega_{nat}$, so

$Q(t) = M\,Re\left[(-\omega^2 + 2i\zeta\omega_{nat}\omega + \omega_{nat}^2)(0.02) \times e^{i(\omega t - 1.5)}\right]$

For $\zeta = 0$ & $\omega = 110$:

$Q = 20\,Re\left[-42\ e^{i(110t - 1.5)}\right]$

$\Rightarrow \quad = -840\cos(110t - 1.5)$ newton

For $\zeta = 0.4$ & $\omega = 110\ rad/s$:

$Q = 20\,Re\left[(-210.0 + 8800i)(0.02)\,e^{i(110t - 1.5)}\right]$

$= Re\left[(-840 + 3520i)\,e^{i(110t - 1.5)}\right]$

$= Re\left[3619\,e^{i\,1.8051}\,e^{i(110t - 1.5)}\right]$

$\Rightarrow \quad = 3619\cos(110t + 0.3051)$ newton

3.11 Solution

$m\ddot{x} + c\dot{x} + Kx = F\cos(\omega t)$

$f = 100\ Hz \longrightarrow x(t) = 4\sin(\omega t)$? meters? not given

$-90°$ out of phase, so $f_n = 100\ Hz$

$f = 105\ Hz \rightarrow$ half power point

→ This part is tedious to re-derive, but we know from the book that:

$\omega_{peak} \approx \omega_n \qquad \omega_{HP} = \omega_{peak} \pm \zeta\omega_n$

so

$105 - 100 = 5\ Hz \cdot 2\pi = \zeta\omega_n$

$\longrightarrow \zeta = \frac{10\pi}{105 \cdot 2\pi} = 0.0476$

a.) $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (F/m)\cos(\omega t) \longrightarrow x(t) = Re(X e^{i\omega t})$

$\longrightarrow \quad X = \frac{(F/m)}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$

$X = (F/m)\frac{1}{\cdots}$ $\qquad \omega = 105\ Hz$, $\omega_n = 100\ Hz$, $\zeta = 0.0476$

$X = (F/m)\cdot(-1.26 - 1.24)\times 10^{-5}$

$X = (F/m)\cdot 1.77\, e^{-2.37}$

$\longrightarrow$ $\boxed{\text{phase lag} = 2.37\ rad = 135.7°}$

b) Amplitude at resonance:

$|X| = \left|\frac{(F/m)}{i\omega_n(2\zeta\omega_n)}\right| \longrightarrow (F/m) = 4\cdot 2\zeta\omega_n^2 = 1.5\times 10^5\ N/Kg$

at $\omega = 110\ Hz$

$X = \frac{F/m}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n} \longrightarrow$

$\boxed{|X| = 1.62 \qquad \measuredangle X = -153°}$

Units not given – m? (Very large!)

3e2 Solution

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$f(t) = F\cos(\omega t)$$

assume: $x_j(t) = Re(X_j e^{i\omega t})$

$\ddot{x}_j(t) = Re(-\omega^2 X_j e^{i\omega t})$

$$-\omega^2 m X_1 + 2K X_1 - K X_2 = 0$$

$$-\omega^2 m X_2 - K X_1 + K X_2 = F$$

$$\therefore X_1 = \frac{K X_2}{(2K - \omega^2 m)} \quad \text{sub into}$$

$$-\omega^2 m X_2 - \frac{K^2 X_2}{(2K - \omega^2 m)} + K X_2 = F$$

$$X_2 = \left(-\omega^2 m - \frac{K^2}{(2K - \omega^2 m)} + K\right)^{-1} F$$

$$\boxed{X_1 = \frac{K}{(2K - \omega^2 m)}\left(-\omega^2 m - \frac{K^2}{(2K - \omega^2 m)} + K\right)^{-1} F}$$

Or → easier method → solve linear equations above

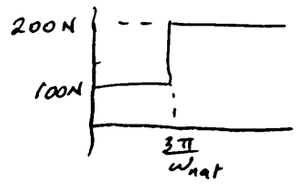
$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} -\omega^2 m + 2K, & -K \\ -K, & -\omega^2 m + K \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$= \frac{\begin{bmatrix} -\omega^2 m + K & K \\ K & -\omega^2 m + 2K \end{bmatrix}}{(-\omega^2 m + 2K)(-\omega^2 m + K) - K^2} \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$= \begin{Bmatrix} KF \\ (2K - \omega^2 m) F \end{Bmatrix}$$

$$\boxed{X_1 = \frac{K}{(2K - \omega^2 m)(K - \omega^2 m) - K^2} F}$$

Exercise 2.52



$$Q(t) = 100\,h(t) + 100\,h\left(t - \frac{3\pi}{\omega_{nat}}\right)$$

$$m\ddot{q} + kq = Q.$$

Assume $q(0) = \dot{q}(0) = 0$

Let $u(t)$ be the unit step response.

$$q = 100\left[u(t) + u\left(t - \frac{3\pi}{\omega_{nat}}\right)\right]$$

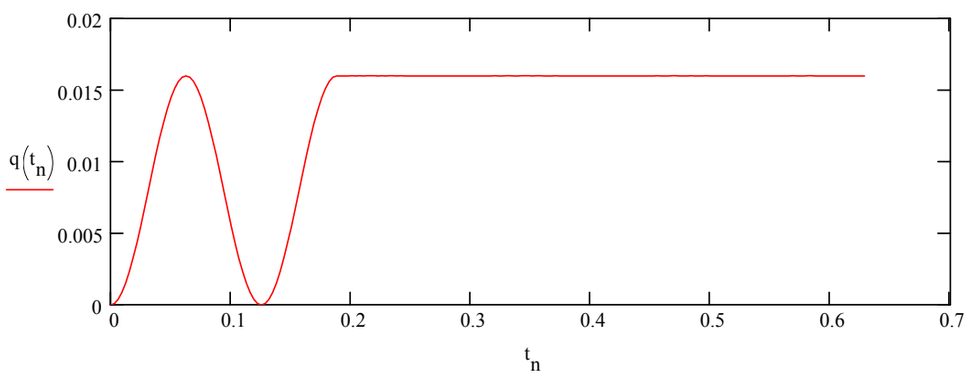
$M := 5$ $\omega_{nat} := 50$

$$u(t) := \frac{1}{M \cdot \omega_{nat}^{2}} \cdot \left(1 - \cos\left(\omega_{nat} \cdot t\right)\right) \cdot (t > 0)$$

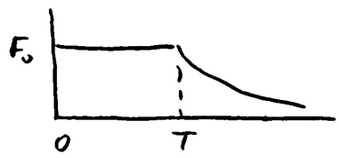
$$q(t) := 100 \cdot \left(u(t) + u\left(t - \frac{3 \cdot \pi}{\omega_{nat}}\right)\right)$$

$N := 201$ $n := 1 .. N$

$$t_n := \frac{n - 1}{N - 1} \cdot \frac{10 \cdot \pi}{\omega_{nat}}$$



Exercise 2.54



$$Q = F_0\left[h(t) - h(t-T)\right] + F_0 \exp\left[-\beta(t-T)\right] h(t-T)$$

$m\ddot{q} + kq = Q$, Assume $q(0) = \dot{q}(0) = 0$

Let u be the unit step response, and x be the unit exponential response. Then

$$q = F_0\, u(t) - F_0\, u(t-T) + F_0\, x(t-T)$$

where $u(t) = \frac{1}{M\omega_{nat}^2}\left[1 - \cos(\omega_{nat} t)\right] h(t)$

$$x(t) = \frac{1}{M(\omega_{nat}^2 - 2\beta\omega_{nat} + \beta^2)}\left\{\exp(-\beta t) - \left[\cos(\omega_{nat} t) - \frac{\beta}{\omega_{nat}}\sin(\omega_{nat} t)\right]\right\} h(t)$$

2.13 HW11

Local contents

2.13.1 problem description

Homework #11
EMA 545, Spring 2013

Problem 1.)
Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{Q\}$$

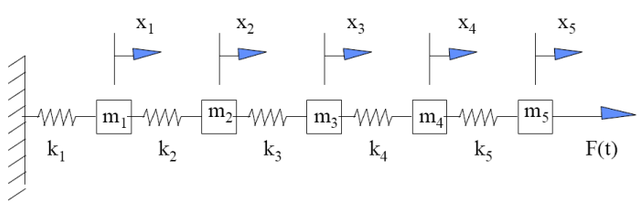
with [C] matrix given by:

$$[C]=[M][\Phi]\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}[\Phi]^T[M].$$

If we transform to normal coordinates using $\{q\}=[\Phi]\{\eta\}$, show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

Problem 2.)
The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs k_i which are drawn, there are viscous dampers c_i which are not shown. The equations of motion for this system are easily found to be:

$$[m]\underline{\ddot{x}}+[c]\underline{\dot{x}}+[k]\underline{x}=\underline{F}$$

where [m] is a diagonal matrix having entries m_i, and [k] is the banded matrix:

$$\begin{bmatrix} (k1+k2) & -k2 & 0 & 0 & 0 \\ -k2 & (k2+k3) & -k3 & 0 & 0 \\ 0 & -k3 & (k3+k4) & -k4 & 0 \\ 0 & 0 & -k4 & (k4+k5) & -k5 \\ 0 & 0 & 0 & -k5 & k5 \end{bmatrix}$$

The damping matrix has the same form as [k], but with viscous damping coefficients c_i in place of stiffness coefficients k_i. The forcing vector $\underline{F}$ is a 5 by 1 vector of zeros, except for the last entry: $\underline{F}=[0 \quad 0 \quad 0 \quad 0 \quad F(t)]^T$.

The numerical value for k_2 through k_5 is 1 N/m, the numerical value of k_1 = 2 N/m, the numerical value for m_1 through m_5 is 1 kg, and the numerical values for the viscous damping coefficients $c_i = 0.1*k_i$, i=1,5 (in units of N-s/m).

(a) Find the natural frequencies and mass-normalized modes of the system.
(b) Find the magnitude and phase of the steady-state response $x_5(t)$ assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to $1.2*\omega_5$. Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
(c) Repeat the analysis in (b), but use the strategy described in Problem 1 to create a [C] matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
(d) Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of γ=0.04.

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters EA and L are as follows: $k_i = N*EA/L$, i=2,N where N is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high, $k_1 = 2N*EA/L$. The lumped masses are equal to the total bar mass divided by N: $m_i = \rho AL/N$, where ρ is the mass density of the bar.

Problem 3.) (40 points)
Use a three-term Ritz series to predict the first 3 natural frequencies and natural modes of a fixed-free bar of length L, elastic modulus E, and constant cross-sectional area A.
(5 pts) a.) Use the potential and kinetic energy expressions in the book (eq. 6.1.1 and 6.1.2) to derive the expressions for the mass and stiffness matrices in eq. 6.1.11 and 6.1.13.
(5 pts) b.) Use the following Ritz basis functions to find the 3x3 mass and stiffness matrices:

$$\psi_1(x) = x/L\,,\quad \psi_2(x) = (x/L)^2\,,\ \text{and}\ \psi_3(x) = (x/L)^3$$

Hint: Use the pattern described in Example Problem 6.1. In that problem, a uniform bar that is fixed at x=0 is studied; however, in that problem, there is an extra spring and dashpot at the right end of the bar, x=L. Note also that Example Problem 6.1 uses a different set of basis functions.
(10 pts) c.) Repeat the analysis using the following basis functions. I suggest using a computer package to estimate the numerical terms in the mass and stiffness matrices.

$$\psi_n = \sin\left(\alpha_n \frac{x}{L}\right), \qquad \alpha_n = \left(\frac{2n-1}{2}\right)\pi, \qquad n = 1,2,3$$

What do you notice about the M and K matrices using this set of basis functions?
(10 pts) d.) Compare the natural frequencies obtained parts (b) and (c) of this problem with those obtained in Problem 2. (Use the relationships given in the problem statement above to find EA/L and ρAL values that agree with those used in problem 2.)
(10 pts) e.) Generate a plot of the mode shapes of the systems based on the models in (b) and (c), and also overlay the mode shapes obtained in problem 2. How do the three sets of results compare?

2.13.2 problem 1

Problem 1.)
Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{Q\}$$

with [C] matrix given by:

$$[C]=[M][\Phi]\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}[\Phi]^T[M].$$

If we transform to normal coordinates using $\{q\}=[\Phi]\{\eta\}$, show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

The columns of matrix [Φ] are orthogonal w.r.t to the mass matrix. Hence the following two relations will be assumed as given in the derivation that follows

$$[\Phi]^T[M][\Phi]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.114}$$

$$[\Phi]^T[K][\Phi]=\begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}$$

Starting with the coupled EOM given, which is

$$[M]\{q''\}+[C]\{q'\}+[K]\{q\}=\{Q\}$$

Since $\{q\}=[\Phi]\{\eta\}$, then $\{q''\}=[\Phi]\{\eta''\}$ and $\{q'\}=[\Phi]\{\eta'\}$. Substituting these in the above EOM gives

$$[M][\Phi]\{\eta''\}+[C][\Phi]\{\eta'\}+[K][\Phi]\{\eta\}=\{Q\}$$

premultiplying by $[\Phi]^T$ both the LHS and RHS results in

$$[\Phi]^T[M][\Phi]\{\eta''\}+[\Phi]^T[C][\Phi]\{\eta'\}+[\Phi]^T[K][\Phi]\{\eta\}=[\Phi]^T\{Q\}$$

Using Eq 3.1 the above simplifies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}\{\eta''\}+[\Phi]^T[C][\Phi]\{\eta'\}+\begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}\{\eta\}=[\Phi]^T\{Q\}$$

Replacing $[C]$ by the expression given in the problem description, the above becomes

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 1\end{bmatrix}\left\{\eta''\right\} + [\Phi]^T\left([M][\Phi]\begin{bmatrix}\omega_1^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_N^2\end{bmatrix}[\Phi]^T[M]\right)[\Phi]\left\{\eta'\right\} + \begin{bmatrix}\omega_1^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_N^2\end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 1\end{bmatrix}\left\{\eta''\right\} + \overbrace{[\Phi]^T[M][\Phi]}^{I}\begin{bmatrix}2\zeta_1\omega_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 2\zeta_N\omega_N\end{bmatrix}\overbrace{[\Phi]^T[M][\Phi]}^{I}\left\{\eta'\right\} + \begin{bmatrix}\omega_1^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_N^2\end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

Since $[\Phi]^T[M][\Phi]$ is the identity matrix, then the above reduces to

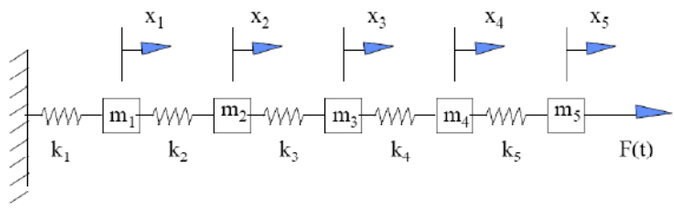
$$\begin{bmatrix}1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 1\end{bmatrix}\left\{\eta''\right\} + \begin{bmatrix}2\zeta_1\omega_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 2\zeta_N\omega_N\end{bmatrix}\left\{\eta'\right\} + \begin{bmatrix}\omega_1^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_N^2\end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

This is decoupled OEM since there is no coupling in the mass matrix, and no coupling in the damping matrix and no coupling in the stiffness matrix.

QED

2.13.3 Problem 2

Problem 2.)
The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs k_i which are drawn, there are viscous dampers c_i which are not shown. The equations of motion for this system are easily found to be:

$$[m]\ddot{\underline{x}} + [c]\dot{\underline{x}} + [k]\underline{x} = \underline{F}$$

where [m] is a diagonal matrix having entries m_i, and [k] is the banded matrix:

$$\begin{bmatrix} (k1+k2) & -k2 & 0 & 0 & 0 \\ -k2 & (k2+k3) & -k3 & 0 & 0 \\ 0 & -k3 & (k3+k4) & -k4 & 0 \\ 0 & 0 & -k4 & (k4+k5) & -k5 \\ 0 & 0 & 0 & -k5 & k5 \end{bmatrix}$$

The damping matrix has the same form as [k], but with viscous damping coefficients c_i in place of stiffness coefficients k_i. The forcing vector $\underline{F}$ is a 5 by 1 vector of zeros, except for the last entry: $\underline{F}=[0 \quad 0 \quad 0 \quad 0 \quad F(t)]^T$.

The numerical value for k_2 through k_5 is 1 N/m, the numerical value of k_1 = 2 N/m, the numerical value for m_1 through m_5 is 1 kg, and the numerical values for the viscous damping coefficients $c_i = 0.1*k_i$, i=1,5 (in units of N-s/m).

(a) Find the natural frequencies and mass-normalized modes of the system.
(b) Find the magnitude and phase of the steady-state response $x_5(t)$ assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to $1.2*\omega_5$. Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
(c) Repeat the analysis in (b), but use the strategy described in Problem 1 to create a [C] matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
(d) Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of γ=0.04.

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters EA and L are as follows: $k_i = N*EA/L$, i=2,N where N is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high, $k_1 = 2N*EA/L$. The lumped masses are equal to the total bar mass divided by N: $m_i = \rho AL/N$, where ρ is the mass density of the bar.

EOM is

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 & 0 \\ 0 & 0 & -c_4 & c_4+c_5 & -c_5 \\ 0 & 0 & 0 & -c_5 & c_5 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{Bmatrix} +$$

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

substituting the numerical values gives $c_1 = 0.2, c_i = 0.1, i = 2,5$, hence EOM becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{Bmatrix} + \frac{1}{10} \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{Bmatrix} + \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

2.13.3.1 part(a)

Natural frequency and mass normalized modes are found by solving the eigenvalue problem to find the natural frequencies and the mass normalized modes.

```
K=[3 -1 0 0 0;-1 2 -1 0 0;0 -1 2 -1 0;0 0 -1 2 -1;0 0 0 -1 1]
M=diag(ones(5,1));
[phi,omega]=eig(K,M);
omega = sqrt(diag(omega));
```

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 & -0.5635 & -0.6247 \\ -0.2871 & 0.6247 & -0.4472 & 0.0989 & 0.5635 \\ -0.4472 & 0.4472 & 0.4472 & 0.4472 & -0.4472 \\ -0.5635 & -0.0989 & 0.4472 & -0.6247 & 0.2871 \\ -0.6247 & -0.5635 & -0.4472 & 0.2871 & -0.0989 \end{bmatrix}$$

$$\begin{aligned} \omega &= \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\} \text{ rad/sec} \\ &= \{0.0498, 0.1445, 0.225, 0.284, 0.314\} \text{ hz} \end{aligned}$$

2.13.3.2 Part(b)

in modal coordinates, EOM is decoupled to become

$$I\{\eta''\} + [\Phi]^T[C][\Phi]\{\eta'\} + [\Phi]^T[k][\Phi]\{\eta\} = [\Phi]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

```
EDU>> C  = 0.1*K;
C  = phi.'*C*phi
K  = phi.'*K*phi
syms f(t);
F  = zeros(5,1); F(5)=1;
F  = phi.'*F
C =
    0.0098    0.0000   -0.0000    0.0000    0.0000
    0.0000    0.0824   -0.0000    0.0000   -0.0000
   -0.0000   -0.0000    0.2000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    0.3176   -0.0000
    0.0000   -0.0000    0.0000   -0.0000    0.3902
K =
    0.0979    0.0000   -0.0000    0.0000    0.0000
    0.0000    0.8244   -0.0000    0.0000   -0.0000
   -0.0000   -0.0000    2.0000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    3.1756   -0.0000
    0.0000         0    0.0000   -0.0000    3.9021
F =
   -0.6247
   -0.5635
   -0.4472
    0.2871
   -0.0989
```

Hence EOM in modal coordinates is

$$[I]\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\\ \eta_4''\\ \eta_5''\end{Bmatrix}+\begin{bmatrix}0.0098 & 0 & 0 & 0 & 0\\ 0 & 0.0824 & 0 & 0 & 0\\ 0 & 0 & 0.2 & 0 & 0\\ 0 & 0 & 0 & 0.3176 & 0\\ 0 & 0 & 0 & 0 & 0.3902\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\\ \eta_3'\\ \eta_4'\\ \eta_5'\end{Bmatrix}+$$

$$\begin{bmatrix}0.0979 & 0 & 0 & 0 & 0\\ 0 & 0.8244 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3.1756 & 0\\ 0 & 0 & 0 & 0 & 3.9021\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix}=\begin{Bmatrix}-0.6247F(t)\\ -0.5635F(t)\\ -0.4472F(t)\\ 0.2871F(t)\\ -0.0989F(t)\end{Bmatrix}$$

Where in the above $F(t)=\cos(\varpi t)$ with ϖ being the forcing frequency in the range 0 to $1.2\omega_5$ where $\omega_5 = 1.9754$ rad/sec.

Since the equations are now decoupled, the 5th equation can solved on its own

$$\eta_5'' + 0.3902\eta_5' + 3.9021\eta_5 = \mathrm{Re}\left\{-0.0989e^{i\varpi t}\right\}$$

Assuming $\eta_5(t) = \mathrm{Re}\left\{Xe^{i\varpi t}\right\}$ and substituting in the above and simplifying gives

$$\left(-\varpi^2 + i\varpi 0.3902 + 3.9021\right)X = -0.0989$$

$$X = \frac{-0.0989}{-\varpi^2 + i\varpi 0.3902 + 3.9021}$$

Hence

$$\eta_5(t) = \mathrm{Re}\left\{\frac{-0.0989}{-\varpi^2 + i\varpi 0.3902 + 3.9021}e^{i\varpi t}\right\}$$

Similarly, all other $\eta_i, i = 1,5$ are found. Hence

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \mathrm{Re}\left(\begin{Bmatrix} \frac{-0.6247}{-\varpi^2+i\varpi 0.0098+0.0979} \\ \frac{-0.5635}{-\varpi^2+i\varpi 0.0824+0.8244} \\ \frac{-0.4472}{-\varpi^2+i\varpi 0.2+2} \\ \frac{0.2871}{-\varpi^2+i\varpi 0.3176+3.176} \\ \frac{-0.0989}{-\varpi^2+i\varpi 0.3902+3.9021} \end{Bmatrix} e^{i\varpi t} \right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\{\eta\}$. Hence

$$\begin{aligned}
x_5 &= \sum_{j=1}^{5} \Phi(5,j)\eta(j) \\
&= \sum_{j=1}^{5} \Phi(5,j)\,\mathrm{Re}\left\{X(j)e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left(\sum_{j=1}^{5} \Phi(5,j)X(j)e^{i\varpi t}\right) \\
&= \mathrm{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\varpi t}\right] \\
&= \mathrm{Re}\left[\left(\begin{array}{c} \frac{(-0.6247)(-0.6247)}{-\varpi^2+i\varpi 0.0098+0.0979} + \frac{(-0.5635)(-0.5635)}{-\varpi^2+i\varpi 0.0824+0.8244} + \frac{(-0.4472)(-0.4472)}{-\varpi^2+i\varpi 0.2+2} + \\ \frac{(0.2871)0.2871}{-\varpi^2+i\varpi 0.3176+3.176} + \frac{(-0.0989)(-0.0989)}{-\varpi^2+i\varpi 0.3902+3.9021} \end{array} \right) e^{i\varpi t}\right] \\
&= \mathrm{Re}\left[\left(\begin{array}{c} \frac{0.390\,25}{-\varpi^2+0.0098i\varpi+0.098} + \frac{0.08243}{-\varpi^2+0.3178i\varpi+3.176} + \frac{0.31753}{-\varpi^2+0.0824i\varpi+0.8244} + \\ \frac{0.00978}{-\varpi^2+0.3902i\varpi+3.9021} + \frac{0.19999}{-\varpi^2+0.2i\varpi+2} \end{array} \right) e^{i\varpi t}\right]
\end{aligned}$$

Therefore

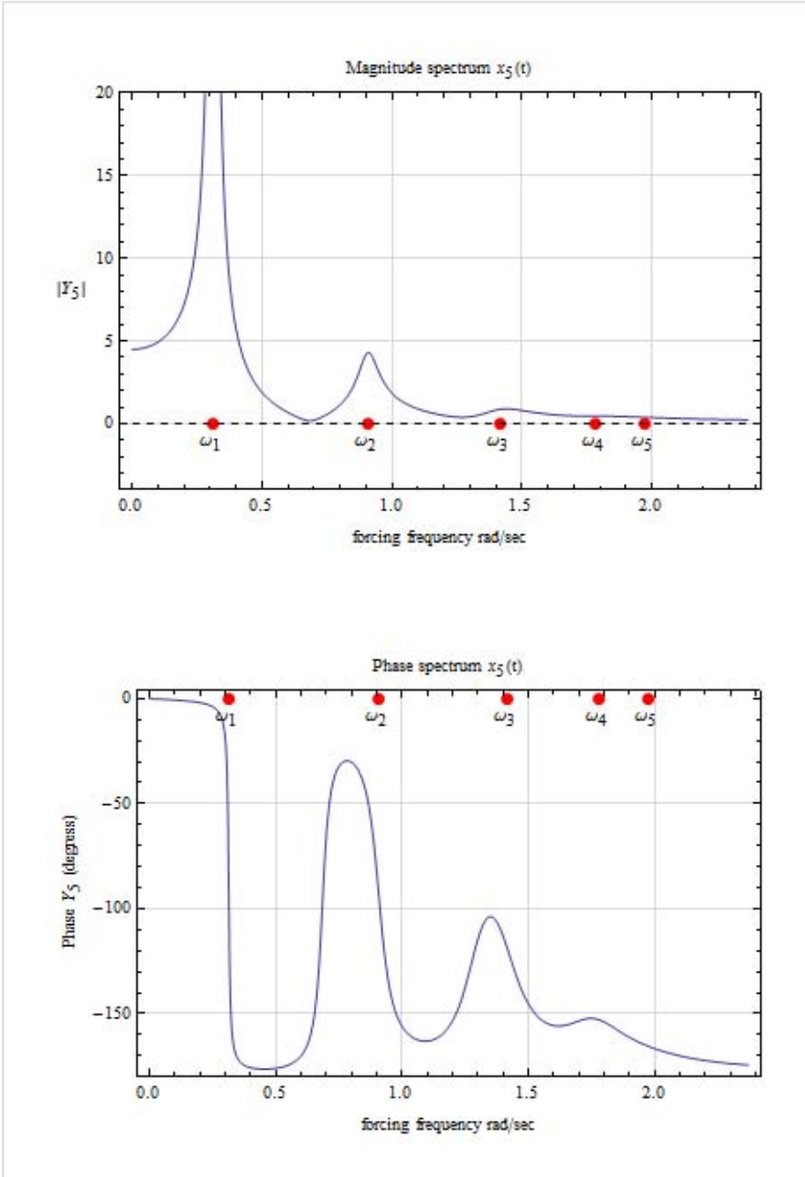
$$x_5 = \mathrm{Re}\left(Y_5 e^{i\varpi t}\right)$$

where

$$Y_5 = \frac{0.390\,25}{-\varpi^2 + 0.0098i\varpi + 0.098} + \frac{0.08243}{-\varpi^2 + 0.3178i\varpi + 3.176} + \frac{0.31753}{-\varpi^2 + 0.0824i\varpi + 0.8244} + \frac{0.00978}{-\varpi^2 + 0.3902i\varpi + 3.9021} + \frac{0.19999}{-\varpi^2 + 0.2i\varpi + 2}$$

Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$. This shows that $x_5(t)$ response will have the largest magnitude when the forcing frequency coincides with the first natural frequency (the fundamental frequency). In otherwords when $\varpi = \omega_1$.

The amplitude of $x_5(t)$ at resonance is smaller for the remaining 4 natural frequencies. For higher order natural frequencies, resonances at those frequencies produces lower amplitudes than lower order natural frequencies.



2.13.3.3 part(c)

Using $\zeta_i = \zeta = 0.02$ for $i = 1,5$ the EOM is

$$[I]\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\\ \eta_4''\\ \eta_5''\end{Bmatrix} + 2\zeta\begin{bmatrix}\omega_1 & 0 & 0 & 0 & 0\\ 0 & \omega_2 & 0 & 0 & 0\\ 0 & 0 & \omega_3 & 0 & 0\\ 0 & 0 & 0 & \omega_4 & 0\\ 0 & 0 & 0 & 0 & \omega_5\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\\ \eta_3'\\ \eta_4'\\ \eta_5'\end{Bmatrix} + \begin{bmatrix}\omega_1^2 & 0 & 0 & 0 & 0\\ 0 & \omega_2^2 & 0 & 0 & 0\\ 0 & 0 & \omega_3^2 & 0 & 0\\ 0 & 0 & 0 & \omega_4^2 & 0\\ 0 & 0 & 0 & 0 & \omega_5^2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix} = \begin{Bmatrix}-0.6247F(t)\\ -0.5635F(t)\\ -0.4472F(t)\\ 0.2871F(t)\\ -0.0989F(t)\end{Bmatrix}$$

Hence the solution

$$\eta_j = \mathrm{Re}\left\{X_j e^{i\varpi t}\right\}$$

where now

$$X_j = \frac{F_j}{-\varpi^2 + 2i\zeta_j\varpi\omega_j + \omega_j^2}$$

Hence, since $\omega = \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\}$ the solutions in modal coordi-

nates is

$$\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix} = \mathrm{Re}\left(\begin{Bmatrix}\dfrac{-0.6247}{-\varpi^2+2i\varpi(0.02)(0.3129)+0.0979}\\ \dfrac{-0.5635}{-\varpi^2+2i\varpi(0.02)(0.9080)+0.8244}\\ \dfrac{-0.4472}{-\varpi^2+2i\varpi(0.02)(1.4142)+2}\\ \dfrac{0.2871}{-\varpi^2+2i\varpi(0.02)(1.7820)+3.176}\\ \dfrac{-0.0989}{-\varpi^2+2i\varpi(0.02)(1.9754)+3.9021}\end{Bmatrix}e^{i\varpi t}\right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\{\eta\}$. Hence

$$\begin{aligned}
x_5 &= \sum_{j=1}^{5}\Phi(5,j)\eta(j)\\
&= \sum_{j=1}^{5}\Phi(5,j)\,\mathrm{Re}\left\{X(j)e^{i\varpi t}\right\}\\
&= \mathrm{Re}\left(\sum_{j=1}^{5}\Phi(5,j)X(j)e^{i\varpi t}\right)\\
&= \mathrm{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\varpi t}\right]\\
&= \mathrm{Re}\left[\left(\begin{array}{c}\dfrac{(-0.6247)(-0.6247)}{-\varpi^2+2i\varpi(0.02)(0.3129)+0.0979} + \dfrac{(-0.5635)(-0.5635)}{-\varpi^2+2i\varpi(0.02)(0.9080)+0.8244} + \dfrac{(-0.4472)(-0.4472)}{-\varpi^2+2i\varpi(0.02)(1.4142)+2} +\\ \dfrac{(0.2871)0.2871}{-\varpi^2+2i\varpi(0.02)(1.7820)+3.176} + \dfrac{(-0.0989)(-0.0989)}{-\varpi^2+2i\varpi(0.02)(1.9754)+3.9021}\end{array}\right)e^{i\varpi t}\right]\\
&= \mathrm{Re}\left[\left(\begin{array}{c}\dfrac{0.390\,25}{-\varpi^2+1.251\,6\times10^{-2}i\varpi+0.097\,9} + \dfrac{0.317\,53}{-\varpi^2+0.036\,32i\varpi+0.824\,4} + \dfrac{0.19999}{-\varpi^2+5.656\,8\times10^{-2}i\varpi+2.0} +\\ \dfrac{8.242\,6\times10^{-2}}{-\varpi^2+0.071\,28i\varpi+3.176} + \dfrac{9.781\,2\times10^{-3}}{-\varpi^2+7.\,901\,6\times10^{-2}i\varpi+3.902\,1}\end{array}\right)e^{i\varpi t}\right]
\end{aligned}$$

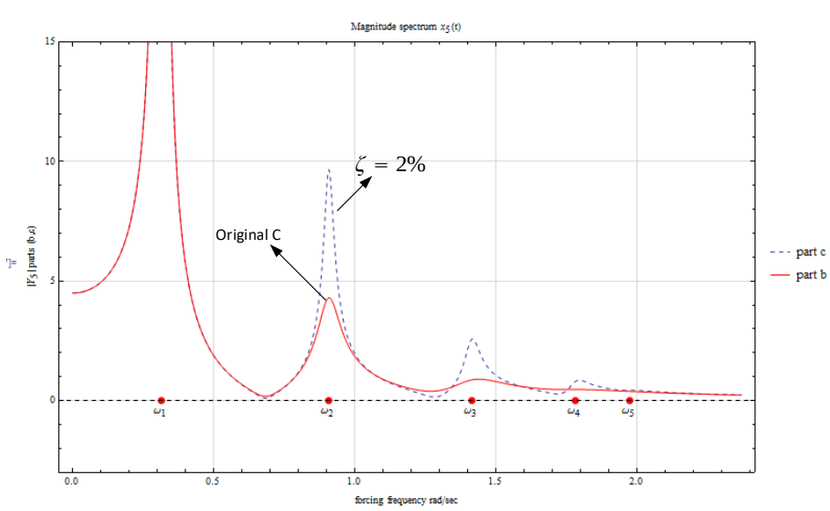
Therefore

$$x_5 = \mathrm{Re}\left(Y_5 e^{i\varpi t}\right)$$

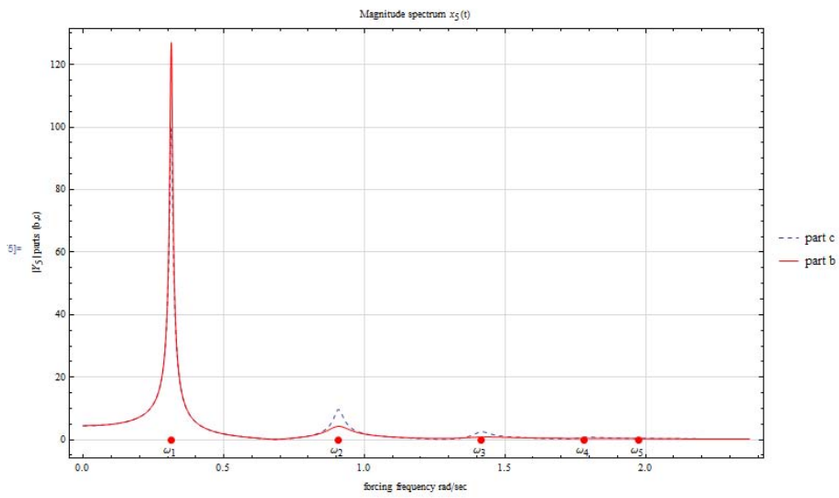
where

$$Y_5 = \frac{0.390\,25}{-\varpi^2 + 1.2516\times10^{-2}i\varpi + 0.0979} + \frac{0.317\,53}{-\varpi^2 + 0.036\,32i\varpi + 0.8244} + \frac{0.19999}{-\varpi^2 + 5.656\,8\times10^{-2}i\varpi + 2.0} + \frac{8.2426\times10^{-2}}{-\varpi^2 + 0.07128i\varpi + 3.176} + \frac{9.7812\times10^{-3}}{-\varpi^2 + 7.9016\times10^{-2}i\varpi + 3.902\,1}$$

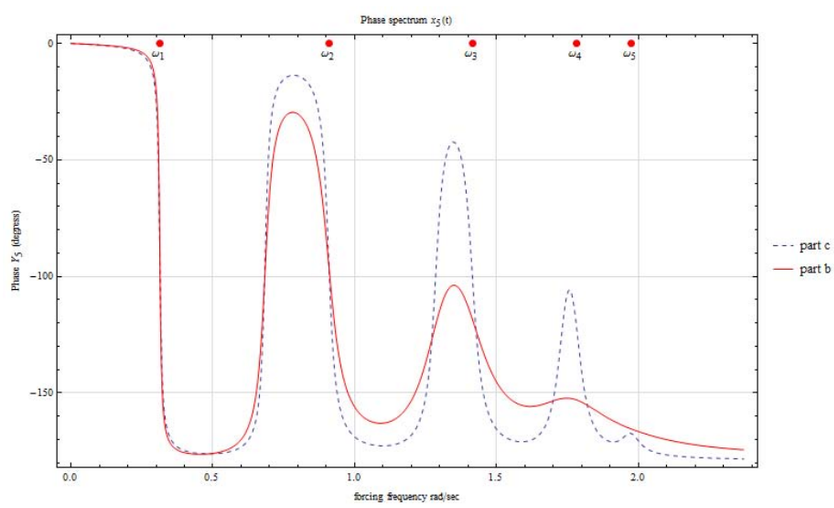
Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$ for both part(b) and (c) on the same plot



When $\zeta = 2\%$ was used, the resonance is seen to be higher (part c) compared to part (b). Here is a full range plot of the above.



Comparing the phase between part(b) and (c) gives



Which shows the effect on the phase spectrum.

2.13.3.4 Part (d)

In structural damping, the damping force is proportional to the elastic force. For example given an EOM $my'' + cy' + ky = f$, and converting to frequency domain to obtain transfer function

$$Y = \frac{F}{-\varpi^2 m + ic\varpi + k}$$

Then structural damping implies replacing $c\varpi$ with γk in the above, giving

$$Y = \frac{F}{-\varpi^2 m + i\gamma k + k} = \frac{F}{-\varpi^2 m + \left(1 + i\gamma\right)k}$$

The above method is now applied to the EOM given, and the resulting transfer function for x_5 is compared to the last results in order to see the effect of using structural damping

on the response. The eigenvalue problem was solved in part (a) where the result was

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 & -0.5635 & -0.6247 \\ -0.2871 & 0.6247 & -0.4472 & 0.0989 & 0.5635 \\ -0.4472 & 0.4472 & 0.4472 & 0.4472 & -0.4472 \\ -0.5635 & -0.0989 & 0.4472 & -0.6247 & 0.2871 \\ -0.6247 & -0.5635 & -0.4472 & 0.2871 & -0.0989 \end{bmatrix}$$

$$\omega = \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\}$$

Hence the modal EOM is now

$$[I]\begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \\ \eta_4'' \\ \eta_5'' \end{Bmatrix} + \left(1 + i\gamma\right)\begin{bmatrix} 0.0979 & 0 & 0 & 0 & 0 \\ 0 & 0.8244 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3.1756 & 0 \\ 0 & 0 & 0 & 0 & 3.9021 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \begin{Bmatrix} -0.6247F(t) \\ -0.5635F(t) \\ -0.4472F(t) \\ 0.2871F(t) \\ -0.0989F(t) \end{Bmatrix}$$

Hence the steady state solution now in modal coordinates is Hence the solution

$$\eta_j = \mathrm{Re}\left\{X_j e^{i\omega t}\right\}$$

where now

$$X_j = \frac{F_j}{-\omega^2 + \left(1 + i\gamma\right)\omega_j^2}$$

The solutions in modal coordinates are (where $\gamma = 0.04$)

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \mathrm{Re}\left(\begin{Bmatrix} \frac{-0.6247}{-\omega^2+\left(1+i\gamma\right)0.0979} \\ \frac{-0.5635}{-\omega^2+\left(1+i\gamma\right)0.8244} \\ \frac{-0.4472}{-\omega^2+\left(1+i\gamma\right)2} \\ \frac{0.2871}{-\omega^2+\left(1+i\gamma\right)3.176} \\ \frac{-0.0989}{-\omega^2+\left(1+i\gamma\right)3.9021} \end{Bmatrix} e^{i\omega t}\right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\left\{\eta\right\}$. Hence

$$\begin{aligned}
x_5 &= \sum_{j=1}^{5} \Phi\left(5, j\right)\eta(j) \\
&= \sum_{j=1}^{5} \Phi\left(5, j\right) \mathrm{Re}\left\{X(j)e^{i\omega t}\right\} \\
&= \mathrm{Re}\left(\sum_{j=1}^{5} \Phi\left(5, j\right) X(j)e^{i\omega t}\right) \\
&= \mathrm{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\omega t}\right] \\
&= \mathrm{Re}\left[\left(\begin{array}{c} \frac{0.390\,25}{-\omega^2+\left(1+i\gamma\right)0.0979} + \frac{0.317\,53}{-\omega^2+\left(1+i\gamma\right)0.8244} + \frac{0.199\,99}{-\omega^2+\left(1+i\gamma\right)2} + \\ \frac{8.242\,6\times10^{-2}}{-\omega^2+\left(1+i\gamma\right)3.176} + \frac{9.781\,2\times10^{-3}}{-\omega^2+\left(1+i\gamma\right)3.9021} \end{array}\right) e^{i\omega t}\right]
\end{aligned}$$

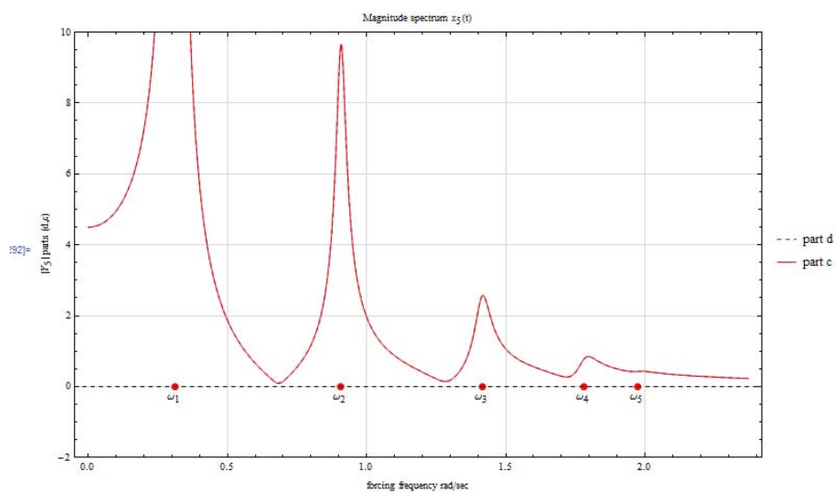
Therefore

$$x_5 = \mathrm{Re}\left(Y_5 e^{i\varpi t}\right)$$

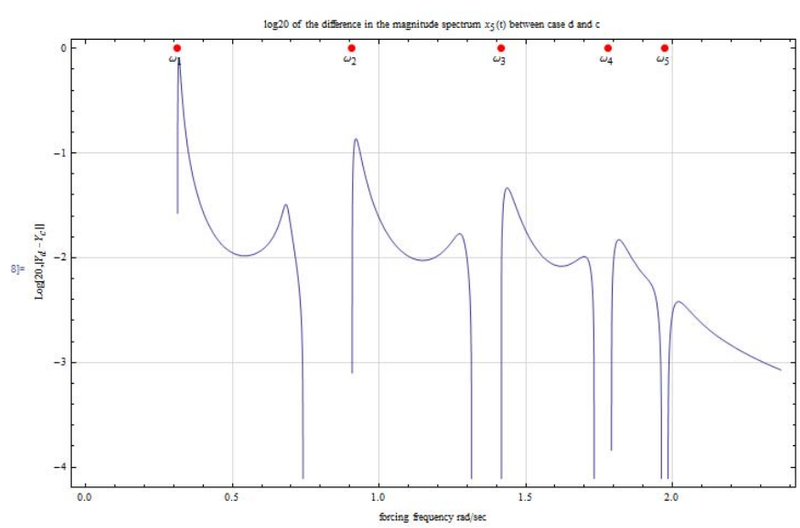
where

$$Y_5 = \frac{0.39025}{-\varpi^2 + \left(1+i\gamma\right)0.0979} + \frac{0.31753}{-\varpi^2 + \left(1+i\gamma\right)0.8244} + \frac{0.19999}{-\varpi^2 + \left(1+i\gamma\right)2} + \frac{8.2426\times 10^{-2}}{-\varpi^2 + \left(1+i\gamma\right)3.176} + \frac{9.7812\times 10^{-3}}{-\varpi^2 + \left(1+i\gamma\right)3.9021}$$

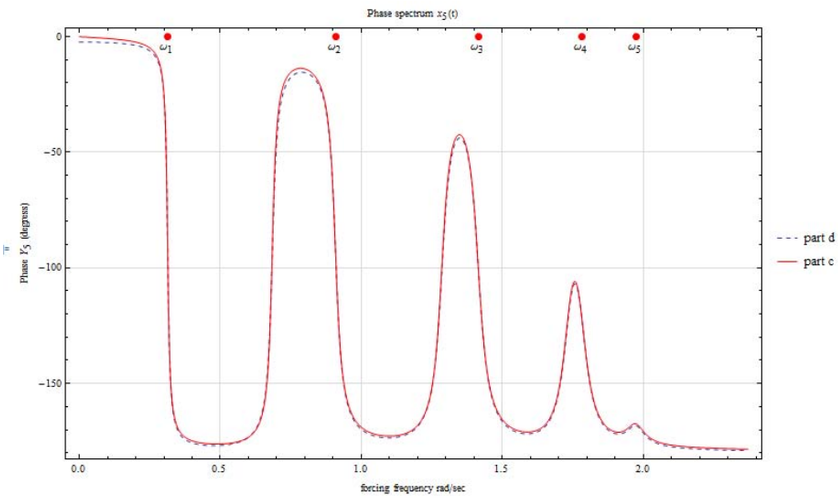
Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$ using the above transfer function, and superimposed on top of part (c). The magnitude spectrum is identical and $\boxed{\text{no difference can be seen}}$. Looking the phase spectrum there is very small change. Here are the plots. In the following plot, part(d) and (c) can not be distinguished. (the x-axis is drawn using dashed as well, not to be confused with the actual response curve).



To better see the difference, the plot was reproduced by taking the difference of the absolute values from part(d) and part(c) and plotting the log to base 20 of this difference. Now the difference can be better seen as very small.



The following the phase difference between case d and c.



The above plots show that using structural damping instead of using the same value of ζ for each EOM made very little difference in the result.

2.13.4 Problem 3

2.13.4.1 part(a)

Given $u(x,t)$ equations 6.1.1 and 6.1.2 in the text are

$$T_{bar} = \frac{1}{2}\int_0^L \dot{u}^2 \rho A dx \tag{2.115}$$

$$V_{bar} = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{2.116}$$

To obtain the mass matrix components T_{bar} is evaluated and each set of quadratic term are used to generate M_{jn} as follows. Using Ritz method, Let $u(x,t) = \sum_{j=1}^{N}\Psi_j(x)q_j(t)$. Substituting this in Eq 2.115 gives

$$\begin{aligned}
T_{bar} &= \frac{1}{2}\int_0^L\left(\frac{\partial}{\partial t}\sum_{j=1}^{N}\Psi_j(x)q_j(t)\right)^2\rho A dx = \frac{1}{2}\int_0^L\left(\sum_{j=1}^{N}\Psi_j(x)q_j'(t)\right)^2\rho A dx\\
&= \frac{1}{2}\int_0^L\left(\sum_{j=1}^{N}\Psi_j(x)q_j'(t)\right)\left(\sum_{n=1}^{N}\Psi_n(x)q_n'(t)\right)\rho A dx\\
&= \frac{1}{2}\int_0^L\left(\sum_{j=1}^{N}\sum_{n=1}^{N}\Psi_j(x)\Psi_n(x)q_j'(t)q_n'(t)\right)\rho A dx
\end{aligned}$$

Replacing order or integration with summation (since both are linear operations) and moving $q_j'(t)q_n'(t)$ outside the integration since it does not depend on x results in

$$T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}\left(\int_0^L \Psi_j(x)\Psi_n(x)\rho A dx\right)q_j'(t)q_n'(t) \tag{2.117}$$

Let

$$M_{jn} = \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx$$

Then eq 2.117 becomes Eq 6.1.11 in the textbook

$$T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}M_{jn}q_j'(t)q_n'(t) \tag{2.118}$$

Now, obtain the components of the stiffness matrix. Starting with eq 2.116 and replacing $u(x,t)$ in this equation gives

$$\begin{aligned}
V_{bar} &= \frac{1}{2}\int_0^L EA\left(\frac{\partial}{\partial x}\sum_{j=1}^{N}\Psi_j(x)q_j(t)\right)^2 dx = \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\frac{d\Psi_j(x)}{dx}q_j(t)\right)^2 dx \\
&= \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\frac{d\Psi_j(x)}{dx}q_j(t)\right)\left(\sum_{n=1}^{N}\frac{d\Psi_n(x)}{dx}q_n(t)\right)dx \\
&= \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\sum_{n=1}^{N}\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}q_j(t)q_n(t)\right)dx
\end{aligned}$$

Replacing order of integration with summation and moving $q_j(t)q_n(t)$ outside the integration since it does not depend on x gives

$$V_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}\left(\int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx\right)q_j(t)q_n(t)$$

Let $K_{jn} = \int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$ then the above becomes

$$V_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}K_{jn}q_jq_n$$

Which is eq 6.1.13 in the book. QED.

2.13.4.2 Part(b)

The basic function to use are $\Psi_1 = \frac{x}{L}, \Psi_2 = \left(\frac{x}{L}\right)^2, \Psi_3 = \left(\frac{x}{L}\right)^3$. Let $u(x,t) = \sum_{j=1}^{3}\Psi_j(x)q_j(t)$. Now eq 6.1.11 and eq. 6.1.13 are used to obtain the mass matrix and the stiffness matrix components based on the power balance method. $T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}M_{jn}q_j'(t)q_n'(t)$ where $M_{jn} = \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx$ hence

$$\begin{aligned}
M_{jn} &= \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx \\
&= \int_0^L \left(\frac{x}{L}\right)^j\left(\frac{x}{L}\right)^n \rho A dx \\
&= \int_0^L \left(\frac{x}{L}\right)^{j+n} \rho A dx \\
&= \frac{\rho A}{L^{j+n}}\int_0^L x^{j+n}dx \\
&= \frac{\rho A}{L^{j+n}}\left[\frac{x^{j+n+1}}{j+n+1}\right]_0^L = \frac{\rho A}{(j+n+1)L^{j+n}}L^{j+n+1} \\
&= \frac{\rho AL}{j+n+1}
\end{aligned}$$

Therefore, the mass matrix is

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \rho AL\begin{bmatrix} \frac{1}{1+1+1} & \frac{1}{1+2+1} & \frac{1}{1+3+1} \\ \frac{1}{2+1+1} & \frac{1}{2+2+1} & \frac{1}{2+3+1} \\ \frac{1}{3+1+1} & \frac{1}{3+2+1} & \frac{1}{3+3+1} \end{bmatrix} = \rho AL\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

and $\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}K_{jn}q_jq_n$ where $K_{jn}=\int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$, hence

$$\begin{aligned}
K_{jn} &= \int_0^L EA\frac{d\left(\frac{x}{L}\right)^j}{dx}\frac{d\left(\frac{x}{L}\right)^n}{dx}dx = \int_0^L EAj\left(\frac{x^{j-1}}{L^j}\right)n\left(\frac{x^{n-1}}{L^n}\right)dx\\
&= \frac{EAjn}{L^jL^n}\int_0^L x^{j-1}x^{n-1}dx\\
&= \frac{EAjn}{L^{j+n}}\int_0^L x^{j+n-2}dx\\
&= \frac{EAjn}{L^{j+n}}\left[\frac{x^{j+n-1}}{j+n-1}\right]_0^L\\
&= \frac{EAjn}{\left(j+n-1\right)L^{j+n}}\left[x^{j+n-1}\right]_0^L\\
&= \frac{EAjn}{\left(j+n-1\right)L^{j+n}}L^{j+n-1}\\
&= \frac{EA}{L}\frac{jn}{j+n-1}
\end{aligned}$$

Hence the stiffness matrix is

$$\begin{aligned}
K &= \begin{bmatrix} K_{11} & K_{12} & K_{13}\\ K_{21} & K_{22} & K_{23}\\ K_{31} & K_{32} & K_{33}\end{bmatrix} = \frac{EA}{L}\begin{bmatrix} \frac{1(1)}{1+1-1} & \frac{1(2)}{1+2-1} & \frac{1(3)}{1+3-1}\\ \frac{2(1)}{2+1-1} & \frac{2(2)}{2+2-1} & \frac{2(3)}{2+3-1}\\ \frac{3(1)}{3+1-1} & \frac{3(2)}{3+2-1} & \frac{3(3)}{3+3-1}\end{bmatrix}\\
&= \frac{EA}{L}\begin{bmatrix} 1 & 1 & 1\\ 1 & \frac{4}{3} & \frac{3}{2}\\ 1 & \frac{3}{2} & \frac{9}{5}\end{bmatrix}
\end{aligned}$$

2.13.4.3 Part(c)

The basic function to use are $\Psi_r=\sin\left(\alpha_r\frac{x}{L}\right)$ where $\alpha_r=\left(\frac{2r-1}{2}\right)\pi$ for $r=1,2,3$.

Let $u(x,t)=\sum_{j=1}^{3}\Psi_j(x)q_j(t)$. Now eq 6.1.11 and eq. 6.1.13 are used to obtain the mass matrix and the stiffness matrix components based on the power balance method. $T_{bar}=\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}M_{jn}q_j'(t)q_n'(t)$ where $M_{jn}=\int_0^L\Psi_j(x)\Psi_n(x)\rho Adx$ hence

$$\begin{aligned}
M_{jn} &= \int_0^L \Psi_j(x)\Psi_n(x)\rho Adx\\
&= \int_0^L \sin\left(\alpha_j\frac{x}{L}\right)\sin\left(\alpha_n\frac{x}{L}\right)\rho Adx\\
&= \int_0^L \sin\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L}\right)\sin\left(\left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right)\rho Adx
\end{aligned}$$

Using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ the above can be solved.

$$
\begin{aligned}
M_{jn} &= \int_0^L \frac{1}{2}\left(\cos\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L} - \left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right) - \cos\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L} + \left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right)\right)\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos\left(\frac{(2j-1)-(2n-1)}{2}\right)\pi\frac{x}{L} - \cos\left(\frac{(2j-1)+(2n-1)}{2}\right)\pi\frac{x}{L}\right]\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos\left(\frac{2(j-n)}{2}\right)\pi\frac{x}{L} - \cos\left(\frac{2(j+n)-2}{2}\right)\pi\frac{x}{L}\right]\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos(j-n)\pi\frac{x}{L} - \cos((j+n)-1)\pi\frac{x}{L}\right]\rho A dx
\end{aligned}
$$

For $j = 1, n = 1$ the above gives

$$
M_{jn} = \int_0^L \frac{1}{2}\left[1 - \cos\pi\frac{x}{L}\right]\rho A dx = \frac{\rho A}{2}\int_0^L 1 - \cos\pi\frac{x}{L}dx = \frac{\rho A}{2}\left(L - \left(\frac{\sin\pi\frac{x}{L}}{\frac{\pi}{L}}\right)_0^L\right) = \frac{\rho A}{2}L
$$

For $j = 1, n = 2$

$$
\begin{aligned}
M_{jn} &= \int_0^L \frac{1}{2}\left[\cos\left(-\pi\frac{x}{L}\right) - \cos 2\pi\frac{x}{L}\right]\rho A dx = \frac{\rho A}{2}\int_0^L \cos\left(\pi\frac{x}{L}\right) - \cos\left(2\pi\frac{x}{L}\right)dx \\
&= \frac{\rho A}{2}\left[\left(\frac{\sin\left(\pi\frac{x}{L}\right)}{\frac{\pi}{L}}\right)_0^L - \left(\frac{\sin\left(2\pi\frac{x}{L}\right)}{\frac{2\pi}{L}}\right)_0^L\right] \\
&= \frac{\rho A}{2}[0-0] = 0
\end{aligned}
$$

The rest of the computation is now done using a small code below to generate the final mass and stiffness matrix

▪ problem 3 part c

```
2]:= Clear[L, x, α, M, ρ , A, Ψ, k];
     Ψ[n_] := Sin[α[n] x/L];
     α[n_] := (2 n - 1)/2 π;
     M[j_, n_] := ∫_0^L Ψ[j] Ψ[n] ρ A dx
     k[j_, n_] := ∫_0^L e A D[Ψ[j], x] D[Ψ[n], x] dx

7]:= MatrixForm @ Table[M[i, j], {i, 1, 3}, {j, 1, 3}]
```

MatrixForm=

$$
\begin{pmatrix} \frac{A L \rho}{2} & 0 & 0 \\ 0 & \frac{A L \rho}{2} & 0 \\ 0 & 0 & \frac{A L \rho}{2} \end{pmatrix}
$$

```
8]:= MatrixForm @ Table[k[i, j], {i, 1, 3}, {j, 1, 3}]
```

MatrixForm=

$$
\begin{pmatrix} \frac{A e \pi^2}{8 L} & 0 & 0 \\ 0 & \frac{9 A e \pi^2}{8 L} & 0 \\ 0 & 0 & \frac{25 A e \pi^2}{8 L} \end{pmatrix}
$$

Therefore, the mass matrix is

$$M = \frac{AL\rho}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

and $\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}K_{jn}q_jq_n$ where $K_{jn} = \int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$, hence

$$K_{jn} = \int_0^L EA\frac{d\left(\Psi_j\right)}{dx}\frac{d(\Psi_n)}{dx}dx$$

From the above code, the result is

$$K = \frac{AE\pi^2}{8L}\begin{bmatrix}1 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 25\end{bmatrix}$$

Using this set of basis functions produces mass and stiffness matrices that are already decoupled. This is good.

2.13.4.4 part (d)

The natural frequencies obtained in problem 2 were

$$problem2 \Rightarrow M = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}, K = \begin{bmatrix}3 & -1 & 0 & 0 & 0\\ -1 & 2 & -1 & 0 & 0\\ 0 & -1 & 2 & -1 & 0\\ 0 & 0 & -1 & 2 & -1\\ 0 & 0 & 0 & -1 & 1\end{bmatrix}$$

$$\begin{aligned}\omega &= (0.3129, 0.9080, 1.4142, 1.7820, 1.9754) \text{ rad/sec}\\ &= (0.0498, 0.1445, 0.225, 0.284, 0.314) \text{ hz}\end{aligned}$$

Now the eigenvalue problem $\det\left([k] - \omega^2[M]\right)$ is solved again using the mass and stiffness matrices in parts b,c above and the natural frequencies are compared with the above result from problem 2. Recall, the M and K from part b were

$$part(b) \Rightarrow M = \rho AL\begin{bmatrix}\frac{1}{3} & \frac{1}{4} & \frac{1}{5}\\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7}\end{bmatrix}, K = \frac{EA}{L}\begin{bmatrix}1 & 1 & 1\\ 1 & \frac{4}{3} & \frac{3}{2}\\ 1 & \frac{3}{2} & \frac{9}{5}\end{bmatrix}$$

$$part(c) \Rightarrow M = \frac{AL\rho}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, K = \frac{AE\pi^2}{8L}\begin{bmatrix}1 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 25\end{bmatrix}$$

First, a numerical values given at end of problem 2 are used, therefore $\rho AL = m = 1$ and

$\frac{EA}{L} = \frac{1}{2}$, hence the K and M for part(b) and c become

$$part(b) \Rightarrow M = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, K = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}$$

$$part(c) \Rightarrow M = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, K = \frac{\pi^2}{16}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

The natural frequencies are found. Here is a summary table

	ω (rad/sec)	f hz
problem 2	$0.3129, 0.9080, 1.4142, 1.7820, 1.9754$	$0.0498, 0.1445, 0.225, 0.284, 0.314$
part(b)	$1.1108, 3.4199, 7.3872$	$0.1768, 0.5443, 1.1757$
part(c)	$1.1107, 3.3322, 5.5536$	$0.1768, 0.5303, 0.8839$

It can be seen that the first three natural frequencies using Ritz basic functions as given for both part b and c are higher than the natural frequencies generated by part b.

The stiffness matrix K for both parts b and c contains much smaller numerical values than the one used in problem 2. Since $\omega^2 = \frac{k}{m}$ then one expects this result.

2.13.4.5 Part(e)

The first 3 mode shapes from problem 2 were

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 \\ -0.2871 & 0.6247 & -0.4472 \\ -0.4472 & 0.4472 & 0.4472 \\ -0.5635 & -0.0989 & 0.4472 \\ -0.6247 & -0.5635 & -0.4472 \end{bmatrix}$$

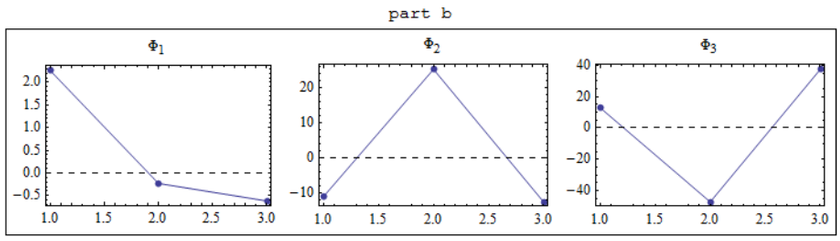
The mode shapes from part(b)

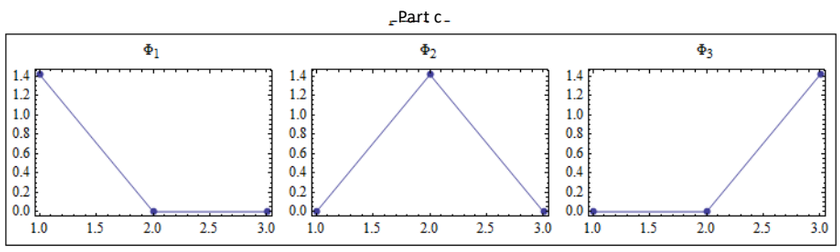
$$[\Phi] = \begin{bmatrix} 2.2642 & -11.2099 & 13.0082 \\ -0.2314 & 25.3536 & -47.2984 \\ -0.6181 & -12.7003 & 37.5941 \end{bmatrix}$$

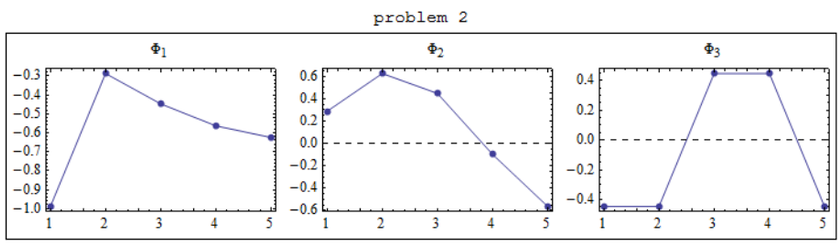
The mode shapes from part(c)

$$[\Phi] = \begin{bmatrix} 1.4142 & 0 & 0 \\ 0 & 1.4142 & 0 \\ 0 & 0 & 1.4142 \end{bmatrix}$$

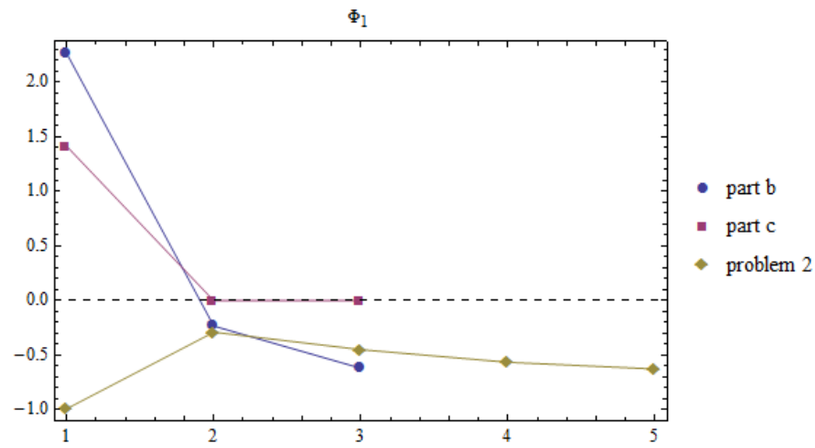
Here is a plot of the above mode shapes

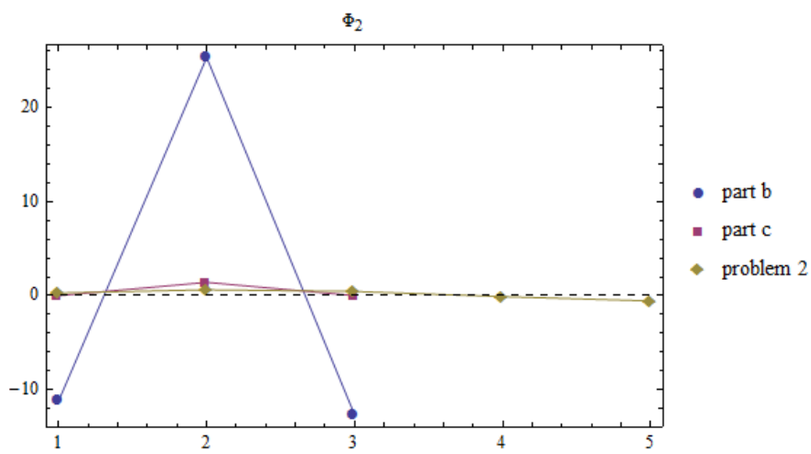


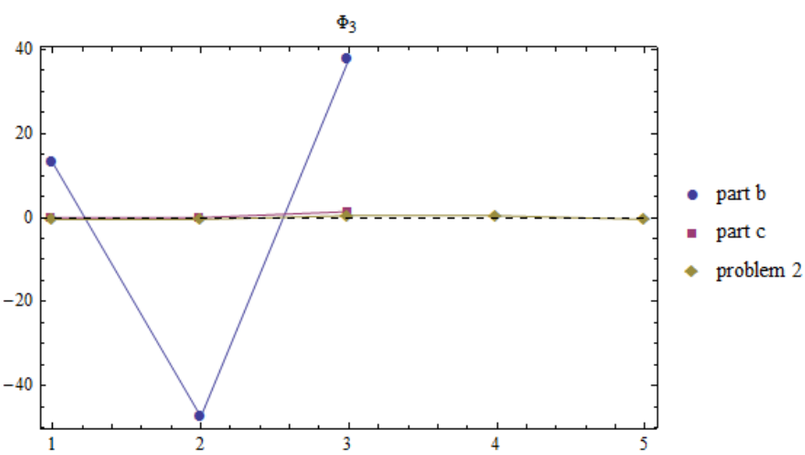




Here is a plot of the mode shapes overlay.


Φ_1
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
1
2
3
4
5
part b
part c
problem 2


Φ_2
20
10
0
−10
1
2
3
4
5
part b
part c
problem 2


Φ_3
40
20
0
−20
−40
1
2
3
4
5
part b
part c
problem 2

2.13.5 Key solution for HW 11

Homework #11
EMA 545, Spring 2013

Problem 1.)
Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{Q\}$$

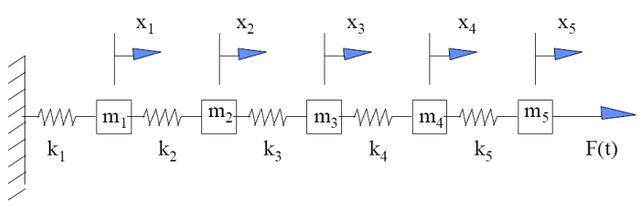
with [C] matrix given by:

$$[C]=[M][\Phi]\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}[\Phi]^T[M].$$

If we transform to normal coordinates using $\{q\}=[\Phi]\{\eta\}$, show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

Problem 2.)
The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs k_i which are drawn, there are viscous dampers c_i which are not shown. The equations of motion for this system are easily found to be:

$$[m]\underline{\ddot{x}}+[c]\underline{\dot{x}}+[k]\underline{x}=\underline{F}$$

where [m] is a diagonal matrix having entries m_i, and [k] is the banded matrix:

$$\begin{bmatrix} (k1+k2) & -k2 & 0 & 0 & 0 \\ -k2 & (k2+k3) & -k3 & 0 & 0 \\ 0 & -k3 & (k3+k4) & -k4 & 0 \\ 0 & 0 & -k4 & (k4+k5) & -k5 \\ 0 & 0 & 0 & -k5 & k5 \end{bmatrix}$$

The damping matrix has the same form as [k], but with viscous damping coefficients c_i in place of stiffness coefficients k_i. The forcing vector $\underline{F}$ is a 5 by 1 vector of zeros, except for the last entry: $\underline{F}=[0 \quad 0 \quad 0 \quad 0 \quad F(t)]^T$.

The numerical value for k_2 through k_5 is 1 N/m, the numerical value of k_1 = 2 N/m, the numerical value for m_1 through m_5 is 1 kg, and the numerical values for the viscous damping coefficients $c_i = 0.1*k_i$, i=1,5 (in units of N-s/m).

(a) Find the natural frequencies and mass-normalized modes of the system.
(b) Find the magnitude and phase of the steady-state response $x_5(t)$ assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to $1.2*\omega_5$. Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
(c) Repeat the analysis in (b), but use the strategy described in Problem 1 to create a [C] matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
(d) Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of γ=0.04.

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters EA and L are as follows: $k_i = N*EA/L$, i=2,N where N is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high, $k_1 = 2N*EA/L$. The lumped masses are equal to the total bar mass divided by N: $m_i = \rho AL/N$, where ρ is the mass density of the bar.

Problem 1) (Creating a damping matrix with any desired modal damping ratios.)

The goal is to show that one can use the modal transformation together with the given damping matrix to obtain uncoupled equations of motion.
We need to show that $[\Phi]^T[C][\Phi]$ is diagonal:

$$[\Phi]^T[C][\Phi]=[\Phi]^T[M][\Phi]\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}[\Phi]^T[M][\Phi]$$

Since $[\Phi]^T[M][\Phi]=[I]$, we have the desired result:

$$[\Phi]^T[C][\Phi]=\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}$$

Now, starting with the coupled equations of motion:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{Q\}$$

Substituting $\{x\}=[\Phi]\{\eta\}$ and pre-multiplying by $[\Phi]^T$ yields

$$[\Phi]^T[M][\Phi]\{\ddot{\eta}\}+[\Phi]^T[C][\Phi]\{\dot{\eta}\}+[\Phi]^T[K][\Phi]\{\eta\}=[\Phi]^T\{Q\}=\{\hat{Q}\}$$

Using the fact that $[\Phi]^T[M][\Phi]=[I]$ and $[\Phi]^T[K][\Phi]$ is a diagonal matrix containing the square of the natural frequencies, we now have N uncoupled equations of motion.

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

Note that we have also used the fact that each row of $[\Phi]^T\{Q\}$ is equal to the *j*th mode vector transposed times $\{Q\}$.

HW 12, Problem 2: Lumped mass approximation for a bar in extension

Part (a):
The mass and stiffness matrices are given, so it is easy to find the natural frequencies and mode shapes in Matlab. Here is the result: (Matlab code given at the end.)

```
wns =
      0.31287
      0.90798
       1.4142
        1.782
       1.9754
PHI =
    -0.098938      0.28713     -0.44721     -0.56352     -0.62467
     -0.28713      0.62467     -0.44721     0.098938      0.56352
     -0.44721      0.44721      0.44721      0.44721     -0.44721
     -0.56352    -0.098938      0.44721     -0.62467      0.28713
     -0.62467     -0.56352     -0.44721      0.28713    -0.098938
```

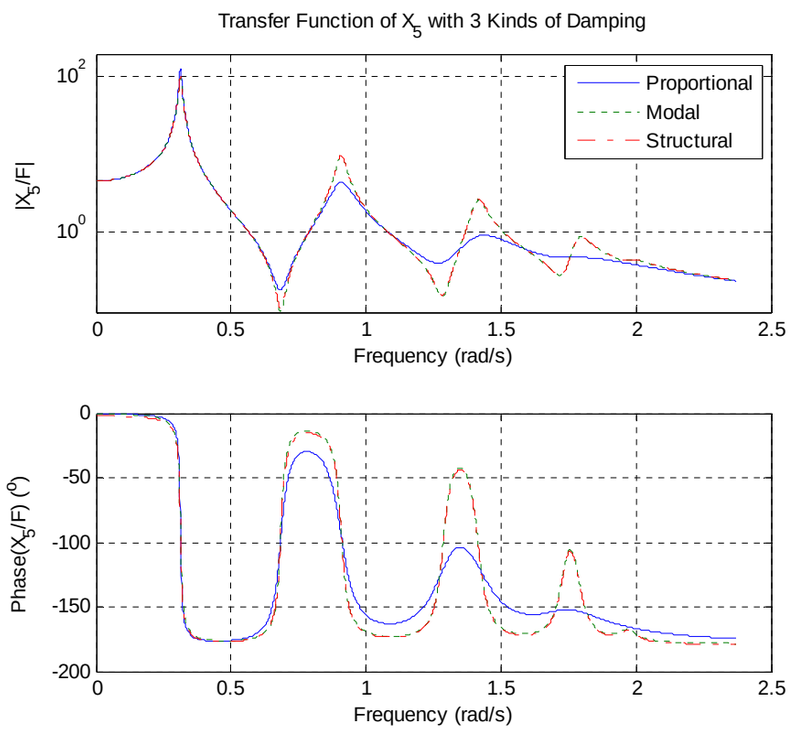
Part (b):
The input is harmonic, so the steady-state response of each mass will also be harmonic. Their complex amplitudes can be found with the equation: inv([K+iωC-ω^2*M]). See the Matlab code for details. The complex amplitude of the 5th DOF is plotted below. Note that this stiffness proportional damping approach gives the following modal damping ratios, which are clearly different than those for part (c) below:

zts =
0.015643
0.045399
0.070711
0.089101
0.098769

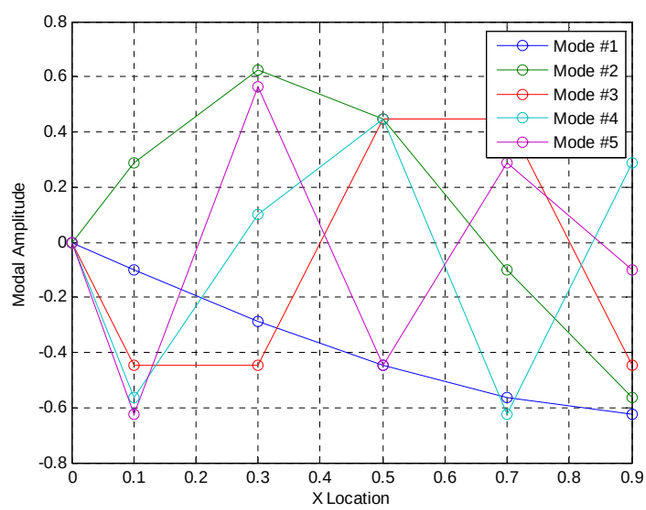
Part (c):
The solution here is the same as for (b), only now we use `C = M*PHI*2*0.02*diag(wns)*PHI.'*M`
Both solutions are plotted below.

Part(d): (not included in Spring 2011)
For the structural damping case, K = K*(1+iγ).

The plot shows that the stiffness proportional damping approach gives heavier damping for the higher frequency modes.



The mode shapes were requested as part of the next problem. They are plotted below:



Matlab Code:

```
% ME 6442 Homework #10, MDOF Systems-LUMPED MASS APPROX TO BAR IN EXTENSION
clear all; close all

M = eye(5); % Identity matrix since each mj=1;
K = eye(5)*2+diag([-1 -1 -1 -1],1)+diag([-1 -1 -1 -1],-1);
    % Above is a fancy way to make the banded stiffness matrix, although
    % the two terms below must be fixed manually.
K(1,1) = 1+2; K(5,5) = 1;
C = 0.1*K;

[phi,lam] = eig(K,M);
wns = sqrt(diag(lam))

% Normalize Eigenvectors
[natfreqs,Isort]=sort(sqrt(diag(lam))); % sort by nat freq
phi=phi(:,Isort);  % sort eigenvectors
scale= phi.'*M*phi;
PHI=real(phi*inv(sqrt(scale))) % normalize e.vectors
zts = (diag(PHI.'*C*PHI)/2)./wns

% Check Orthogonality
check_m = norm(PHI.'*M*PHI-eye(size(PHI)))
check_k = norm(PHI.'*K*PHI-lam)

% Plot Mode Shapes
% Create x-vector.  Remember that the masses are at the center of each
% element, so the first node is at 0.5*(L/N)
xs = [0, 0.5:1:4.5]/5;
    % Will have to stack zeros above PHI below for the displacement at x=0.
figure(1)
plot(xs.',[0 0 0 0 0; PHI],'o-'); grid on;
xlabel('X Location'); ylabel('Modal Amplitude');
legend('Mode #1', 'Mode #2','Mode #3','Mode #4','Mode #5');

ws = [0:max(wns)*1.2/2000:max(wns)*1.2];
% TF Using Proportional C-Matrix
for ii = 1:1:length(ws);
    Gn(:,ii) = [K+i*ws(ii)*C-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gb = Gn(5,:);

% TF Using modal damping ratios
Cc = M*PHI*2*0.02*diag(wns)*PHI.'*M
for ii = 1:1:length(ws);
    Gn(:,ii) = [K+i*ws(ii)*Cc-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gc = Gn(5,:);

% TF Using Structural Damping
Kd = K*(1+i*0.04);
for ii = 1:1:length(ws);
    Gn(:,ii) = [Kd-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gd = Gn(5,:);

figure(2)
subplot(211);
semilogy(ws,abs(Gb),ws,abs(Gc),':',ws,abs(Gd),'-.'); grid on;
xlabel('Frequency (rad/s)'); ylabel('|X_5/F|');
legend('Proportional','Modal','Structural');
title('Transfer Function of X_5 with 3 Kinds of Damping');
axis([0 2.5 0.09 200]);
```

```
subplot(212);
plot(ws,angle(Gb)*180/pi,ws,angle(Gc)*180/pi,':',ws,angle(Gd)*180/pi,'-.');
grid on;
xlabel('Frequency (rad/s)'); ylabel('Phase(X_5/F) (^o)');
```

Parts (a & b):

First, note that since k_2 in Problem 3 is 1 N/m, this suggests EA/L = 1/5 N/m. Also, since m_1 = 1 kg, ρAL = 5 kg. The kinetic energy of the rod is given by:

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{4.1}$$

We use the following Ritz series to approximate the displacement firld for the rod:

$$u(x,t) = \sum_{i=1}^{3} q_i(t)\psi_i(x) \tag{4.2}$$

Substitution of (4.2) into (4.1) yields:

$$T = \frac{1}{2}\int_0^L \rho A\left(\sum_{i=1}^{3}\dot{q}_i(t)\psi_i(x)\right)\left(\sum_{j=1}^{3}\dot{q}_j(t)\psi_j(x)\right)dx \tag{4.3}$$

$$T = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\dot{q}_i\dot{q}_j\int_0^L \rho A\psi_i\,\psi_j\;dx \equiv \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\dot{q}_i\dot{q}_j m_{ij} \tag{4.4}$$

where the elements of the mass matrix are identified to be:

$$m_{ij} = \int_0^L \rho A\left(\frac{x}{L}\right)^i\left(\frac{x}{L}\right)^j dx = \frac{\rho A}{L^{i+j}}\int_0^L x^{i+j}dx = \rho AL\frac{1}{i+j+1} \tag{4.5}$$

The potential energy of the rod is given by

$$V = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{4.6}$$

The derivative of the displacement field (4.2) with respect to x is given by:

$$\frac{\partial u}{\partial x}(x,t) = \sum_{i=1}^{3} q_i(t)\frac{\partial\psi_i(x)}{\partial x} \tag{4.7}$$

Substitution of (4.7) into (4.6) yields:

$$V = \frac{1}{2}\int_0^L EA\left(\sum_{i=1}^{3} q_i(t)\frac{\partial\psi_i}{\partial x}\right)\left(\sum_{j=1}^{3} q_j(t)\frac{\partial\psi_j}{\partial x}\right)dx \tag{4.8}$$

$$V = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} q_i q_j \int_0^L EA \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x}\, dx \equiv \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} q_i q_j k_{ij} \tag{4.9}$$

where the elements of the stiffness matrix are identified to be:

$$k_{ij} = \int_0^L EA\left(\frac{i}{L^i}\right)\left(\frac{j}{L^j}\right) x^{i+j-2}\, dx = \frac{EA\, i\, j}{L^{i+j}}\int_0^L x^{i+j-2} dx = \frac{EA}{L}\frac{i\, j}{i+j-1} \tag{4.10}$$

The three-term Ritz approximation for the rod is given by the following 3DOF system:

$$\rho AL \begin{bmatrix} 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \\ 1/5 & 1/6 & 1/7 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \frac{EA}{L}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4/3 & 6/4 \\ 1 & 6/4 & 9/5 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4.11}$$

Note that the dimensional factors can be pulled out so that the eigenvalue problem can be solved in Matlab (for part (d)). To do so, we define:

$$\lambda^2 = \omega^2 \frac{\rho L^2}{E}$$

Part (c):

The integrals are not convenient to evaluate analytically using these basis functions, so the solution is carried out only in Matlab. The resulting matrices are:

```
>> M
M =
          0.5            0            0
            0          0.5            0
            0            0          0.5
>> K
K =
       1.2337            0            0
            0       11.103            0
            0            0       30.843
```

Notice that the matrices are diagonal. This occurs because the chosen basis functions happen to be the eigenfunctions for this continuous system. So, the coordinate governing each basis function is independent of all of the others and the natural frequencies can be found by inspection, for example: ω_1=sqrt(1.2337/0.5)=1.5708, etc…

Part (d):

COMPARISON OF NATURAL FREQUENCIES:

First we observe that since ρAL/(EA/L) = 25, we must multiply the natural frequencies found in Problem 2 by (25)^(1/2) to compare with the results from the nondimensional Ritz analysis. The results are summarized in the table below, where the natural frequencies for other Ritz Series lengths are also shown FYI. Notice that as the series length increases, new modes appear at higher frequencies, and the lower natural frequencies decrease slightly, converging towards the true values. The lumped parameter method in Problem 2 gives similar results although the frequencies are sometimes lower than the true values, while the Ritz method always over predicts the natural frequencies.

Ritz Series Length	Natural Frequencies (nondimensional)				
	ω1	ω2	ω3	ω4	ω5
2	1.5767	5.6728	-	-	-
3	1.5709	4.8365	10.4471	-	-
4	1.5708	4.7246	8.3309	16.3036	-
5	1.5708	4.7132	7.9390	12.1739	23.3614
Results for Lumped Parameter Approximation with N=5 below					
N=5	1.5643	4.539	7.0711	8.9101	9.8769

With the second set of basis functions from part (c), one obtains the true analytical natural frequency for any length Ritz Series.

Ritz Series Length	Natural Frequencies (nondimensional)				
	ω1	ω2	ω3	ω4	ω5
2	1.5708	4.7124	-	-	-
3	1.5708	4.7124	7.8540	-	-
4	1.5708	4.7124	7.8540	10.9956	-
5	1.5708	4.7124	7.8540	10.9956	14.1372

MODE SHAPES

Using the polynomials in part (b):

```
PHI =

   -2.2642  -11.2099  -13.0082
    0.2314   25.3536   47.2984
    0.6181  -12.7003  -37.5941
```

Each column gives the proportion of each Ritz vector in the approximation for the eigenfunction. For example, the approximation for the first modeshape is given by:

$$-2.2642\left(\frac{x}{L}\right)+0.2314\left(\frac{x}{L}\right)^2+0.6181\left(\frac{x}{L}\right)^3$$

The second eigenfunction is approximated by:

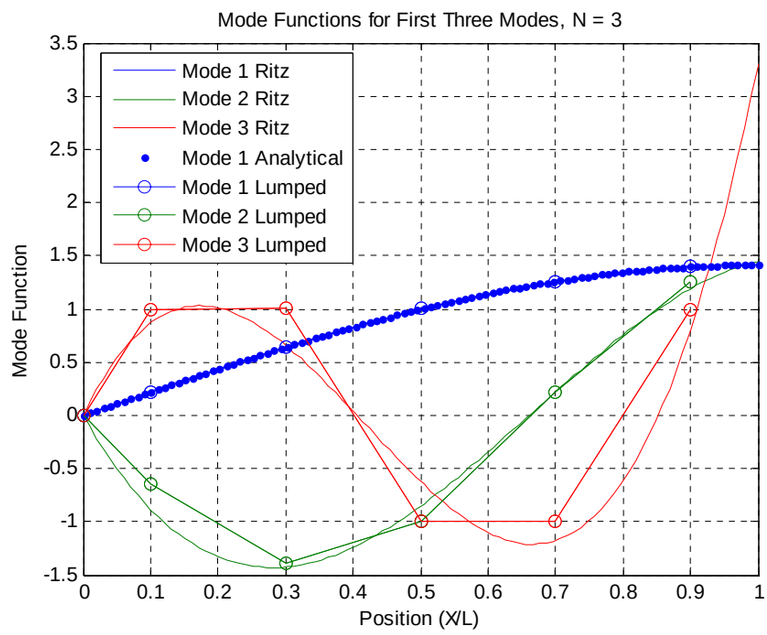
$$-11.2099\left(\frac{x}{L}\right)+25.3536\left(\frac{x}{L}\right)^2-12.7003\left(\frac{x}{L}\right)^3$$

The third eigenfunction is approximated by:

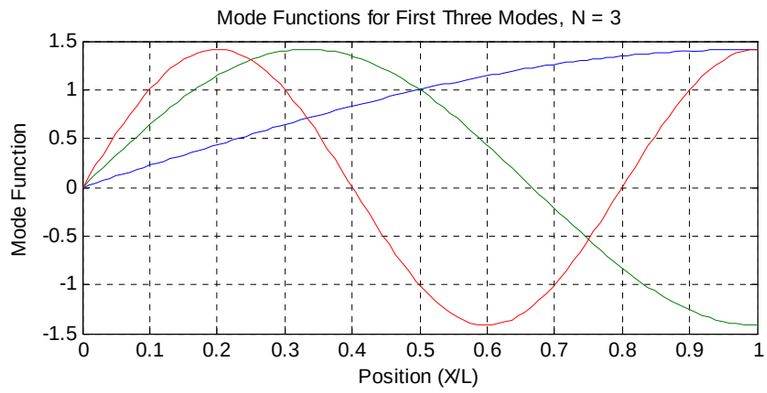
$$-13.0082\left(\frac{x}{L}\right)+47.2984\left(\frac{x}{L}\right)^2-37.5941\left(\frac{x}{L}\right)^3$$

The three modeshapes obtained from the Ritz method are shown in Figure 4.1 It is seen that the first mode closely resembles a "1/4 sine," which is the exact modeshape for a fixed-free rod. The second mode resembles a "3/4 sine." The theoretical prediction of the third mode is a "5/4 sine." It is seen that there is considerable inaccuracy in the Ritz-method prediction for this mode. The Matlab code given at the back of the solution set gives a concise way of finding these approximate modeshapes given the modal matrix PHI.

The three mode shapes obtained from the Ritz method for part (b) are shown below. The first mode closely resembles a “1/4 sine,” which is the exact mode shape for a fixed-free rod. The second resembles a “3/4 sine.” The theoretical prediction of the third mode is a “5/4 sine,” but one can see that there is considerable error in the Ritz approximation for that mode since the basis is inadequate to describe it.



When using the basis functions from part (c), the modes are exact and are given in the figure below.



The Matlab code used for these calculations follows. Symbolic variables were used to check the answer for part c), and to compute the M and K matrices for part d).

Part b.)

```
% EMA 545, HW12
% Ritz Series solution for Clamped-free rod in extension
%
% M.S. Allen, May 2011

clear all; close all
syms x jj kk real;
tic
N = 5;

% Create basis functions as symbolic functions in Matlab - Symbolic
% variables x and jj and kk defined above
for jj = 1:N;
    psi(jj) = x^jj;
end
% Loop to create (jj,kk) terms of mass and stiffness matrices
for jj = 1:N;
    for kk = 1:jj;
        % Usig Matlab symbolics
        MM(jj,kk) = int(psi(jj)*psi(kk),x,0,1);% *rho*A*L
        KK(jj,kk) = int((diff(psi(jj),x,1)*diff(psi(kk),x,1)),x,0,1); % *EA/L
        % Using analytically derived formulas for M and K
        M1(jj,kk) = 1/(jj+kk+1);
        K1(jj,kk) = (jj*kk)/(jj+kk-1);
        % note - matrices are symmetric, so we can fill the rest of the
        % matrix in with the same terms.
        MM(kk,jj) = MM(jj,kk); KK(kk,jj) = KK(jj,kk);
        M1(kk,jj) = M1(jj,kk); K1(kk,jj) = K1(jj,kk);
    end
end
M = double(MM); % convert symbolic expressions to numbers.
C = 0; K = double(KK);
toc
% display the difference, which is on the order of numerical round off
% error.
M-M1
K-K1
```

```
% Use Embedding Property to find Eigensolutions for 2 <= N <= 5.  For
% example, the EVP for N = 3 uses the 3x3 matrix in the upper left quadrant
% of M and K.
for p = 2:1:N;
    [phi,lambda] = eig(K(1:p,1:p),M(1:p,1:p));
    wnd(p-1,1:p) = sqrt(diag(lambda))'
end

% Finding Mode Functions
Ns = [3]; % number of basis functions to use for mode shape plots.
delta = 1/(100-1); % spacing for y-axis on mode shapes.
yd = [0:delta:1]';
for q = 1:length(Ns);
    psi_vals = 0; phi = 0; lambda = 0;
    [phi lambda] = eig(K(1:Ns(q),1:Ns(q)),M(1:Ns(q),1:Ns(q)));
    wns = sqrt(diag(lambda));
    for p = 1:1:100;
        for n = 1:Ns(q);
            % evaluate each basis function 'n' at each point 'p'.  (or use
yd(p)^n)
            psi_vals(p,n) = subs(psi(n),'x',yd(p)); % Matlab symbolics - sub
yd(p) for 'y'
        end
    end
% mode shapes for plotting are psi_vals*phi
psi_c(1:100,1:Ns(q),q) = psi_vals(:,1:Ns(q))*phi;
end

% Analytical Mode Shape
psi_an = sin((2*1-1)/2*pi*yd);
psi_an = psi_an/max(psi_an)*max(psi_c(:,1,1)); % scale to same amplitude as
psi_c

% Plotting
figure(3)
plot(yd, psi_c(:,1,1),yd, psi_c(:,2,1), yd, psi_c(:,3,1),...
    yd, psi_an,'b.'); grid on;%, yd, psi_c(:,1,2), '.:', yd,
psi_c(:,3,2),'.:'); grid;
legend('Mode 1 Ritz','Mode 2 Ritz','Mode 3 Ritz','Mode 1 Analytical');
xlabel('Position (X/L)'); ylabel('Mode Function');
title(['Mode Functions for First Three Modes, N = ',num2str(Ns)]);

return
%% Plot the solution to Problem 2 on top:
M = eye(5); % Identity matrix since each mj=1;
K = eye(5)*2+diag([-1 -1 -1 -1],1)+diag([-1 -1 -1 -1],-1);
    % Above is a fancy way to make the banded stiffness matrix, although
    % the two terms below must be fixed manually.
K(1,1) = 1+2; K(5,5) = 1;
M = M/5; K = K*5; % change to non-dimensional

[phi,lam] = eig(K,M);
PHI=real(phi*inv(sqrt(phi.'*M*phi)));
xs = [0, 0.5:1:4.5]/5;

% To get a good plot, have to manually adjust the sign of some of the mode
% vectors (the sign of a mode vector is arbitrary).  Make all end values
% positive:
PHI = PHI*diag(sign(PHI(end,:)));
hold on; plot(xs.',[0 0 0; PHI(:,1:3)],'o-'); hold off;
legend('Mode 1 Ritz','Mode 2 Ritz','Mode 3 Ritz','Mode 1 Analytical','Mode 1
Lumped','Mode 2 Lumped','Mode 3 Lumped');
```

Part (c):

```
% EMA 545, HW12
% Ritz Series solution for Clamped-free rod in extension
%
% M.S. Allen, April 2008

clear all; close all
syms x jj kk real;
tic
N = 3;

% Create basis functions as symbolic functions in Matlab - Symbolic
% variables x and jj and kk defined above
for jj = 1:N;
    psi(jj) = sin((2*jj-1)/2*pi*x);
end
% Loop to create (jj,kk) terms of mass and stiffness matrices
for jj = 1:N;
    for kk = 1:jj;
        % Usig Matlab symbolics
        MM(jj,kk) = int(psi(jj)*psi(kk),x,0,1);% *rho*A*L
        KK(jj,kk) = int((diff(psi(jj),x,1)*diff(psi(kk),x,1)),x,0,1); % *EA/L
        % note - matrices are symmetric, so we can fill the rest of the
        % matrix in with the same terms.
        MM(kk,jj) = MM(jj,kk); KK(kk,jj) = KK(jj,kk);
    end
end
M = double(MM); % convert symbolic expressions to numbers.
C = 0; K = double(KK);
toc

% Use Embedding Property to find Eigensolutions for 2 <= N <= 5.  For
% example, the EVP for N = 3 uses the 3x3 matrix in the upper left quadrant
% of M and K.
for p = 2:1:N;
    [phi,lambda] = eig(K(1:p,1:p),M(1:p,1:p));
    wnd(p-1,1:p) = sqrt(diag(lambda))'
end

% Finding Mode Functions
Ns = [3]; % number of basis functions to use for mode shape plots.
delta = 1/(100-1); % spacing for y-axis on mode shapes.
yd = [0:delta:1]';
for q = 1:length(Ns);
    psi_vals = 0; phi = 0; lambda = 0;
    [phi lambda] = eig(K(1:Ns(q),1:Ns(q)),M(1:Ns(q),1:Ns(q)));
    wns = sqrt(diag(lambda));
    for p = 1:1:100;
        for n = 1:Ns(q);
            % evaluate each basis function 'n' at each point 'p'.  (or use
yd(p)^n)
            psi_vals(p,n) = subs(psi(n),'x',yd(p)); % Matlab symbolics - sub
yd(p) for 'y'
        end
    end
% mode shapes for plotting are psi_vals*phi
psi_c(1:100,1:Ns(q),q) = psi_vals(:,1:Ns(q))*phi;
end
```

Chapter 3

Design project

Local contents

3.1 Design project, team 4

Local contents

3.2 Introduction

By Nasser M. Abbasi, Donny Kuettel III and Paul Frisch.

This report outlines a simple passive vibration isolation system design for use in the first class cabin of a Boeing 757-200 airplane with the goal of reducing the vibrations felt by the passengers in the first class cabin. This was done by simulation in order to select suitable design parameters that produced an acceptable absolute acceleration time history compared the rest of the airplane during a turbulent flight.

3.2.1 Discussion and results

3.2.1.1 Notations used in the report

M	mass of first class cabin
k	spring constant
ζ	critical damping constant
r	ratio of external load frequency to the natural frequency of first class cabin $\frac{\omega}{\omega_{natural}}$
r_n	ratio of external load n^{th} harmonic frequency to the natural frequency of first class cabin $\frac{\omega_n}{\omega_{natural}}$
T_r	Transmissibility. The ratio of cabin absolute displacement to base absolute displacement
ω_{nat}	Natural frequency of first class cabin
ω_1	Fundamental frequency of the external load frequency.
EOM	Equation Of Motion
c	damping constant for damper under first class cabin
Z_n^{acc}	the complex amplitude of the term associated with the nth harmonic of the frequency $z''(t)$
Z_n^{disp}	the complex amplitude of the term associated with the nth harmonic of the displacement $z(t)$
Y_n	the complex amplitude of the term associated with the nth harmonic of the displacement of $y(t)$

Table 1. Description of mathematical notations used in report

3.2.1.2 Mathematical model

Reducing the vibration effect felt by the passengers in the first class cabin was based on reducing the transmissibility ratio (T_r) of the *absolute acceleration* of the airplane to

that of the first class cabin. A passive vibration isolation system was used for its ease of implementaion and its low cost. The model is based on figure 1 below

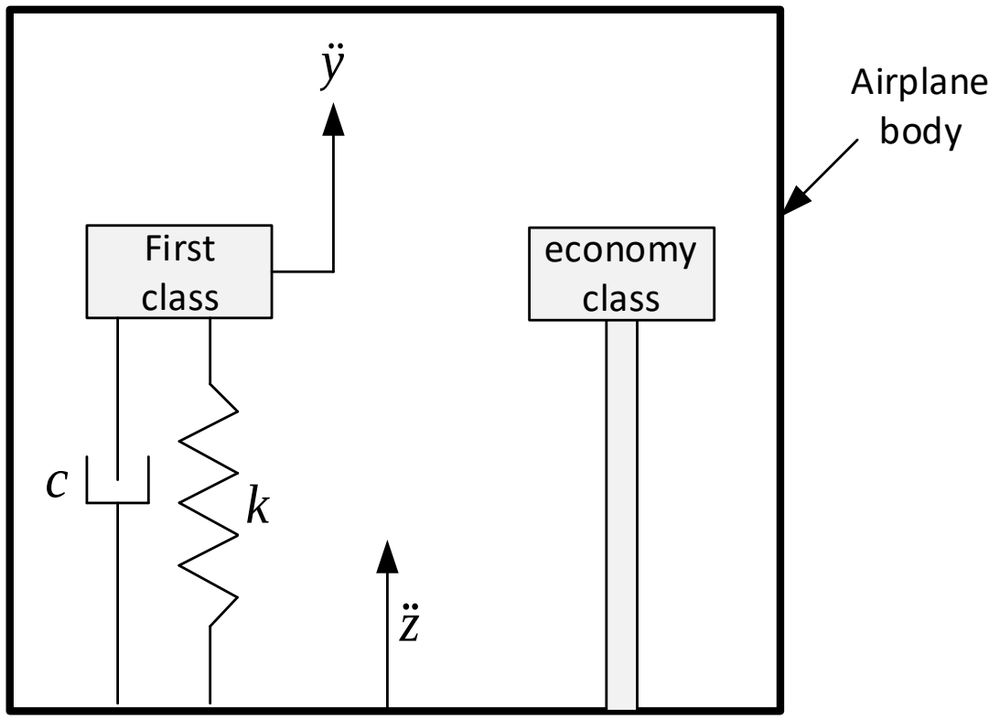


Figure 1. Mechanical model view of vibration isolation system in place.

The absolute acceleration of the first class cabin, $y''(t)$, was calculated with the vibration isolation system in place and then compared to the absolute acceleration, $z''(t)$, of the rest of the airplane. The goal was to produce a smooth absolute acceleration time history when compared to the rest of the airplane. This was done by adjusting M, ζ and K and running a simulation of the motion of the plane with our vibration isolation system in place. A plot of T_r vs. r was also used to insure that the maximum T_r remained small as the frequency ratio r was increased.

Assuming the mass of cabin is M, which includes the live load (passengers), then applying Newton's laws the the first class cabin results in the equation of motion

$$my'' + c\left(y' - z'\right) + k\left(y - z\right) = 0$$
$$my'' + cy' + ky = cz' + kz$$

The transfer function between $y(t)$ and $z(t)$ in the frequency domain can now be derived (Appendix contains complete derivation) resulting in

$$T(r) = \left|\frac{Y_n}{Z_n^{disp}}\right| = \frac{\sqrt{1 + (2\zeta r_n)^2}}{\sqrt{\left(1 - r_n^2\right)^2 + (2\zeta r_n)^2}}$$

To compare the absolute acceleration of the first class cabin with the rest of the airplane, the absolute acceleration, $y''(t)$, is now found from Y_n. Since $y(t) = \mathrm{Re}\left\{Y_n e^{i\varpi_n t}\right\}$ then $y''(t) = \mathrm{Re}\left\{-\varpi_n^2 Y_n e^{i\varpi_n t}\right\}$.

3.2.1.3 Design results

$z''(t)$ (given) and $y''(t)$ (computed) are now plotted on the same plot in order to compare the effect of our vibration isolation system to the comfort of the first class passengers. The final design parameters used are (Appendix 5.1)

M Mass of first class cabin (dead+live)	3050 kg
ζ	0.7
k	9700 N/M

Table 3. Final values of design parameters

Figure 2 below shows the result using the above parameters

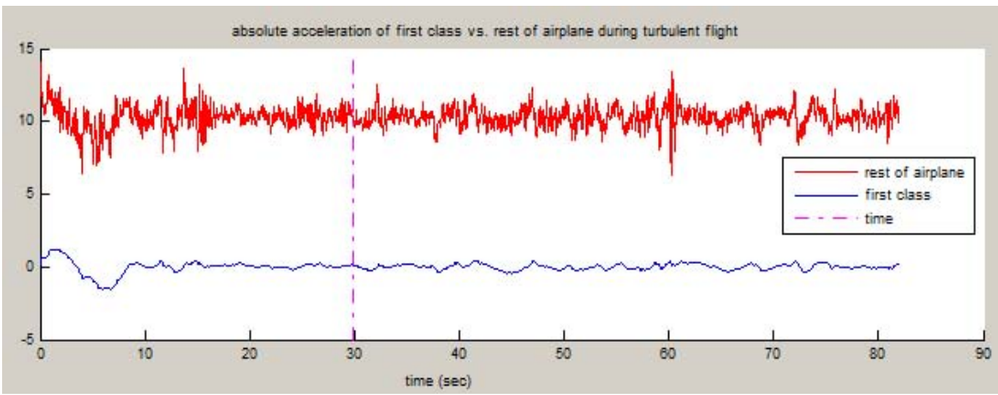


Figure 2. First class cabin absolute acceleration compared to rest of airplane.

We see from figure 2 that the absolute acceleration of the first class cabin has much less variation and is much smoother than the absolute acceleration of the rest of the airplane. From this we can see that the first class passengers experience a much more comfortable flight than the rest of the airplane. In addition, the transmissibility plot was found to be acceptable since T_r decreases with increasing r

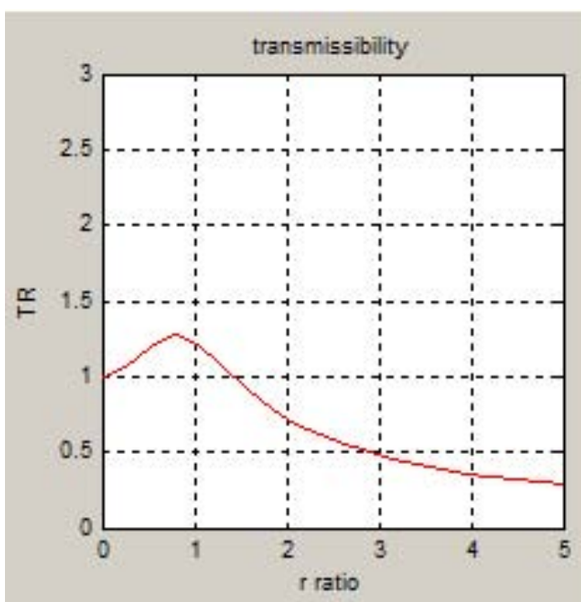


Figure 3. Transmissibility plot of first class cabin.

In addition to producing a smooth absolute acceleration time history, the goal was also to insure that T_r decreased as r increased. This implies that at higher external acceleration relative to the natural frequency, our vibration isolation system remained effective. The simulation program generated a mechanical view showing the absolute position of the first class cabin, with an offset, and the absolute position of the airplane during the flight as shown in figure 4 below.

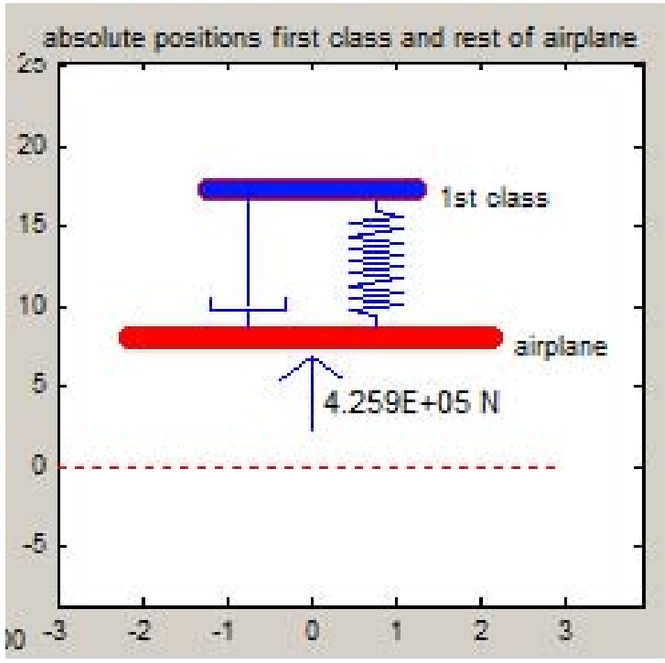


Figure 4. Animation of vibration isolation during flight.

The force shown in Figure 4. below the airplane is the numerical value of $-Mz''$ where z'' is the absolute acceleration of the airplane and M is the total mass of the first class cabin.

3.2.2 Implementation of the vibration isolation system

The vibration dampening system proposed for the first class cabin is a simple spring dashpot system that utilizes the additive properties of springs and dashpots to dampen the vibration of the first class cabin in the Boeing 757-200.

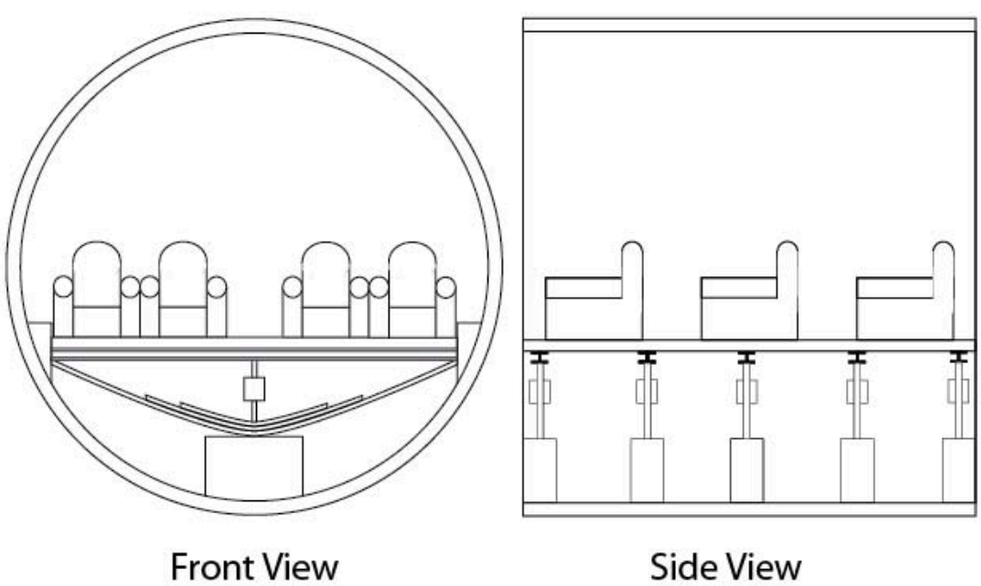


Figure 5. Schematic diagram of vibration isolation system in place

The design of our passive vibration isolation system is simple and effective with a minimal costs. It starts by defining the area that represents the first class cabin, which is at the front of the plane right behind the cockpit.

The cabin spans the entire inside width of the airplane body, which is 3.53 m (11.58 ft),

and then extends down the body of the plane roughly 3.35 m (11 ft) giving the first class cabin a total area of 3.53×3.35 m^2 (11.58×11 ft^2).

The next step in our design is to define the area that will actually be part of the vibration isolation system. We cannot use the whole floor of the first class cabin because the rounded body of the airplane would not allow the floor to travel up and down rendering our whole system ineffective. To solve this problem we started at the center of the plane's cross section and went out 1.524 m (5 ft) in either direction giving a total area of the platform used in our vibration isolation system 3.048×3.35 m^2 (10×11 ft^2) as seen above in figure 5.

To begin the actual design, additional support must be given to the aluminum floor of the cabin. The use of 6061 T6 Aluminum I-beams (specifications are given in appendix 5) spanning the width of the platform provides the needed support. In addition the I-beams provide a sturdy surface for the spring and dashpot system to contact the cabin floor.

The key component of the vibration dampening system is the use of carbon fiber leaf springs. We chose carbon fiber leaf springs in place of steel for several reasons. They provide a softer ride at a lower noise level and excellent stability due to better damping characteristics than steel. Placed in series, the use of 5 carbon fiber leaf springs provides the spring constant required (9700 N/m) and a low increase in weight.

The dashpots needed for our design, 2K325 Dashpots, can be purchased from many manufactures. When added in parallel they provided the necessary damping coefficient of 7425 N*s/m needed when the first class cabin is full and 5800 N/s*m when the cabin is empty.

Our design for this passive vibration isolation system works whether the first class cabin is full, empty, or half way in-between. The system works best when the cabin is fully loaded with passengers, and has almost identical results with no passengers on board. Even though the results are slightly diminished with fewer passengers, the system still creates a noticeably smoother flight.

3.2.3 Cost estimate of the vibration isolation system

The total cost of our vibration isolation system is around $16500 (appendix 5). The cost of the aluminum support beams, dashpots and carbon fiber leaf springs make up the majority of the material cost totaling only about $3000. The majority of the total cost comes from the additional weight of the system and the resulting price of fuel used during the planes lifetime. The additional weight results in an expected cost of about $13500 over the lifetime of the plane.

The damping effects of the system could be improved if weight were added to the cabin. However the additional cost of the added weight over the lifetime of the plane would outweigh the benefits for the passengers. If however some heavy components of the plane were to be attached to the first class cabin, the system could be redesigned for an even better ride. This would require further investigation into the balance of the plane, flight dynamics and a deeper knowledge of the various components of the plane so it falls out of the scope of this project.

3.2.4 Appendix

3.2.4.1 Design values

weight

This table shows the design values based on weight

Description of item	Mass (kg)
15 Chairs @ 100 kg/chair	1500
10 ft by 11 ft aluminum flooring	200
5 Aluminum I-beams @ 20 kg/beam	100
5 Carbon Fiber leaf springs @ 5 kg/spring	25
Miscellaneous weight	25
15 Passengers @ 80 kg	1200
Weight of First Class Cabin before vibration isolation system	1700
Weight of First Class Cabin after vibration isolation system	1850
Weight of First Class Cabin with maximum passengers	3050

Table 3. Mass of items used in design calculations

The total mass M has the value of 3050 kg. For our ζ value we choose the value 0.7 as it worked well in simulations to provide a smooth ride for the passengers while still keeping T_r small.

Leaf springs and spring K value

The most important aspect of picking a k value is the total allowed clearance the first class cabin floor has to move. The first class cabin's floor has a displacement relative to the body of the aircraft and if that gets too large the floor will make contact with the body of the airplane. The lower the k value we choose, the larger the displacement of the first class cabin relative to the body of the airplane will become. The maximum travel distance of the first class cabin is around 20 cm (7.87 in) and we can use this value to pick an appropriate k value. A k value around 10000 N/m keeps the first class cabin floor within this tolerance. The following plot shows the absolute acceleration of the cabin vs. the rest of the airplane during the turbulent flight [1].

[1] absolute position of the first class cabin was computed from the absolute acceleration of the cabin in the frequency domain. Hence the average value was not used due to the division by zero problem with this method. We do not have another method to find absolute position from absolute acceleration (unless we use more advanced numerical integration method in time domain, which is beyond the scope of this course)

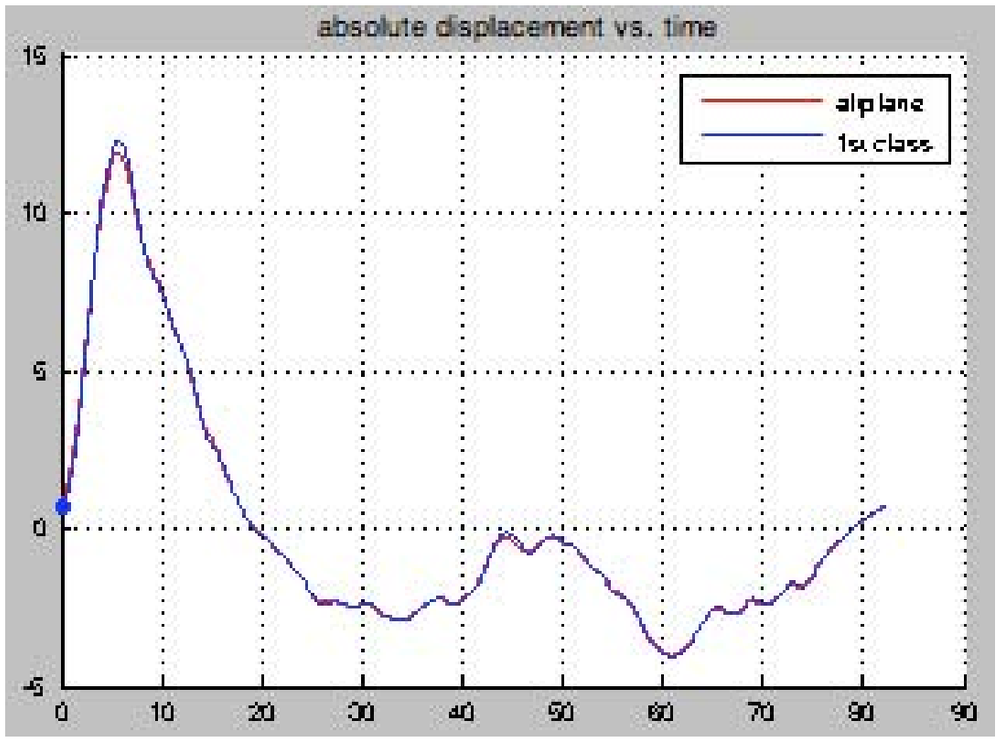


Figure 6. absolute acceleration of first class cabin compared to rest of airplane

To keep the weight of our vibration isolation system as small as possible we opted to use carbon fiber leaf springs. The k value of any leaf spring system can be calculated by the equation

$$k = \frac{8Enbt^3}{3L^3}$$

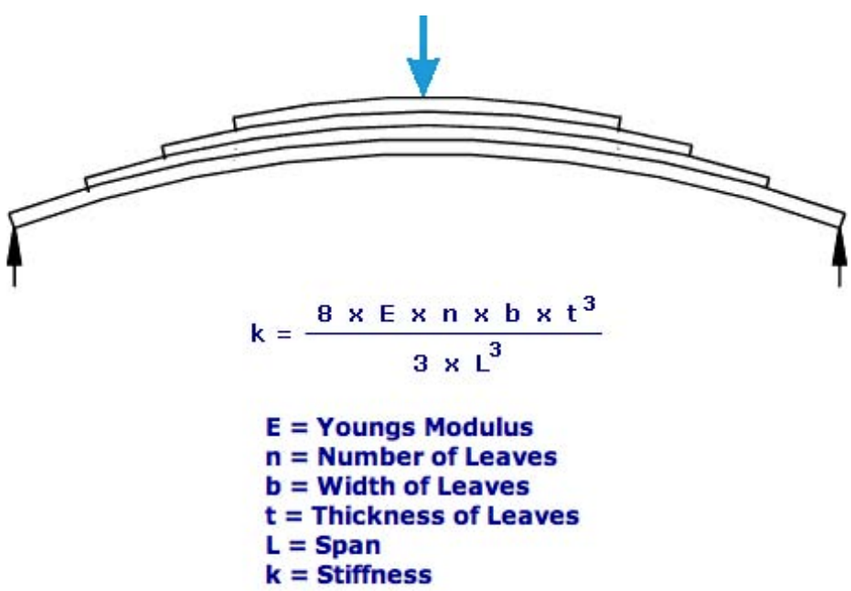


Figure 7. Leaf spring design used in vibration isolation system

Since our springs are in parallel, the k values add to give a total equivalent k. We took the k value selected (10000 N/m) and divided it by 5 giving us an individual k value of 2000 N/m. Using the following dimensions for the leaf spring resulted in a k value of 1940 N/m for a total k value of 9700 N/m.

- $E = 17$ Gpa
- $n = 3$

- L = 3.048 meter (10 ft)
- b = 0.1016 meter (4 in)
- t = 0.015875 meter (5/8 in)

3.2.4.2 Cost values

After finding the materials we needed, the following describes how we calculated the total cost of our vibration isolation system.

- 5 @ 10 ft 6061 T6 Aluminum I-beams @ $180/beam results in $900.
  Width 6 in, Flange 4 in, Web 0.19 in, Thickness 0.28 in.
- 5 Carbon Fiber leaf Springs @ $300/spring results in $1500.
- 150 kg of extra weight, total weight of the airplane is 59350 kg.
  Fuel costs for this aircraft was estimated to be $\$3,500$/hour and a typical aircraft operates 3000 hours per year. An increase of 1% in the weight of the aircraft is expected to increase fuel costs by 0.5%
  $\frac{150\text{ kg}}{2*59350\text{ kg}} \times \$3500 \times 3000 = \$13270$
- 5 2k325 Dashpots @ $100/dashpot = $500
  Needed c is around 2000 N*s/m. These dashpots have an adjustable c from 0 to 7000 N*s/m
- Total cost estimate $16500

3.2.4.3 Simulation program description

The simulation program was a GUI program written in Matlab version 2013a, which made it easier to determine the parameters to use for the design. The following is a screen shot of the program. The program can be downloaded from the project web site

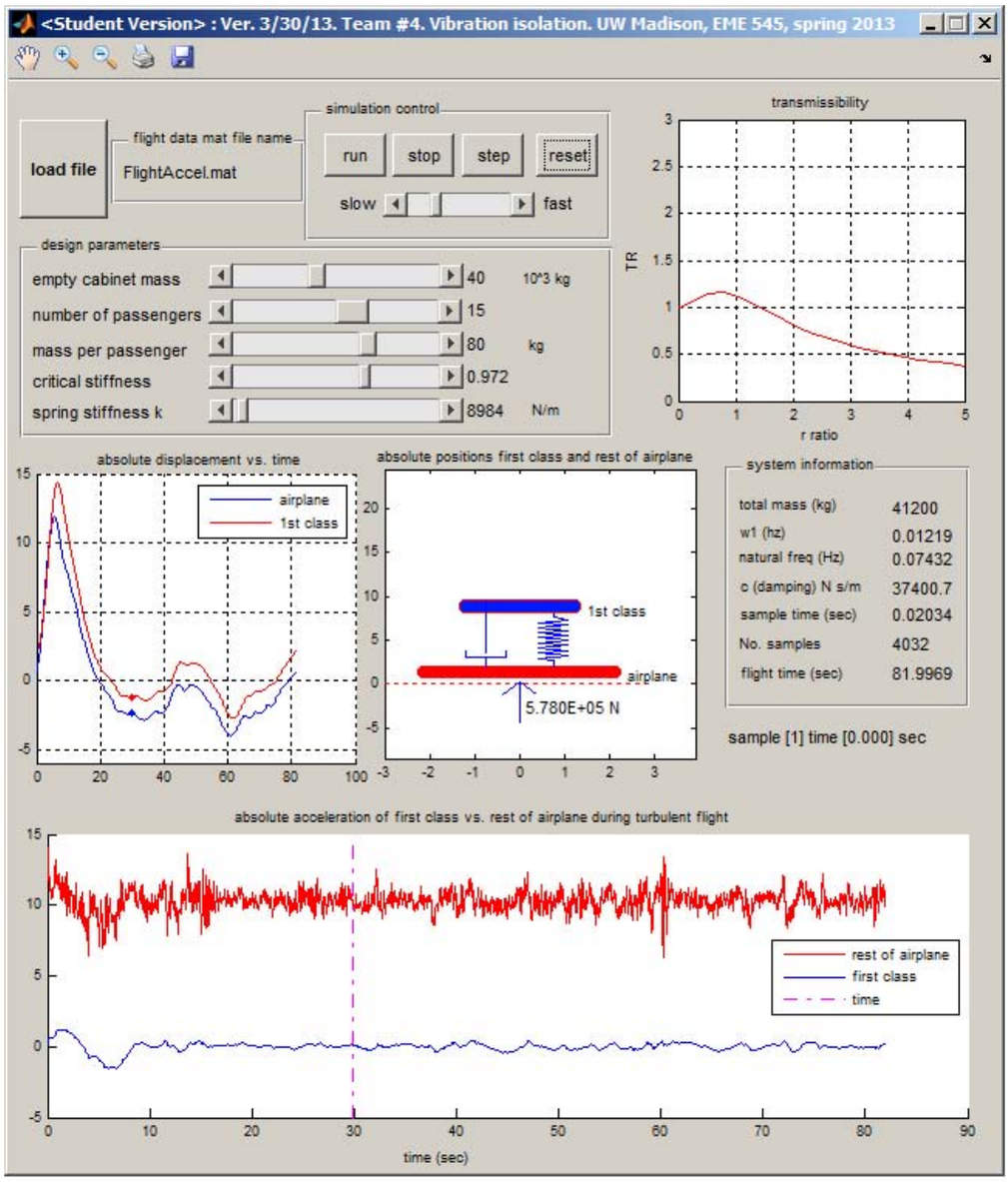


Figure 6. Simulation Matlab program used for obtaining the design parameters.

The first step is to load the Matlab .mat file which contains the acceleration time history. Then one can use the sliders to adjust the system parameters and see the effect on the absolute acceleration of the first class cabin. Computation was done in the FFT domain using the functions $fft\_easy()$ and $ifft\_easy()$ in the class web site. The absolute displacement was found from the absolute acceleration in the frequency domain. Due to the problem of division by zero for the first component in the frequency vector, this was set to zero before using $ifft\_easy()$.

3.2.4.4 Derivation of the transfer function

Assuming the mass of cabinet is M which includes passengers weight, by applying Newton's laws the EOM for the first class cabin is

$$\begin{aligned} my'' + c\left(y' - z'\right) + k\left(y - z\right) &= 0 \\ my'' + cy' + ky &= cz' + kz \end{aligned} \tag{3.1}$$

The time history of the turbulent acceleration $z''(t)$ was given to us in the matlab mat file. Therefore in the frequency domain, and assuming the time history represents one period we can write

$$z'' = \mathrm{Re}\left\{Z_n^{acc} e^{i(\omega_1 n)t}\right\}$$

Substituting back into Eq 3.1 and simplifying, the magnitude of the absolute displacement of the first class cabin relative to absolute displacement of airplane is found to

be

$$\left|\frac{Y_n}{Z_n^{disp}}\right| = \frac{\sqrt{1+(2\zeta r_n)^2}}{\sqrt{\left(1-r_n^2\right)^2+(2\zeta r_n)^2}}$$

Where Z_n^{acc} is the complex amplitude of the nth harmonic component in the acceleration data. Letting $\omega_1 n \equiv \varpi_n$ then in the frequency domain Eq 3.1 becomes

$$\begin{aligned}
\mathrm{Re}\left\{\left(-m\varpi_n^2 + i\varpi_n c + k\right)Y_n e^{i\varpi_n t}\right\} &= \mathrm{Re}\left\{\left(c\frac{Z_n^{acc}}{i\varpi_n} + k\frac{Z_n^{acc}}{-\varpi_n^2}\right)e^{i\varpi_n t}\right\} \\
Y_n &= \left(\frac{\frac{c}{i\varpi_n} - \frac{k}{\varpi_n^2}}{-m\varpi_n^2 + i\varpi_n c + k}\right)Z_n^{acc} \\
&= -\frac{Z_n^{acc}}{\varpi_n^2}\frac{1+i2\zeta r_n}{\left(1-r_n^2\right)+2i\zeta r_n}
\end{aligned}$$

Where

$$r_n = \frac{\varpi_n}{\omega_{natural}}$$

But $-\frac{Z_n^{acc}}{\varpi_n^2}$ is the absolute displacement of the airplane, say Z_n^{disp}, hence the transfer function between the absolute displacement of first class cabin and the absolute displacement of the airplane is

$$Y_n = \frac{1+i2\zeta r_n}{\left(1-r_n^2\right)+2i\zeta r_n}Z_n^{disp}$$

The magnitude of the absolute displacement of first class cabinet relative to absolute displacement of the airplane is

$$\left|\frac{Y_n}{Z_n^{disp}}\right| = \frac{\sqrt{1+(2\zeta r_n)^2}}{\sqrt{\left(1-r_n^2\right)^2+(2\zeta r_n)^2}}$$

3.2.4.5 References

1. Aluminum data `http://www.onlinemetals.com`
2. Airpot Dashpot Performance Specifications. N.p., n.d. Web. 15 Apr. 2013.

   `http://www.airpot.com/html/dashpot.html`
3. Boeing Commercial Airplanes. 757 Program. n.d. Web. 5 Apr. 2013.

   `http://www.boeing.com/boeing/commercial/757family/index.page`
4. Engineering ToolBox. Young's Modulus. Fabrication Extrusion Company, n.d. Web. 9 Apr. 2013.
5. Ginsberg, Jerry H. Mechanical and structural vibrations: theory and applications. New York: Wiley, 2001.
6. Online Metal Store Metal Product Guides at OnlineMetals.com. Metal Product Guides at OnlineMetals.com. N.p., n.d. Web. 15 Apr. 2013. `http://www.onlinemetals.com/merchant.cfm?id=980`
7. 7575-200 Airliner flugzeuginfo.net-the aircraft encyclopedia. N.p., n.d. Web. 10 Apr. 2013 `http://www.flugzeuginfo.net/acdata_php/acdata_7572_en.php`

3.2.5 software

The following zip file contains the current version of Matlab software to use to design the vibration isolation system.

Chapter 4

Exams

Local contents

4.1 Practice exam Spring 2013

Name: _____

EMA 545 – Practice Exam #1
Spring 2013 - Prof. M. S. Allen

Honor Pledge: On my honor, I pledge that I have neither given nor received inappropriate aid in the preparation of this exam.

_____
Signature

Calculators are allowed but not really needed. You may use one sheet of notes (one side).
Formulas:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Transient Response
The general solution to an underdamped SDOF system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

is:

$$x(t) = \mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{\mathrm{i}\omega_d t}\right)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and A is a complex constant.
Forced Transient Response:

$$x(t) = x_{IC}(t) + x_F(t)$$

where $x_F(t)$ can be found in Appendix B

Forced Steady-State Response:

$$f(t) = \mathrm{Re}\left(Fe^{\mathrm{i}\omega t}\right) \quad \rightarrow \quad x(t) = \mathrm{Re}\left(Xe^{\mathrm{i}\omega t}\right)$$

Page 1 of 5

Name: _____

Appendix B from Ginsberg, Wiley, 2001:

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = \frac{F(t)}{M} \quad \zeta < 1, \quad \omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1-\zeta^2}$$

- Free vibration: $F(t) = 0$

$$q = \exp(-\zeta\omega_{\text{nat}}t)\left[q(0)\cos(\omega_{\text{d}}t) + \frac{\dot{q}(0) + \zeta\omega_{\text{nat}}q(0)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]$$

- Impulse excitation: $F(t) = \delta(t)$

$$q = \frac{1}{M\omega_{\text{d}}}\exp(-\zeta\omega_{\text{nat}}t)\sin(\omega_{\text{d}}t)h(t)$$

- Step excitation: $F(t) = h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^2}\left\{1 - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Ramp excitation: $F(t) = th(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^3}\left\{(\omega_{\text{nat}}t) - 2\zeta + \exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\cos(\omega_{\text{d}}t) - (1-2\zeta^2)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Quadratic excitation: $F(t) = t^2h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^4}\left\{(\omega_{\text{nat}}t)^2 - 4\zeta(\omega_{\text{nat}}t) - 2(1-4\zeta^2) + \exp(-\zeta\omega_{\text{nat}}t) \times \left[2(1-4\zeta^2)\cos(\omega_{\text{d}}t) + \left(6\zeta - 8\zeta^3\right)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Exponential excitation:

$$F(t) = \exp(-\beta t)\,h(t)$$

$$q = \frac{1}{M(\omega_{\text{nat}}^2 - 2\zeta\omega_{\text{nat}}\beta + \beta^2)}\left\{\exp(-\beta t) - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}} - \beta}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient sinusoidal excitation:

$$F(t) = \sin(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\sin(\omega t) - 2\zeta\omega_{\text{nat}}\omega\cos(\omega t) + \omega\exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\omega_{\text{nat}}\cos(\omega_{\text{d}}t) - \frac{(1-2\zeta^2)\omega_{\text{nat}}^2 - \omega^2}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient co-sinusoidal excitation:

$$F(t) = \cos(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\cos(\omega t) + 2\zeta\omega_{\text{nat}}\omega\sin(\omega t) - \exp(-\zeta\omega_{\text{nat}}t)\left[(\omega_{\text{nat}}^2 - \omega^2)\cos\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}(\omega_{\text{nat}}^2 + \omega^2)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

Name: _____

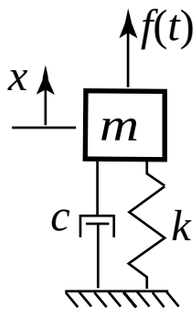
Short Answer Questions: (10 pts each)

1.) A very lightly damped single-degree-of-freedom system is observed to oscillate at 200 Hz after being excited by an impulsive force at $t = 0$. The oscillations are observed to fall below 1% of the initial amplitude after $t = 1.5$ seconds. What is the damping ratio, ζ? (***Show your work! An unjustified answer will not receive credit.***)

2.) A certain single-degree of freedom system is excited by a force $f(t) = -\cos(\omega t)-1.5\sin(\omega t)$, resulting in a response $x(t) = 1.2\cos(\omega t)+0.226\sin(\omega t)$. By what angle does the response lag the force? (***Show your work! An unjustified answer will not receive credit.***) Is this possible? If it is, would this require that the excitation frequency be smaller or larger than the natural frequency?

Name: _____

Problem #3 (40 pts)



A single-degree-of-freedom system is initially at rest in its static equilibrium position when, at t=0, a force, $f(t) = \mathrm{Re}(F_0 e^{i\omega t})$ is applied, where the drive frequency, ω, is one fourth of the natural frequency of the system and F_0 is a real constant. The stiffness of the system is 10 *N/m*, its mass is 0.1 kg, and the system can be modeled as undamped over the time interval of interest, so c = 0.

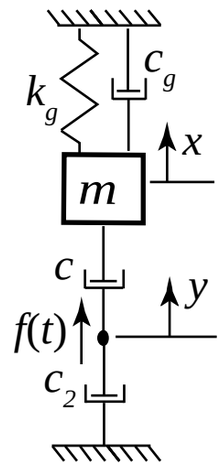
a.) Find an expression for the response of the system for $t > 0$.

b.) Sketch the response noting any important features.

Name: _____

Problem #4 (40 pts)

The schematic shows a simplified model for an automotive suspension, where the mass represents a wheel and the springs and dashpots represent the suspension and tire. The equations of motion for the system shown are:



$$\begin{bmatrix} m & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} c+c_g & -c \\ -c & c+c_2 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} + \begin{bmatrix} k_g & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

All of the system parameters, m, c, c_g, c_2, and k_g are known. A harmonic force $f(t)$ excites the system and steady state conditions have been reached so the motion of the massless point, $y(t)$, is harmonic, $y(t) = \mathrm{Re}(Ye^{i\omega t})$. This motion has been measured so the complex amplitude, Y, and frequency, ω, are known.

(a) (20 pts) Derive the transfer function relating the complex amplitude of $x(t)$ to that of $y(t)$.

(b) (10 pts) What would the ***amplitude*** of $x(t)$ be for very small and very large drive frequencies? You may express your answer in terms of m, c, c_g, c_2, and k_g, Y and ω.

4.2 first practice exam for finals. 2011

EMA 545 – Review Problems for Final Exam - Prof. M. S. Allen
Spring 2011

Problem #1:

4.34 The mass matrix, excitation, and modal properties of a two-degree-of-freedom system are known to be

$$[m] = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \text{kg}, \quad \{Q\} = \begin{Bmatrix} 0 \\ 20t \end{Bmatrix} N$$

$$\omega_1 = 5.42 \text{ rad/s}, \quad \omega_2 = 13.00 \text{ rad/s},$$

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 0.414 \end{Bmatrix}, \quad \{\phi_2\} = \begin{Bmatrix} 1 \\ \alpha \end{Bmatrix}$$

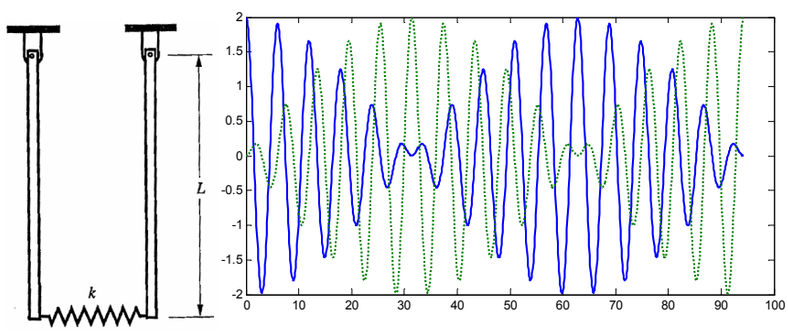
where α is an unspecified value. The system was initially at rest in the static equilibrium position. Determine the response.

Problem #2:

The system below consists of two pendulums on frictionless pins, connected at their tips by a soft spring $k = \alpha mg(L/2)$ where α is a small constant. The equations of motion are the following, where θ_1 corresponds to the left bar.

$$\frac{1}{3}mL^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + mg\frac{L}{2} \begin{bmatrix} \alpha + 0.5 & -\alpha \\ -\alpha & \alpha + 0.5 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The bar on the left is given an initial displacement of 2 degrees with the bar on the right vertical and the response is recorded and is shown below. The blue line corresponds to θ_1 while the dashed green line corresponds to θ_2. The amplitude of the motion of the left beam decays and then grows with time as the vibration energy transfers from one pendulum to the other and then back again. The beat period is 63 seconds. What is the value of α? Justify your answer, but you need only do those parts of the analysis that are critical to understand what is happening.



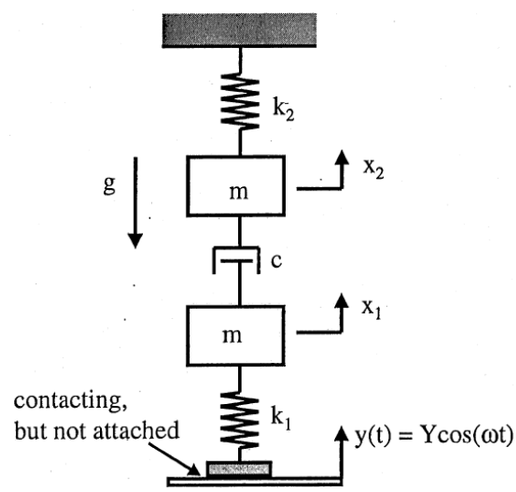
FYI: You can see a video of a system such as this at:
http://www.youtube.com/watch?v=RoSYKPTdlxs

Problem #3:
The two degree-of-freedom system shown below is forced by means of ground excitation. The lower spring rests on the moving base, but is not attached to it. When y(t)=0, the system is in static equilibrium when $x_1 = x_2 = 0$. The equations of motion for the system are given by:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} k_1 y \\ 0 \end{Bmatrix}$$

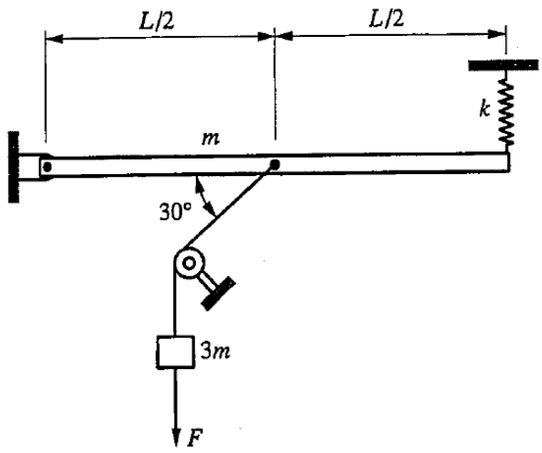
For all of the following you may assume that the base of spring k_1 never looses contact with the platform.

a.) Find the steady-state response $x_1(t)$ and $x_2(t)$ when ω is equal to the first natural frequency of the system. Use $k_1 = k$ and $k_2 = 2k$.
b.) Are there any frequencies of excitation for which the steady-state response amplitude for x_1 is identically zero? Use $k_1 = k$ and $k_2 = 2k$.
c.) If $k_1 = k_2 = k$, what is the steady-state response amplitude of each mass when the system is excited at a resonant frequency? Can you explain the result?
d.) For arbitrary k_1 and k_2, if y(t) is such that $x_1(t) = A\cos(\omega t)$, what is the response $x_2(t)$?



Extra credit – under what conditions does the base of the spring k_1 loose contact with the platform?

Problem #4
Find the linearized equations of motion for the system pictured below. Model the cable as a massless spring with stiffness *k*. The system is in static equilibrium in the position shown when the dynamic force F is not present.



Problem #5
Sorry, I didn't have time to write a 5th problem. This final probably isn't quite long enough.

TRANSIENT RESPONSES FOR UNDERDAMPED ONE-DEGREE-OF-FREEDOM SYSTEMS

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = \frac{F(t)}{M} \quad \zeta < 1, \quad \omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1-\zeta^2}$$

- Free vibration: $F(t) = 0$

$$q = \exp(-\zeta\omega_{\text{nat}}t)\left[q(0)\cos(\omega_{\text{d}}t) \frac{\dot{q}(0) + \zeta\omega_{\text{nat}}q(0)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]$$

- Impulse excitation: $F(t) = \delta(t)$

$$q = \frac{1}{M\omega_{\text{d}}}\exp(-\zeta\omega_{\text{nat}}t)\sin(\omega_{\text{d}}t)h(t)$$

- Step excitation: $F(t) = h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^2}\left\{1 - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Ramp excitation: $F(t) = th(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^3}\left\{(\omega_{\text{nat}}t) - 2\zeta + \exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\cos(\omega_{\text{d}}t) - (1-2\zeta^2)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Quadratic excitation: $F(t) = t^2h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^4}\left\{(\omega_{\text{nat}}t)^2 - 4\zeta(\omega_{\text{nat}}t) - 2(1-4\zeta^2) + \exp(-\zeta\omega_{\text{nat}}t) \times \left[2(1-4\zeta^2)\cos(\omega_{\text{d}}t) + (6\zeta - 8\zeta^3\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Exponential excitation:

$$F(t) = \exp(-\beta\, t)\, h(t)$$

$$q = \frac{1}{M(\omega_{\text{nat}}^2 - 2\zeta\omega_{\text{nat}}\beta + \beta^2)}\left\{\exp(-\beta t) - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}} - \beta}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient sinusoidal excitation:

$$F(t) = \sin(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\sin(\omega t) - 2\zeta\omega_{\text{nat}}\omega\cos(\omega t) + \omega\exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\omega_{\text{nat}}\cos(\omega_{\text{d}}t) - \frac{(1-2\zeta^2)\omega_{\text{nat}}^2 - \omega^2}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient co-sinusoidal excitation:

$$F(t) = \cos(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\cos(\omega t) + 2\zeta\omega_{\text{nat}}\omega\sin(\omega t) - \exp(-\zeta\omega_{\text{nat}}t)\left[(\omega_{\text{nat}}^2 - \omega^2)\cos\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}(\omega_{\text{nat}} + \omega^2)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

key solution

(1) Practice Midterm
4.34

1:35 find $\alpha \rightarrow \phi_1^T M \phi_2 = 0$ Modes must be $\perp$

$$[1 \quad 0.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \end{bmatrix} =$$

$$= [1 \quad 0.414]\begin{bmatrix} 3+\alpha \\ 1+\alpha \end{bmatrix} = 3+\alpha+0.414+0.414\alpha$$

$$= 3.414 + 1.414\alpha$$

$$\boxed{\alpha = -\frac{3.414}{1.414} = -2.414}$$

Mass Normalize: (Wouldn't have all This Arithmatic on final)

$$[1 \quad 0.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.414 \end{bmatrix} = [1 \quad 0.414]\begin{bmatrix} 3.414 \\ 1.414 \end{bmatrix}$$

$$= 3.414 + 1.414 \cdot 0.414 = \mu_1$$

$$[1 \quad -2.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2.414 \end{bmatrix} = [1 \quad -2.414]\begin{bmatrix} 3-2.414 \\ 1-2.414 \end{bmatrix} = \mu_2$$

$$\{\Phi_1\} = \{\phi_1\}\frac{1}{\sqrt{\mu_1}} \quad \text{etc...}$$

$$\ddot{\eta}_1 + \omega_n^2 \eta_1 = \{\Phi_1\}^T \begin{Bmatrix} 0 \\ 20t \end{Bmatrix} = \frac{0.414 \cdot 20t}{\sqrt{\mu_1}} = c_1 t$$

$$\ddot{\eta}_2 + \omega_n^2 \eta_2 = \{\Phi_2\}^T \begin{Bmatrix} 0 \\ 20t \end{Bmatrix} = c_2 t$$

Ramp

$$\eta_1(t) = \frac{c_1}{\omega_1^3}\left[\omega_1 t - \sin(\omega_1 t)\right] h(t) \quad - \text{App. B for } \zeta = 0$$

$$\eta_2(t) = \frac{c_2}{\omega_2^3}\left[\omega_2 t - \sin(\omega_2 t)\right] h(t)$$

2:47 $$q = [\Phi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \quad \ldots$$

(2) Practice Midterm

2:47

Nondimensionalize $\rightarrow$ $t' = \omega_n t$, $\omega_n = \sqrt{\dfrac{\cancel{m}g\frac{\cancel{L}}{2}}{\frac{1}{3}\cancel{m}L^{\cancel{2}}}} = \sqrt{\dfrac{3g}{2L}}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix} + \begin{bmatrix} \alpha+0.5 & -\alpha \\ -\alpha & \alpha+0.5 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

EVP

$$\begin{bmatrix} \alpha+0.5-\omega^2 & -\alpha \\ -\alpha & \alpha+0.5-\omega^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0$$

nontrivial if matrix is singular — det $= 0$

$$(\alpha+0.5-\omega^2)(\alpha+0.5-\omega^2) - \alpha^2 = 0$$

$$\cancel{\alpha^2} + 0.5\alpha - \underline{\alpha\omega^2} + 0.5\alpha + 0.25 - \underline{0.5\omega^2} - \underline{\alpha\omega^2} - \underline{0.5\omega^2} + \omega^4 - \cancel{\alpha^2} = 0$$

$$\omega^4 - 2(\alpha+0.5)\omega^2 + 0.25 + \alpha = 0$$

$$\omega^2 = \frac{2(\alpha+0.5) \pm \sqrt{4(\alpha+0.5)^2 - 4(1)(0.25+\alpha)}}{2}$$

$$\omega^2 = (\alpha+0.5) \pm \sqrt{(\alpha+0.5)^2 - (\alpha+0.25)}$$

$$= (\alpha+0.5) \pm \sqrt{\alpha^2 + \cancel{\alpha} + \cancel{0.25} - \cancel{\alpha} - \cancel{0.25}}$$

$$= (\alpha+0.5) \pm \alpha =$$

$$\boxed{\omega_1^2 = 0.5 \quad , \omega_2^2 = 0.5 + 2\alpha}$$

for small α — $\omega_1 \approx \omega_2$

Modes $\rightarrow$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix}\phi_1 = 0 \qquad \phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix}\phi_2 = 0 \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Given an initial condition,

$$\left.\begin{aligned} \eta_1(t) &= Re(N_1 e^{i\omega_1 t}) \\ \eta_2(t) &= Re(N_2 e^{i\omega_2 t}) \end{aligned}\right\} \text{No decay} \rightarrow \text{damping neglected}$$

$$\{\delta\} = [\Phi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \rightarrow$$

(2-cont.)

$$\xi_1 = \frac{1}{\sqrt{\mu_1}}\eta_1(t) + \frac{1}{\sqrt{\mu_2}}\eta_2(t)$$

$$\xi_2 = \frac{1}{\sqrt{\mu_1}}\eta_1(t) + \frac{-1}{\sqrt{\mu_2}}\eta_2(t)$$

$\xi_1(t)$ is the sum of 2 harmonics with nearly equal amplitudes and nearly the same frequency, so beating occurs

$$\xi_1 \approx Re(Ae^{i\Delta\omega t})\,Re(Ae^{i\omega_{ave}t})$$ (see below)

3:05

$$\omega_{ave} = \left(\sqrt{0.5} + \sqrt{0.5+2\alpha}\right)\frac{1}{2}$$ (α has little effect here.)

$$\Delta\omega = \left(\sqrt{0.5+2\alpha} - \sqrt{0.5}\right)\frac{1}{2}$$ depends on α

Could solve this on calculator iteratively to find α for known $\Delta\omega$

This part is little intricate for an exam

for small α ~ Taylor Series

$$\frac{\partial\Delta\omega}{\partial\alpha} = (0.5+2\alpha)^{-1/2}\left(\frac{1}{2}\right)(2)\Big|_{\alpha=0} = \frac{1}{\sqrt{0.5}}$$

$$\Delta\omega = 0 + \frac{\partial\Delta\omega}{\partial\alpha}\alpha = \sqrt{2}\,\alpha$$

$$\Delta\omega = \frac{2\pi}{T} = \frac{2\pi}{(63/2)}$$ ~ 2 beats per period ✓

$$\Delta\omega = \frac{\pi}{63} = \sqrt{2}\,\alpha$$

$$\boxed{\alpha \approx \frac{\pi}{63\sqrt{2}} \approx 0.035}$$ (True $\alpha = 0.036605$)

3:15

(23 min)

Or, if you need to derive the beating equation above:

$$\xi_1 = Re(c_1 e^{i\omega_1 t}) + Re(c_2 e^{i\omega_2 t}) \quad \text{for } c_1 = c_2$$

$$\xi_1 = c_1\left[Re\left(e^{i\omega_{ave}t}e^{-i\Delta\omega t} + e^{i\omega_{ave}t}e^{i\Delta\omega t}\right)\right]$$

$$= \frac{1}{2}c_1\left[e^{i\omega_{ave}t}e^{-i\Delta\omega t} + e^{-i\omega_{ave}t}e^{i\Delta\omega t} + e^{i\omega_{ave}t}e^{i\Delta\omega t} + e^{-i\omega_{ave}t}e^{-i\Delta\omega t}\right]$$

$$\xi_1 = \frac{1}{2}c_1\left[e^{i\omega_{ave}t}\underbrace{\left(e^{i\Delta\omega t} + e^{-i\Delta\omega t}\right)}_{2\cos(\Delta\omega t)\text{ - Factor out}} + e^{-i\omega_{ave}t}\left(e^{i\Delta\omega t} + e^{-i\Delta\omega t}\right)\right]$$

→ equation for beating

③

8:26

a) $\rightarrow \{x(t)\} = Re(\{X\} e^{i\omega t}) \longrightarrow y(t) = Re(Y e^{i\omega t}) \longrightarrow$

$$\{X\} = [K + i\omega c - \omega^2 M]^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

$$= \begin{bmatrix} K + i\omega c - \omega^2 m & -i\omega c \\ -i\omega c & 2K + i\omega c - \omega^2 m \end{bmatrix}^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

use $\omega = \omega_1 \longrightarrow$ find ω_1

$$\left| [K - \omega^2 M] \right| = \left| \begin{bmatrix} K - m\omega^2 & \\ & 2k - m\omega^2 \end{bmatrix} \right|$$

by inspection (diagonal system) $\rightarrow \omega_1 = \sqrt{K/m}, \quad \omega_2 = \sqrt{\frac{2K}{m}}$

$$= \begin{bmatrix} \cancel{K} + i\omega c - \cancel{(K/m)} \cdot m & -i\omega c \\ -i\omega c & K + i\omega c \end{bmatrix}^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

$$= \begin{bmatrix} K + i\omega c & i\omega c \\ i\omega c & i\omega c \end{bmatrix} \frac{1}{(K + i\omega c) i\omega c - (i\omega c)^2} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

$$= \begin{bmatrix} (K + i\omega c) K \\ K(i\omega c) \end{bmatrix} \frac{1}{K(i\omega c)} Y$$

$$= \begin{bmatrix} \frac{K^2 + i\omega c K}{i\omega c K} \\ 1 \end{bmatrix} Y$$

b) General case:

$$\{X\} = \begin{bmatrix} 2K + i\omega c - \omega^2 m & i\omega c \\ i\omega c & K + i\omega c - \omega^2 m \end{bmatrix} \frac{1}{\det(\,)} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

$$= \begin{Bmatrix} 2K + i\omega c - \omega^2 m \\ i\omega c \end{Bmatrix} \frac{K}{\det(\,)} Y$$

x_2 cannot $= 0$ unless $c = 0$

$x_1 \rightarrow 2K + i\omega c - \omega^2 m = 0$

$$\omega = \frac{-ic \pm \sqrt{-c^2 - 4(-m)(2K)}}{-2m}$$

solution for ω is imaginary unless $c = 0$

$\boxed{\rightarrow \text{so there is no real } \omega \text{ for which } x_1 = 0}$

5:37

3-cont.

8:37 c) from previous

$$\begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix} = \begin{bmatrix} i\omega c & i\omega c \\ i\omega c & i\omega c \end{bmatrix} \frac{1}{0} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

$\bar{X}_1, \bar{X}_2 \rightarrow \infty$

Explanation $\rightarrow$ equal nat freq's

$$[[K]-\omega^2[M]]\{\phi\} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\{\phi\} = 0 \rightarrow \phi \text{ is arbitrary!}$$

if $\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ Then $\{\phi_1\}^T[C]\{\phi_1\} =$

$$[1 \ 1]\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = [1 \ 1]\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$\boxed{\begin{array}{l} \text{so, } 2\zeta_1\omega_1 = 0, \ \zeta_1 = 0 \\ \text{so this mode is undamped and it's} \\ \text{response approaches } \infty \text{ when it is forced} \\ \text{at it's nat. freq.} \end{array}}$$

d) given $x_1 = Re(A e^{i\omega t}) \rightarrow$ use 2nd EOM

x_2 must also be harmonic $\rightarrow x_2 = Re(X_2 e^{i\omega t})$

$$m\ddot{x}_2 + c\dot{x}_2 - c\dot{x}_1 + K_2 x_2 = 0$$

$$\bar{X}_2 = \frac{i\omega c \bar{X}_1}{K + i\omega c - \omega^2 m}$$

$$\boxed{\bar{X}_2 = \frac{i\omega c A}{K + i\omega c - \omega^2 m}} \qquad x_2(t) = Re(\bar{X}_2 e^{i\omega t})$$

$\boxed{E.C}$ preload ~

$\downarrow mg$

K_2

$\uparrow K_2\Delta_0$

$K_2\Delta_0 - mg = 0$

$\Delta_0 = \frac{mg}{K_2}$

looses contact if $\Delta > \Delta_0$

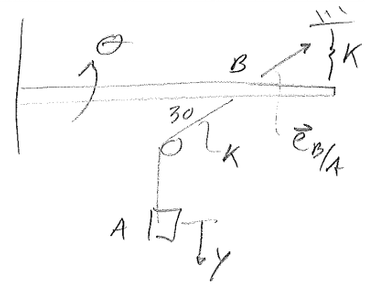
$\Delta = x_1 - y$

$$\boxed{x_1 - y > \frac{mg}{K}}$$

3:48

(4)

9:50 Find EOM $q_1 = \theta$ $q_2 = y$



$$T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 + \frac{1}{2}3m\dot{y}$$

I about pin

→ ok for bar in pure rotation, otherwise use $\frac{1}{2}mV_g^2$ and $\frac{1}{2}I_g\dot{\theta}^2$

$$M = \begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & 3m \end{bmatrix} \text{ for } \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix}$$

$$V = \frac{1}{2}K\Delta_1^2 + \frac{1}{2}K\Delta_2^2$$

$$\Delta_1 = L\theta$$

$$\Delta_2 = u_a - u_B \quad \text{— still spring approx / Linearization}$$

$$= y - (\cos 30\hat{i} + \sin 30\hat{j})\cdot\left(\frac{L}{2}\theta\hat{j}\right)$$

$$= y - \frac{L}{2}\sin 30\,\theta$$

$$V = \frac{1}{2}KL^2\theta^2 + \frac{1}{2}K\left(y^2 + 2\cdot\frac{L}{2}\sin 30\,\theta y + \frac{L^2}{4}\sin^2 30\,\theta^2\right)$$

$$K = \begin{bmatrix} KL^2 & -\frac{KL}{2}\sin 30 \\ -\frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}$$

$$\checkmark \sim J_o\ddot{\theta} + KL^2\theta - \frac{KL}{2}\sin 30\, y = 0$$

$\theta = 0,\ y > 0$ ~ oops — sign error on y → not consistent w $\vec{e}_{B/A}$

$$K = \begin{bmatrix} KL^2 & +\frac{KL}{2}\sin 30 \\ +\frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}$$

$$P_{in} = \dot{y}\cdot F$$

$$Q = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \Rightarrow \boxed{\begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & 3m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} KL^2 & \frac{KL}{2}\sin 30 \\ \frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}}$$

0:00

4.3 extra one problem practice

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Formulas:

Stiff spring approx:
$\dot{\Delta} \approx \dot{u}_B - \dot{u}_A = (\bar{v}_B - \bar{v}_A) \bullet \bar{e}_{B/A}$

Newton's Laws (2D):
$\sum \bar{F} = m\bar{a}$
$\sum M_g = I_g \ddot{\theta}$ or
$\sum M_A = I_A \ddot{\theta}$ if $\bar{a}_A = 0$

moment of inertia of a thin rod about its center of mass:
$I_g = (1/12)mL^2$
about its end:
$I_{end} = (1/3)mL^2$

Appendix B from Ginsberg, Wiley, 2001: (Corrected)

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = \frac{F(t)}{M} \quad \zeta < 1, \quad \omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1-\zeta^2}$$

- Free vibration: $F(t) = 0$

$$q = \exp(-\zeta\omega_{\text{nat}}t)\left[q(0)\cos(\omega_{\text{d}}t) + \frac{\dot{q}(0) + \zeta\omega_{\text{nat}}q(0)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]$$

- Impulse excitation: $F(t) = \delta(t)$

$$q = \frac{1}{M\omega_{\text{d}}}\exp(-\zeta\omega_{\text{nat}}t)\sin(\omega_{\text{d}}t)h(t)$$

- Step excitation: $F(t) = h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^2}\left\{1 - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Ramp excitation: $F(t) = th(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^3}\left\{(\omega_{\text{nat}}t) - 2\zeta + \exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\cos(\omega_{\text{d}}t) - (1-2\zeta^2)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Exponential excitation:
$F(t) = \exp(-\beta t)\, h(t)$

$$q = \frac{1}{M(\omega_{\text{nat}}^2 - 2\zeta\omega_{\text{nat}}\beta + \beta^2)}\left\{\exp(-\beta t) - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}} - \beta}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient sinusoidal excitation:
$F(t) = \sin(\omega t)h(t), \ \omega \neq \omega_{\text{nat}}$ if $\zeta \neq 0$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\sin(\omega t) - 2\zeta\omega_{\text{nat}}\omega\cos(\omega t) + \omega\exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\omega_{\text{nat}}\cos(\omega_{\text{d}}t) - \frac{(1-2\zeta^2)\omega_{\text{nat}}^2 - \omega^2}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

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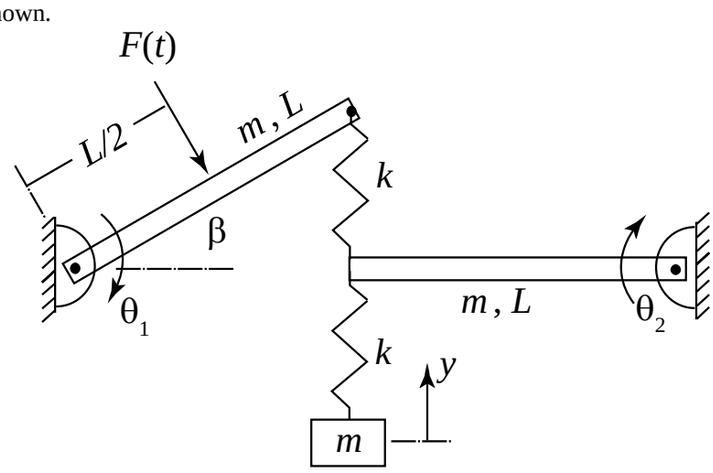
Name: _____

Problem #1 (20 pts)
The system pictured below consists of two identical bars of mass m and length L. The bars are connected as shown with stiff springs, so that the left bar is inclined at an angle β while the right bar is horizontal. This position is the static equilibrium position. Both springs have stiffnesses k. The generalized coordinates are the position of the mass, denoted y, and the angular deflections of the bars from static equilibrium, denoted θ_1 and θ_2. The angles θ_1 and θ_2 are positive in the directions shown.



A co-worker asserts that one of the equations of motion for this system is the following.

$$\frac{1}{3}mL\ddot{\theta}_1 + kL\cos\beta\left(L\theta_2 + L\theta_1\cos\beta\right) - kLy\cos\beta = \frac{FL}{2}$$

Consider the physics of the problem and check the **sign** and **units** of each term. Does each term produce the expected effect? Explain your reasoning.

4.4 practice exam 2

practice exam 2 . checking validity of K matrix, derive EOM for 2 DOF system, bar, spring, damper full modal analysis, find solution due to impulse.

4.4.1 questions

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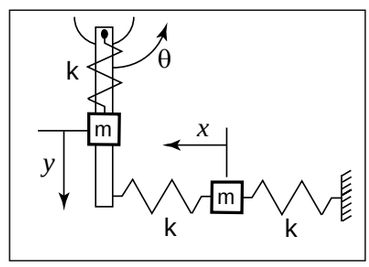
One (1) 8.5x11" double-sided sheet of notes allowed and must be turned in with your exam.

Problem #1 (10 pts)
a.) A colleague asserts that the linearized equations of motion for this system are as given below, where ×'s denote terms that are not given to you, which may be zero or constant. $[]|_{springs}$ denotes the portion of the stiffness matrix due to the springs and $[]|_{gravity}$ denotes that portion due to gravity. Check the ***units*** and the ***sign*** on the $K_{12}|_{springs}$ term. If incorrect, please provide the corrected term and ***explain your reasoning***. (The left mass is constrained so that it slides along the bar as the bar rotates.)



$$\begin{bmatrix} I & 0 & \times \\ 0 & m & \times \\ \times & \times & \times \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} kL^2 & k & \times \\ k & \times & \times \\ \times & \times & \times \end{bmatrix}_{springs} \begin{Bmatrix} \theta \\ x \\ y \end{Bmatrix} + \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}_{gravity} \begin{Bmatrix} \theta \\ x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

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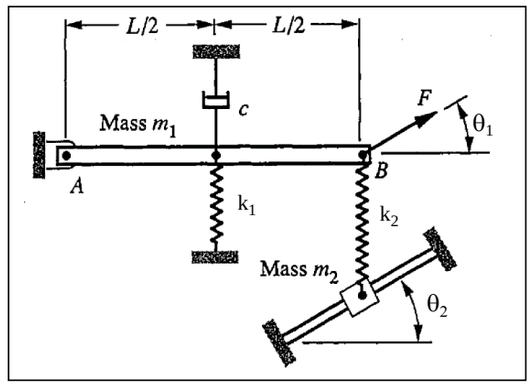
Name: _____

Problem #2 (45 pts)

Gravity acts downward (along dashpot c), and the initial lengths of the springs are such that the position shown corresponds to the static equilibrium when the applied dynamic force $F(t)$ is not present. The moment of inertia of a rod about its mass center is $I_g = (1/12)mL^2$ and about its end is $I_{end} = (1/3)mL^2$.

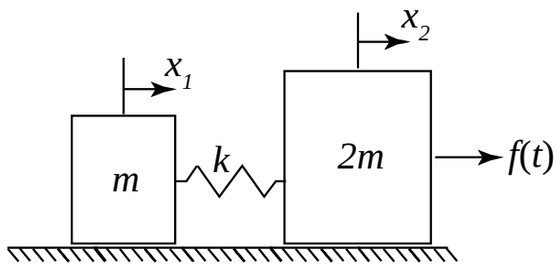


a.) Identify generalized coordinates and derive the corresponding equations of motion. Employ the stiff-spring approximation to simplify your analysis. Friction is negligible in the pin joint A and the friction force between the guide and m_2 is equal to $f=c_2v$, where v is the speed of the mass. (30 pts)

b.) Check that your answers make sense. Explain each check that you perform and why it shows that your EOM are/are not correct. (15 pts)

Name: _____

Problem #3 (45 pts)
The system pictured is initially at rest when an impulsive force $f(t) = F_0\delta(t\text{-}T)$ is applied to the mass on the right. The masses are constrained so that they only translate in the horizontal direction, and there is no friction between the masses and ground.



The equations of motion of this system are:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Find the response of the first mass, $x_1(t)$, as a function of time.

APPENDIX B TRANSIENT RESPONSES FOR UNDERDAMPED ONE-DEGREE-OF-FREEDOM SYSTEMS

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = \frac{F(t)}{M} \quad \zeta < 1, \quad \omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1-\zeta^2}$$

- Free vibration: $F(t) = 0$

$$q = \exp(-\zeta\omega_{\text{nat}}t)\left[q(0)\cos(\omega_{\text{d}}t) \frac{\dot{q}(0) + \zeta\omega_{\text{nat}}q(0)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]$$

- Impulse excitation: $F(t) = \delta(t)$

$$q = \frac{1}{M\omega_{\text{d}}}\exp(-\zeta\omega_{\text{nat}}t)\sin(\omega_{\text{d}}t)h(t)$$

- Step excitation: $F(t) = h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^2}\left\{1 - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Ramp excitation: $F(t) = th(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^3}\left\{(\omega_{\text{nat}}t) - 2\zeta + \exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\cos(\omega_{\text{d}}t) - (1-2\zeta^2)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Quadratic excitation: $F(t) = t^2h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^4}\left\{(\omega_{\text{nat}}t)^2 - 4\zeta(\omega_{\text{nat}}t) - 2(1-4\zeta^2) + \exp(-\zeta\omega_{\text{nat}}t) \times \left[2(1-4\zeta^2)\cos(\omega_{\text{d}}t) + (6\zeta - 8\zeta^3\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Exponential excitation:

$$F(t) = \exp(-\beta\, t)\, h(t)$$

$$q = \frac{1}{M(\omega_{\text{nat}}^2 - 2\zeta\omega_{\text{nat}}\beta + \beta^2)}\left\{\exp(-\beta t) - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}} - \beta}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient sinusoidal excitation:

$$F(t) = \sin(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\sin(\omega t) - 2\zeta\omega_{\text{nat}}\omega\cos(\omega t) + \omega\exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\omega_{\text{nat}}\cos(\omega_{\text{d}}t) - \frac{(1-2\zeta^2)\omega_{\text{nat}}^2 - \omega^2}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

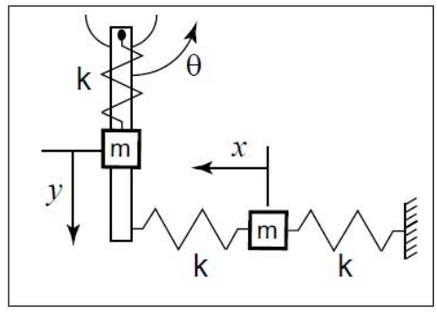
- Transient co-sinusoidal excitation:

$$F(t) = \cos(\omega t)h(t), \ \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\cos(\omega t) + 2\zeta\omega_{\text{nat}}\omega\sin(\omega t) - \exp(-\zeta\omega_{\text{nat}}t)\left[(\omega_{\text{nat}}^2 - \omega^2)\cos\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}(\omega_{\text{nat}} + \omega^2)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

4.4.2 Problem 1

Problem #1 (10 pts)
a.) A colleague asserts that the linearized equations of motion for this system are as given below, where ×'s denote terms that are not given to you, which may be zero or constant. $[]|_{springs}$ denotes the portion of the stiffness matrix due to the springs and $[]|_{gravity}$ denotes that portion due to gravity. Check the ***units*** and the ***sign*** on the $K_{12}|_{springs}$ term. If incorrect, please provide the corrected term and ***explain your reasoning***. (The left mass is constrained so that it slides along the bar as the bar rotates.)



$$\begin{bmatrix} I & 0 & \times \\ 0 & m & \times \\ \times & \times & \times \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} kL^2 & k & \times \\ k & \times & \times \\ \times & \times & \times \end{bmatrix}_{springs} \begin{Bmatrix} \theta \\ x \\ y \end{Bmatrix} + \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix}_{gravity} \begin{Bmatrix} \theta \\ x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Taking x as positive as shown, and y as positive as shown, then the middle spring is in compression with change of length $\Delta = (x + L\theta)$ and the right most spring is in tension with change of length $\Delta = x$, hence

$$\begin{aligned} V_{spring} &= \frac{1}{2}ky^2 + \frac{1}{2}k(x + L\theta)^2 + \frac{1}{2}kx^2 \\ &= \frac{1}{2}ky^2 + \frac{1}{2}k\left(x^2 + L^2\theta^2 + 2xL\theta\right) + \frac{1}{2}kx^2 \\ &= \theta^2\left(\frac{1}{2}kL^2\right) + x^2(k) + y^2\left(\frac{1}{2}k\right) + x\theta(kL) \end{aligned}$$

Compare to quadratic form

$$V_{spring} = \frac{1}{2}K_{11}\theta^2 + \frac{1}{2}K_{22}x^2 + \frac{1}{2}K_{33}y^2 + K_{12}x\theta + K_{13}\theta y + K_{23}xy$$

Then

$$\begin{aligned} K_{11} &= kL^2 \\ K_{22} &= k \\ K_{33} &= k \\ K_{12} &= kL \\ K_{13} &= 0 \\ K_{23} &= 0 \end{aligned}$$

Hence the K matrix due to stiffness is

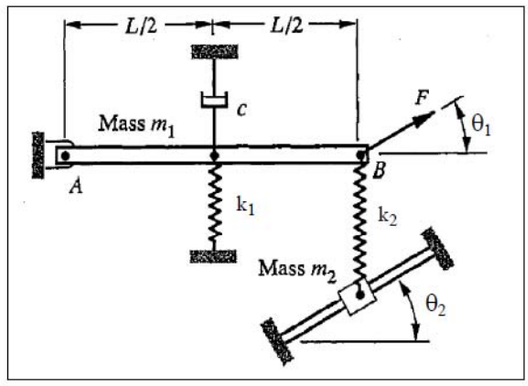
$$\begin{pmatrix} kL^2 & kL & 0 \\ kL & k & 0 \\ 0 & 0 & k \end{pmatrix} \begin{pmatrix} \theta \\ x \\ y \end{pmatrix}$$

Therefore, K_{12} had the wrong units. This reason is as follows: result of multiplying the first row of the K_{spring} matrix with the column $\begin{pmatrix} \theta \\ x \\ y \end{pmatrix}$ should have units of torque. Therefore the units should be $force \times meter$ and hence $K_{12}x$ should come out as Nm units. But as given in the problem, it has units N only, ie. units of force. But now, the units will come out to be Nm.

Similarly, the second row of the K matrix when multiplied by $\begin{pmatrix} \theta \\ x \\ y \end{pmatrix}$ should have units of force only (not torque). We can see this this is the case with this correction. So the sign was correct, but the units did not match before.

4.4.3 Problem 2

Problem #2 (45 pts)
Gravity acts downward (along dashpot c), and the initial lengths of the springs are such that the position shown corresponds to the static equilibrium when the applied dynamic force $F(t)$ is not present. The moment of inertia of a rod about its mass center is $I_g = (1/12)mL^2$ and about its end is $I_{end} = (1/3)mL^2$.

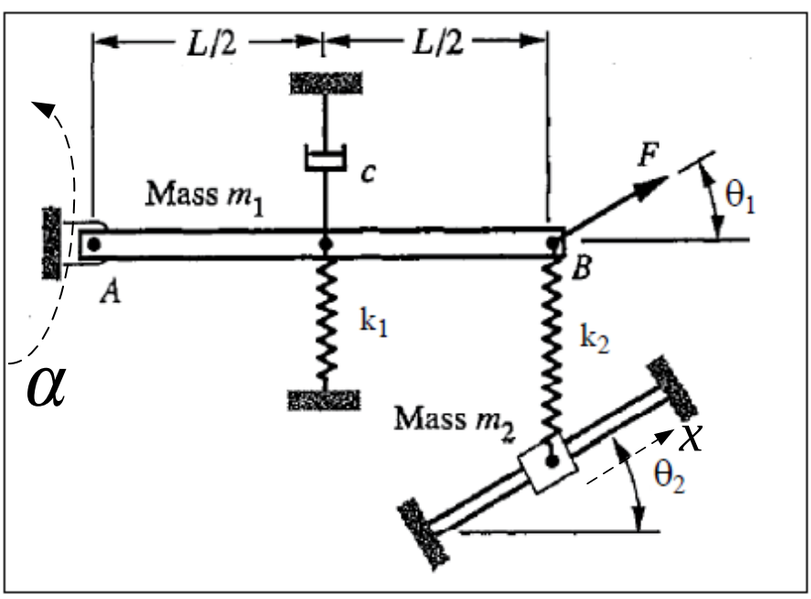


a.) Identify generalized coordinates and derive the corresponding equations of motion. Employ the stiff-spring approximation to simplify your analysis. Friction is negligible in the pin joint A and the friction force between the guide and m_2 is equal to $f=c_2v$, where v is the speed of the mass. (30 pts)

b.) Check that your answers make sense. Explain each check that you perform and why it shows that your EOM are/are not correct. (15 pts)

4.4.3.1 Part a

This is a 2 degrees of freedom system. The first generalized coordinate is taken as α which the angle of rotation of the top bar around joint A. The second degree of freedom is taken as x which is the sliding distance that mass m_2 moves as it slides over the lower bar



Static equilibrium is at $\alpha = 0$ and $x = 0$.

We start by finding the kinetic energy. Since bar m_1 is fixed at one point to inertial space, then only its rotational kinetic energy is added to the system kinetic energy

$$T = \frac{1}{2}\left(\frac{1}{12}m_1L^2\right)(\alpha')^2 + \frac{1}{2}m_2(x')^2$$

Now we find the potential energy, assuming springs remain straight. Spring k_1 will extend by amount

$$\Delta_1 = \frac{L}{2}\alpha$$

and spring k_2 will extend by amount

$$\Delta_2 = L\alpha - x\sin\theta_2$$

Hence potential energy of the system is

$$V = \frac{1}{2}k_1(\Delta_1)^2 + \frac{1}{2}k_2(\Delta_2)^2 + m_1 g\frac{L}{2}\sin\alpha + m_2 g x\sin\theta_2$$

Therefore the Lagrangian Φ is

$$\begin{aligned}
\Phi &= T - V \\
&= \frac{1}{2}\left(\frac{1}{12}m_1L^2\right)(\alpha')^2 + \frac{1}{2}m_2(x')^2 - \left(\frac{1}{2}k_1(\Delta_1)^2 + \frac{1}{2}k_2(\Delta_2)^2 + m_1 g\frac{L}{2}\sin\alpha + m_2 g x\sin\theta_2\right) \\
&= \frac{1}{2}\left(\frac{1}{12}m_1L^2\right)(\alpha')^2 + \frac{1}{2}m_2(x')^2 - \left(\frac{1}{2}k_1\left(\frac{L}{2}\alpha\right)^2 + \frac{1}{2}k_2(L\alpha - x\sin\theta_2)^2 + m_1 g\frac{L}{2}\sin\alpha + m_2 g x\sin\theta_2\right) \\
&= \frac{1}{24}m_1L^2(\alpha')^2 + \frac{1}{2}m_2(x')^2 - k_1\frac{L^2}{8}\alpha^2 - \frac{1}{2}k_2\left(L^2\alpha^2 + x^2\sin^2\theta_2 - 2L\alpha x\sin\theta_2\right) - m_1 g\frac{L}{2}\sin\alpha - m_2 g x\sin\theta_2
\end{aligned}$$

EOM for x is

$$\frac{d}{dt}\left(\frac{\partial\Phi}{\partial x'}\right) - \frac{\partial\Phi}{\partial x} = Q_x$$

where Q_x is the generalized for for the x coordinate. To find Q_x we make virtual displacement δx while fixing all other coordinates and obtain virtual work done by non-conservative forces. Only non-conservative force acting on m_2 is the friction force $f = c_2 v$ where v is the speed of the mass m_2. The speed of the mass m_2 is the vertical direction is $v = x'\sin\theta_2$, hence the non-conservative force acting on m_2 is $c_2(\dot{x}\sin\theta_2)$ and is acting in negative direction. Hence taking projection of this force along x gives

$$\delta W = -c_2(x'\sin\theta_2)\sin\theta_2\delta x$$

Therefore

$$Q_x = -c_2 x'\sin^2\theta_2$$

Hence

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\Phi}{\partial x'}\right) - \frac{\partial\Phi}{\partial x} &= -c_2 x'\sin^2\theta_2 \\
\frac{d}{dt}(m_2 x') - \left(-k_2 x\sin^2\theta_2 + 2k_2 L\alpha\sin\theta_2 - m_2 g\sin\theta_2\right) &= -c_2 x'\sin^2\theta_2 \\
m_2 x'' + c_2 x'\sin^2\theta_2 + k_2 x\sin^2\theta_2 - 2k_2 L\alpha\sin\theta_2 &= -m_2 g\sin\theta_2
\end{aligned}$$

EOM for α is

$$\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\alpha'}\right) - \frac{\partial\Phi}{\partial\alpha} = Q_\alpha$$

where Q_α is the generalized for for the α coordinate. To find Q_α we make virtual displacement $\delta\alpha$ while fixing all other coordinates and obtain virtual work done by non-conservative forces. We see that the work is

$$\begin{aligned}
\delta W &= -c(L\alpha')\frac{L}{2}\delta\alpha + (F\sin\theta_1)L\delta\alpha \\
&= \left(FL\sin\theta_1 - \frac{cL^2}{2}\alpha'\right)\delta\alpha
\end{aligned}$$

Hence

$$Q_\alpha = FL\sin\theta_1 - \frac{cL^2}{2}\alpha'$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \alpha'}\right) - \frac{\partial \Phi}{\partial \alpha} &= FL\sin\theta_1 - \frac{cL^2}{2}\alpha' \\
\frac{d}{dt}\left(\frac{1}{12}m_1L^2\alpha'\right) - \left(-k_1\frac{L^2}{4}\alpha - k_2L^2\alpha + 2Lx\sin\theta_2 - m_1g\frac{L}{2}\cos\alpha\right) &= FL\sin\theta_1 - \frac{cL^2}{2}\alpha' \\
\frac{1}{12}m_1L^2\alpha'' + k_1\frac{L^2}{4}\alpha + k_2L^2\alpha - 2k_2Lx\sin\theta_2 + m_1g\frac{L}{2}\cos\alpha &= FL\sin\theta_1 - \frac{cL^2}{2}\alpha' \\
\frac{1}{12}m_1L^2\alpha'' + \frac{cL^2}{2}\alpha' + \left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha - 2k_2Lx\sin\theta_2 &= FL\sin\theta_1 - m_1g\frac{L}{2}\cos\alpha
\end{aligned}$$

Hence the 2 EOM are

$$\begin{aligned}
m_2x'' + c_2x'\sin^2\theta_2 + k_2x\sin^2\theta_2 - 2k_2L\alpha\sin\theta_2 &= -m_2g\sin\theta_2 \\
\frac{1}{12}m_1L^2\alpha'' + \frac{cL^2}{2}\alpha' + \left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha - 2k_2Lx\sin\theta_2 &= FL\sin\theta_1 - m_1g\frac{L}{2}\cos\alpha
\end{aligned}$$

Linearize around static equilibrium, $\alpha = 0, x = 0$ then we obtain

$$\begin{aligned}
m_2x'' + c_2x'\sin^2\theta_2 + k_2x\sin^2\theta_2 - 2k_2L\alpha\sin\theta_2 &= -m_2g\sin\theta_2 \\
\frac{1}{12}m_1L^2\alpha'' + \frac{cL^2}{2}\alpha' + \left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha - 2k_2Lx\sin\theta_2 &= FL\sin\theta_1 - m_1g\frac{L}{2}
\end{aligned}$$

In Matrix form

$$\begin{pmatrix} \frac{1}{12}m_1L^2 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \alpha'' \\ x'' \end{pmatrix} + \begin{pmatrix} \frac{cL^2}{2} & 0 \\ 0 & c_2\sin^2\theta_2 \end{pmatrix}\begin{pmatrix} \alpha' \\ x' \end{pmatrix} + \begin{pmatrix} k_1\frac{L^2}{4} + k_2L^2 & -2k_2L\sin\theta_2 \\ -2k_2L\sin\theta_2 & k_2 \end{pmatrix}\begin{pmatrix} \alpha \\ x \end{pmatrix} = \begin{pmatrix} FL\sin\theta_1 - m_1g\frac{L}{2} \\ -m_2g\sin\theta_2 \end{pmatrix}$$

I think the weight contributions should be zero. So I need to look more into this, but I think the OEM should be as follows

$$\begin{pmatrix} \frac{1}{12}m_1L^2 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \alpha'' \\ x'' \end{pmatrix} + \begin{pmatrix} \frac{cL^2}{2} & 0 \\ 0 & c_2\sin^2\theta_2 \end{pmatrix}\begin{pmatrix} \alpha' \\ x' \end{pmatrix} + \begin{pmatrix} k_1\frac{L^2}{4} + k_2L^2 & -2k_2L\sin\theta_2 \\ -2k_2L\sin\theta_2 & k_2 \end{pmatrix}\begin{pmatrix} \alpha \\ x \end{pmatrix} = \begin{pmatrix} FL\sin\theta_1 \\ 0 \end{pmatrix}$$

4.4.4 Part b

Checking the Damping matrix units. First row of $C\begin{pmatrix} \alpha' \\ x' \end{pmatrix}$ should give units of torque. looking at $\frac{cL^2}{2}\alpha'$. viscous damping coefficient c has units of $N\frac{T}{L}$, hence the units of the expression $\frac{cL^2}{2}\alpha'$ are $N\frac{T}{L}(L)^2\frac{1}{T} = NL$, in other words, a torque. (in here, L stands for length units, T stands for time units and N stands for force units). Now to verify the second row of C. We see it is $c_2\sin^2\theta_2x'$ which has units of force (given in the problem). Since the second must have units of force, this is verified.

Now checking the stiffness matrix units. First row of $K\begin{pmatrix} \alpha \\ x \end{pmatrix}$ should have units of torque. But $\left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha$ has units of torque since k has units of force per unit length. and $2k_2L\sin\theta_2x$ has units of torque also (note α has no units as it is an angle).

For the second row of K, it should have units of force, which it does, since k_2x has units of force and $-2k_2L\sin\theta_2\alpha$ has units of force. Hence verified.

Check signs on the x EOM:

$$m_2x'' + c_2x'\sin^2\theta_2 + k_2x\sin^2\theta_2 - 2k_2L\alpha\sin\theta_2 = 0$$
$$m_2x'' + c_2x'\sin^2\theta_2 + k_2x\sin^2\theta_2 = 2k_2L\alpha\sin\theta_2$$

$x'' > 0, x' > 0, x > 0$ then $\alpha > 0$,checks OK, since when $x > 0$ then the top bar will be rotating in the positive direction and $\alpha > 0$, i.e. the top bar will be above the horizontal.

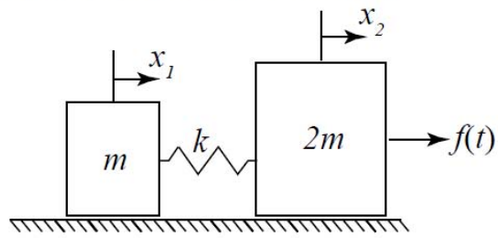
Check signs on the α EOM:

$$\frac{1}{12}m_1L^2\alpha'' + \frac{cL^2}{2}\alpha' + \left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha - 2k_2Lx\sin\theta_2 = FL\sin\theta_1$$
$$\frac{1}{12}m_1L^2\alpha'' + \frac{cL^2}{2}\alpha' + \left(k_1\frac{L^2}{4} + k_2L^2\right)\alpha = FL\sin\theta_1 + 2k_2Lx\sin\theta_2$$

$\alpha'' > 0, \alpha' > 0, \alpha > 0$ then $x > 0$,checks OK, since when $\alpha > 0$ then the top bar will be rotating in the positive direction and $x > 0$, means the lower mass m_2 is moving upwards.

4.4.5 Problem 3

Problem #3 (45 pts)
The system pictured is initially at rest when an impulsive force $f(t) = F_0\delta(t\text{-}T)$ is applied to the mass on the right. The masses are constrained so that they only translate in the horizontal direction, and there is no friction between the masses and ground.



The equations of motion of this system are:

$$\begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Find the response of the first mass, $x_1(t)$, as a function of time.

We solve this in modal coordinates so to de-couple the EOM's. First find the 2 natural frequencies

$$\left| k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 m\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \omega^2 \frac{m}{k}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right| = 0$$

Let $\omega^2 \frac{m}{k} = \eta^2$ then

$$\left| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} - \eta^2\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 1-\eta^2 & -1 \\ -1 & 1-2\eta \end{pmatrix} \right| = 0$$

$$\left(1-\eta^2\right)\left(1-2\eta^2\right) - 1 = 0$$

Hence taking positive roots $\boxed{\eta = 1.2247, \eta = 0}$. When $\eta = 0$

$$\begin{pmatrix} 1-\eta^2 & -1 \\ -1 & 1-2\eta^2 \end{pmatrix}\begin{Bmatrix} \varphi_{11} \\ \varphi_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{Bmatrix} 1 \\ \varphi_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Hence $1 - \varphi_{12} = 0$ or $\varphi_{12} = 1$, therefore $\varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

When $\eta = 1.2247$

$$\begin{pmatrix} 1-\eta^2 & -1 \\ -1 & 1-2\eta^2 \end{pmatrix}\begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{pmatrix} -0.5 & -1 \\ -1 & -2 \end{pmatrix}\begin{Bmatrix} 1 \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Hence $-0.5-\varphi_{22}=0$ or $\varphi_{22}=-0.5$, therefore $\varphi_2=\begin{Bmatrix}1\\-0.5\end{Bmatrix}$. Now do mass normalization

$$\begin{aligned}\mu_1&=\{\varphi\}_1^T[M]\{\varphi\}_1\\&=\begin{Bmatrix}1\\1\end{Bmatrix}^T\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}\\&=3\end{aligned}$$

and

$$\begin{aligned}\mu_2&=\{\varphi\}_2^T[M]\{\varphi\}_2\\&=\begin{Bmatrix}1\\-0.5\end{Bmatrix}^T\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{Bmatrix}1\\-0.5\end{Bmatrix}\\&=1.5\end{aligned}$$

Hence

$$\{\Phi\}_1=\frac{\{\varphi\}_1}{\sqrt{\mu_1}}=\frac{\begin{Bmatrix}1\\1\end{Bmatrix}}{\sqrt{3}}=\begin{Bmatrix}0.57735\\0.57735\end{Bmatrix}$$

$$\{\Phi\}_2=\frac{\{\varphi\}_2}{\sqrt{\mu_2}}=\frac{\begin{Bmatrix}1\\-0.5\end{Bmatrix}}{\sqrt{1.5}}=\begin{Bmatrix}0.81650\\-0.40825\end{Bmatrix}$$

Hence

$$\boxed{[\Phi]=\begin{pmatrix}0.57735&0.81650\\0.57735&-0.40825\end{pmatrix}}$$

Then the modal EOM are

$$\begin{aligned}[\Phi]^T[M][\Phi]+[\Phi]^T[K][\Phi]&=[\Phi]^T\{F\}\\\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{Bmatrix}\ddot{\eta}_1\\\ddot{\eta}_2\end{Bmatrix}+\begin{pmatrix}\eta_1^2&0\\0&\eta_2^2\end{pmatrix}\begin{Bmatrix}\eta_1\\\eta_2\end{Bmatrix}&=\begin{pmatrix}0.57735&0.57735\\0.81650&-0.40825\end{pmatrix}\begin{Bmatrix}0\\F_0\delta(t)\end{Bmatrix}\\\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{Bmatrix}\ddot{\eta}_1\\\ddot{\eta}_2\end{Bmatrix}+\begin{pmatrix}0&0\\0&1.5\end{pmatrix}\begin{Bmatrix}\eta_1\\\eta_2\end{Bmatrix}&=\begin{Bmatrix}0.57735F_0\delta(t)\\-0.40825F_0\delta(t)\end{Bmatrix}\end{aligned}$$

For the first mass, EOM is

$$\begin{aligned}\ddot{\eta}_1&=0.57735F_0\delta(t)\\\dot{\eta}_1&=\int_0^t 0.57735F_0\delta(t)dt+C_1\\&=0.57735F_0\left(h(t)-\frac{1}{2}\right)+C_1\\\eta_1(t)&=\int_0^t\left(0.57735F_0\left(h(t)-\frac{1}{2}\right)+C_1\right)dt+C_2\\&=0.57735F_0t\left(h(t)-\frac{1}{2}\right)+tC_1+C_2\end{aligned}$$

Now initial conditions are zero since $\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and also $\begin{Bmatrix} x_1'(0) \\ x_2'(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ then $\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and also $\begin{Bmatrix} \dot{\eta}_1(0) \\ \dot{\eta}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Initial conditions $\eta_1(0) = 0$ implies

$$C_2 = 0$$

while and $\dot{\eta}_1(0) = 0$ implies

$$C_1 = -0.57735F_0\left(h(t) - \frac{1}{2}\right)$$

Hence the solution is

$$\begin{aligned}
\eta_1(t) &= 0.57735F_0 t\left(h(t) - \frac{1}{2}\right) + tC_1 + C_2 \\
&= 0.57735F_0 t\left(h(t) - \frac{1}{2}\right) - 0.57735F_0\left(h(t) - \frac{1}{2}\right) \\
&= 0.57735F_0\left(h(t) - \frac{1}{2}\right)(t-1)
\end{aligned}$$

Now the second EOM is solved.

$$\ddot{\eta}_2 + 1.5\eta_2 = -0.40825F_0\delta(t)$$

Which has solution (using appendix B) and using $M = 1$ and $\omega_D = \omega_n = \sqrt{1.5} = 1.2247$ since $\zeta = 0$, hence

$$\eta_2(t) = \frac{-0.40825F_0}{1.2247}\sin(1.2247t)$$

Now to obtain the solution in normal coordinates

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [\Phi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$$

Then

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{pmatrix} 0.57735 & 0.81650 \\ 0.57735 & -0.40825 \end{pmatrix}\begin{Bmatrix} 0.57735F_0\left(h(t) - \frac{1}{2}\right)(t-1) \\ \frac{-0.40825F_0}{1.2247}\sin(1.2247t) \end{Bmatrix}$$

So

$$\begin{aligned}
x_1(t) &= 0.57735\left[0.57735F_0\left(h(t) - \frac{1}{2}\right)(t-1)\right] - 0.81650\left[\frac{0.40825F_0}{1.2247}\sin(1.225t)\right] \\
x_2(t) &= 0.57735\left[0.57735F_0\left(h(t) - \frac{1}{2}\right)(t-1)\right] + 0.40825\left[\frac{0.40825F_0}{1.2247}\sin(1.225t)\right]
\end{aligned}$$

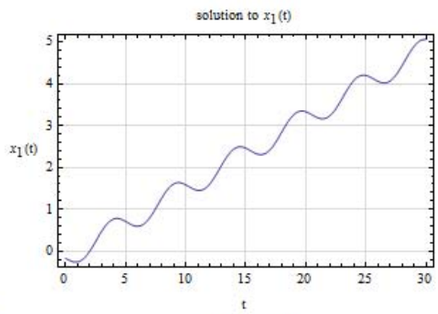
For example, if $F_0 = 1$ then

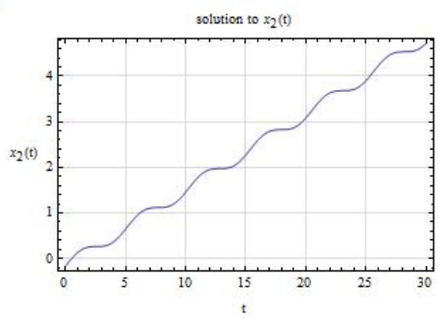
$$\begin{aligned}
x_1(t) &= 0.57735\left[0.57735\left(h(t) - \frac{1}{2}\right)(t-1)\right] - 0.81650\left[\frac{0.40825}{1.2247}\sin(1.2247t)\right] \\
x_2(t) &= 0.57735\left[0.57735\left(h(t) - \frac{1}{2}\right)(t-1)\right] + 0.40825\left[\frac{0.40825}{1.2247}\sin(1.2247t)\right]
\end{aligned}$$

Here is a plot of the solution $x_1(t)$ and $x_2(t)$. The 2 masses move to the right after the impulse, while in sinusoidal motion at the same frequency, but different amplitudes.

```
a0 = 0.57735;
b0 = 0.8165;
c0 = 0.40825;
wm = 1.2247;
x1[t_] := a0 (a0 (UnitStep[t] - 1/2) (t - 1)) - b0 (c0/wm Sin[wm t])
x2[t_] := a0 (a0 (UnitStep[t] - 1/2) (t - 1)) + c0 (c0/wm Sin[wm t])
Grid[
 {{Plot[x1[t], {t, 0, 30}, Frame → True, FrameLabel → {{"x1(t)", None}, {"t", "solution to x1(t)"}},
    GridLines → Automatic, GridLinesStyle → LightGray, RotateLabel → False, ImageSize → 300]},
  {Plot[x2[t], {t, 0, 30}, Frame → True, FrameLabel → {{"x2(t)", None}, {"t", "solution to x2(t)"}},
    GridLines → Automatic, GridLinesStyle → LightGray, RotateLabel → False, ImageSize → 300]}}]
```

Out[42]=





4.5 finals 2nd practice exam

4.5.1 questions

Name: _____

EMA 545 – Final Exam - Prof. M. S. Allen
Spring 2011

Honor Pledge: On my honor, I pledge that this exam represents my own work, and that I have neither given nor received inappropriate aid in the preparation of this exam.

_____
Signature

Problem 1 (20)	_____
Problem 2 (20)	_____
Problem 3 (30)	_____
Problem 4 (10)	_____
Problem 5 (10)	_____
Problem 6 (10)	_____
Total (100)	_____

You are allowed one sheet of notes for this exam, front and back. Staple your note sheet to the back of your exam when you turn it in. Calculators are allowed, but you must show all of your work to receive credit.

Page 1 of 9

Name: _____

Appendix B from Ginsberg, Mechanical & Structural Vibration, Wiley, 2001: **(corrected)**

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = \frac{F(t)}{M} \quad \zeta < 1, \quad \omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1-\zeta^2}$$

- Free vibration: $F(t) = 0$

$$q = \exp(-\zeta\omega_{\text{nat}}t)\left[q(0)\cos(\omega_{\text{d}}t) + \frac{\dot{q}(0) + \zeta\omega_{\text{nat}}q(0)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]$$

- Impulse excitation: $F(t) = \delta(t)$

$$q = \frac{1}{M\omega_{\text{d}}}\exp(-\zeta\omega_{\text{nat}}t)\sin(\omega_{\text{d}}t)h(t)$$

- Step excitation: $F(t) = h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^2}\left\{1 - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Ramp excitation: $F(t) = th(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^3}\left\{(\omega_{\text{nat}}t) - 2\zeta + \exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\cos(\omega_{\text{d}}t) - (1-2\zeta^2)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Quadratic excitation: $F(t) = t^2h(t)$

$$q = \frac{1}{M\omega_{\text{nat}}^4}\left\{(\omega_{\text{nat}}t)^2 - 4\zeta(\omega_{\text{nat}}t) - 2(1-4\zeta^2) + \exp(-\zeta\omega_{\text{nat}}t) \times \left[2(1-4\zeta^2)\cos(\omega_{\text{d}}t) + (6\zeta - 8\zeta^3)\frac{\omega_{\text{nat}}}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Exponential excitation:

$$F(t) = \exp(-\beta t)\,h(t)$$

$$q = \frac{1}{M(\omega_{\text{nat}}^2 - 2\zeta\omega_{\text{nat}}\beta + \beta^2)}\left\{\exp(-\beta t) - \exp(-\zeta\omega_{\text{nat}}t)\left[\cos(\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}} - \beta}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient sinusoidal excitation:

$$F(t) = \sin(\omega t)h(t), \quad \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\sin(\omega t) - 2\zeta\omega_{\text{nat}}\omega\cos(\omega t) + \omega\exp(-\zeta\omega_{\text{nat}}t)\left[2\zeta\omega_{\text{nat}}\cos(\omega_{\text{d}}t) - \frac{(1-2\zeta^2)\omega_{\text{nat}}^2 - \omega^2}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

- Transient co-sinusoidal excitation:

$$F(t) = \cos(\omega t)h(t), \quad \omega \neq \omega_{\text{nat}} \text{ if } \zeta \neq 0$$

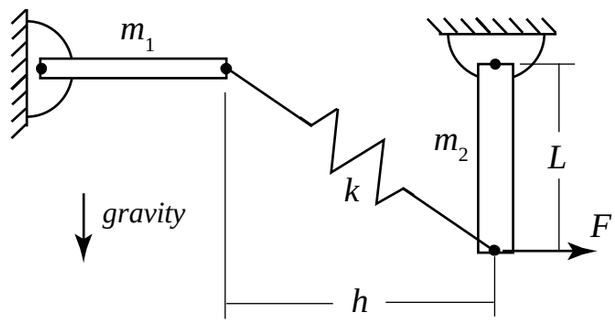
$$q = \frac{1}{M[(\omega_{\text{nat}}^2 - \omega^2)^2 + 4\zeta^2\omega_{\text{nat}}^2\omega^2]} \times \left\{(\omega_{\text{nat}}^2 - \omega^2)\cos(\omega t) + 2\zeta\omega_{\text{nat}}\omega\sin(\omega t) - \exp(-\zeta\omega_{\text{nat}}t)\left[(\omega_{\text{nat}}^2 - \omega^2)\cos\omega_{\text{d}}t) + \frac{\zeta\omega_{\text{nat}}(\omega_{\text{nat}}^2 + \omega^2)}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]\right\}h(t)$$

Name: _____

Problem #1 (20 pts)

Two rigid beams are pinned at their ends and arranged as shown below with a stiff spring connecting their tips. Gravity acts in the direction indicated. The position shown corresponds to the static equilibrium position. The masses of the two beams are m_1 and m_2 and they both have the same length, L. They are separated by a distance h. A dynamic force is applied to the tip of the right beam as shown. The moment of inertia of a bar is I_g=(1/12)mL^2 about its center and I_{end}= (1/3)mL^2 about its end.

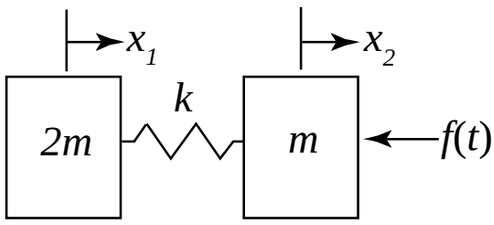
Find the linearized equation(s) of motion for this system and <u>check that your equation(s) are physically reasonable</u>.

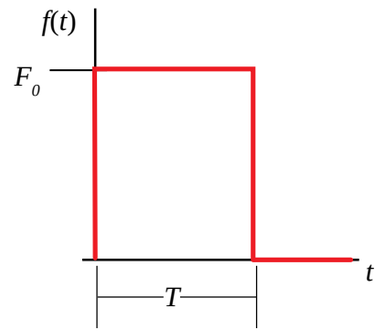


Name: _____

Name: _____

Problem #2 (20 pts)
The impact of a tennis ball with a racquet can be modeled using the two degree-of-freedom system shown below to represent the ball (the masses are only permitted to move in the horizontal direction). A ball is initially traveling to the right at speed v_0, (i.e. with $\dot{x}_1 = \dot{x}_2 = v_0$) when it strikes a racquet. Suppose that the impact force is known and is modeled as a square pulse whose duration is T. Damping is negligible.





The equations of motion of this system are:

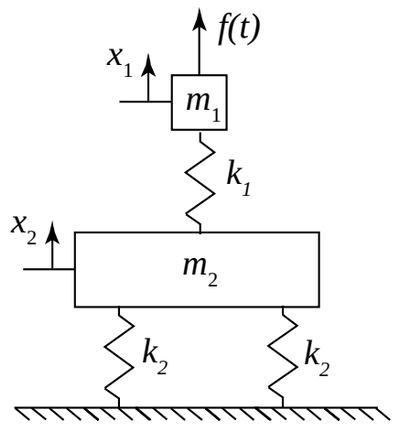
$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix}$$

a.) (10 pts) Find the natural frequencies and mass-normalized mode shapes of the system.
b.) (10 pts) Find two uncoupled, second-order differential equations that could be solved to find the response of the tennis ball. Be sure to substitute all known quantities into each of the equations.
c.) (3 pts extra credit) Use the result from (b) to sketch the response of the first mass, $x_1(t)$, qualitatively for t > T, explaining any important features.

Name: _____

Problem #3 (30 pts)

The system below is a simplified model of an aircraft with an engine mounted on its tail.



The equations of motion for certain values of the k_1, m_1, etc…, are known except for the mass matrix,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

so M_{11}, M_{12} and M_{22} are unknown constants. The <u>mass normalized</u> modes are also known and are:

$$\{\Phi\}_1 = \begin{Bmatrix} 0.85 \\ 0.65 \end{Bmatrix} \quad \{\Phi\}_2 = \begin{Bmatrix} 1.1 \\ -0.5 \end{Bmatrix}$$

The second natural frequency is ω_2= 16.9 rad/s. Suppose the system is initially at rest when the engine starts exerting a force $f(t) = A\cos(\omega t)h(t)$ where $h(t)$ is the unit step function.

a.) (10 pts) What is the first natural frequency ω_1?

b.) (10 pts) <u>How long</u> will it take for the system's response to settle to within approximately 1% of its steady state value? (Think carefully about what is being asked here and only answer the question that was asked.)

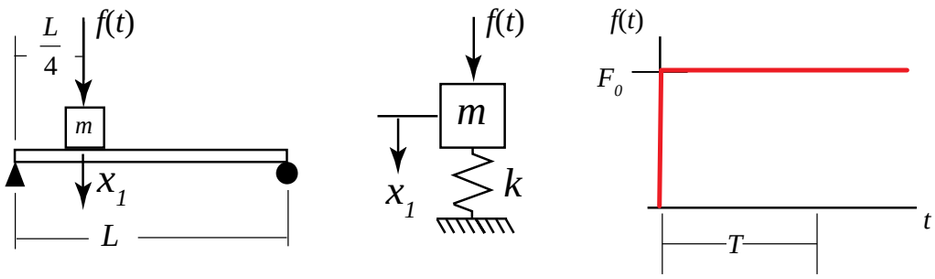
c.) (10 pts) Find an expression for the steady state response of the first mass $x_1(t)$ in terms of the forcing frequency ω.

Name: _____

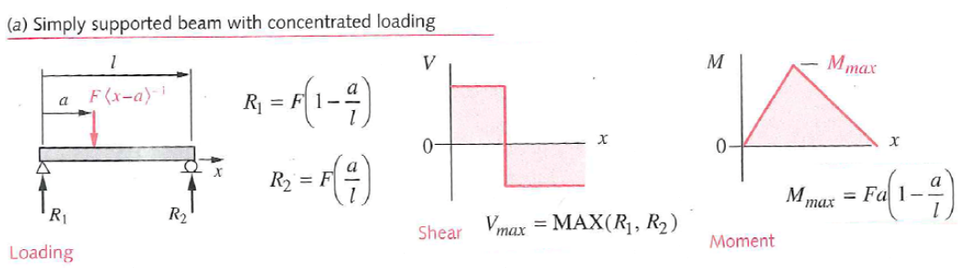
Name: _____

Problem #4 (10 pts)

The system shown consists of a beam with a large mass mounted one fourth of the distance from its end. This can be represented with the undamped spring-mass system shown to the right, with $k=85EI/L^3$. The system is initially in its static equilibrium position when a step force, $f(t)=F_0h(t)$, is applied to the mass.



The following information is available from a static analysis of the beam. When a static load, F, is applied to a beam, the maximum bending stress occurs in the outer fiber of the beam is given by $\sigma_{max}=-M_{max}c/I$, where M_{max} is the maximum bending moment in the beam, c is the (known) distance to the outer fiber and I is the area moment of inertia (also known). See the figure below for additional details regarding a static loading scenario.



<u>What is the amplitude of the load, F_0, that causes the beam to exceed its yield stress, σ_y?</u>

Name: _____

Problem #5 (10 pts)

A single degree-of-freedom system's response is given by $x(t) = \mathrm{Re}\left(Xe^{i\omega t}\right)$, with $X=e^{i2\pi/3}$.

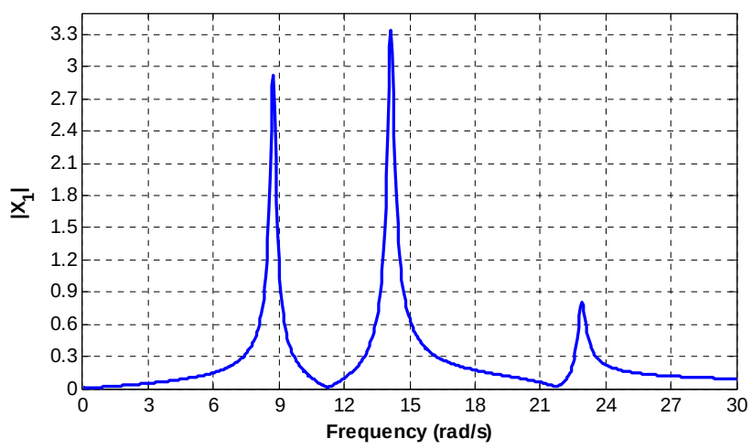
Sketch the complex amplitude, X, in the complex plane and sketch the corresponding time function $x(t)$ over at least one cycle.

Problem #6 (10 pts)

A three degree-of-freedom system is excited by a sinusoidal force, $f(t)=\cos(\omega t)$.

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}\,f(t)$$

The frequency response was computed using $\{X\}=\left(-\omega^2[M]+i\omega[C]+[K]\right)^{-1}\{F\}$ and $|X_1|$ from that calculation is plotted below.



Suppose that the input, $f(t)$, is replaced with a periodic function that can be expressed as follows,

$$f(t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{100-n}{n}\right)e^{in\omega_1 t}$$

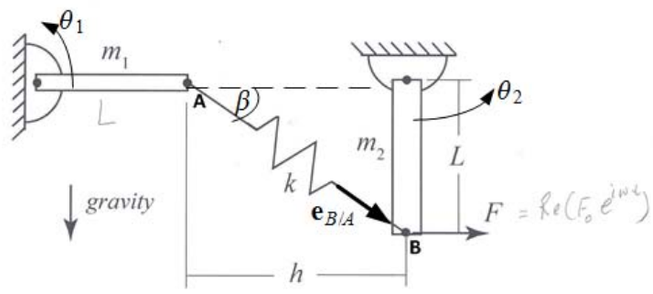
with ω_1 = 3.0 rad/s. What frequencies would be present in the steady-state response $x_1(t)$? Which of those would be dominant (i.e. have the largest amplitude)?

4.5.2 Problem 1

Problem #1 (20 pts)

Two rigid beams are pinned at their ends and arranged as shown below with a stiff spring connecting their tips. Gravity acts in the direction indicated. The position shown corresponds to the static equilibrium position. The masses of the two beams are m_1 and m_2 and they both have the same length, L. They are separated by a distance h. A dynamic force is applied to the tip of the right beam as shown. The moment of inertia of a bar is I_g=(1/12)mL^2 about its center and I_{end}= (1/3)mL^2 about its end.

Find the linearized equation(s) of motion for this system and check that your equation(s) are physically reasonable.



This is a 2 D.O.F. system. The degrees of freedom are θ_1 and θ_2 shown above in the positive sense. The method of power balance is used to obtain the EOM.

The system kinetic energy is $T = \frac{1}{2}m_1\frac{L^2}{3}\left(\theta_1'\right)^2 + \frac{1}{2}m_1\frac{L^2}{3}\left(\theta_2'\right)^2$, hence by comparing term to the quadratic form, the mass matrix part of the EOM is obtained

$$\frac{L^2}{3}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix}$$

To find spring stiffness, the spring deformation is found using stiff spring approximation.

$$\begin{aligned} \Delta' &= (V_B - V_A)\cdot e_{B/A} \\ &= \left(L\theta_2'\mathbf{i} - L\theta_1'\mathbf{j}\right)\cdot\left(\cos\beta\mathbf{i} - \sin\beta\mathbf{j}\right) \end{aligned}$$

Where $e_{B/A}$ is unit vector oriented to B from A and $\tan\beta = \frac{L}{h}$. The above becomes

$$\Delta' = L\theta_2'\cos\beta + L\theta_1'\sin\beta$$

Hence, integrating, squaring and collecting terms gives

$$\begin{aligned} \Delta &= L\theta_2\cos\beta + L\theta_1\sin\beta \\ \Delta^2 &= L^2\theta_2^2\cos^2\beta + L^2\theta_1^2\sin^2\beta + 2L^2\theta_1\theta_2\sin\beta\cos\beta \\ &= \theta_1^2\left(L^2\sin^2\beta\right) + \theta_2^2\left(L^2\cos^2\beta\right) + \theta_1\theta_2\left(2L^2\sin\beta\cos\beta\right) \end{aligned}$$

Using the quadratic form of the power balance method, the spring stiffness matrix part of the EOM is found from $V_{spring} = \frac{1}{2}k\left(\Delta^2\right)$ and by comparing quadratic terms, which leads to

$$V_{spring} = kL^2\begin{bmatrix} \sin^2\beta & 2\sin\beta\cos\beta \\ 2\sin\beta\cos\beta & \cos^2\beta \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

But $\sin\beta\cos\beta = \frac{1}{2}\left(\sin 2\beta\right)$ hence

$$V_{spring} = kL^2\begin{bmatrix} \sin^2\beta & \sin 2\beta \\ \sin 2\beta & \cos^2\beta \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

Stiffness due to gravity V_g is now found. Let datum for zero potential energy be at the horizontal level of the top bar, hence $V_g = m_1 g\frac{L}{2}\sin\theta_1 - m_2 g\frac{L}{2}\cos\theta_2$. Since the derivatives are evaluated at static equilibrium $\theta_1 = 0$ and $\theta_2 = 0$, the only term that remains is $m_2 g\frac{L}{2}$ which is now added to the k_{22} term of the stiffness matrix. FL is the generalized force for θ_2 since work done by F in making virtual $\delta\theta_2$ is $FL\delta\theta_2$. Therefore, the EOM becomes

$$\frac{L^2}{3}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix} + kL^2\begin{bmatrix} \sin^2\beta & \sin 2\beta \\ \sin 2\beta & \cos^2\beta + m_2 g\frac{L}{2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ FL \end{Bmatrix}$$

To check units of the above EOM, looking at the first EOM from above

$$\frac{L^2}{3}m_1\theta_1'' + kL^2\left(\sin^2\beta\right)\theta_1 + kL^2\left(\sin 2\beta\right)\theta_2 = 0$$

Let $\theta_1 = 0$. Hence $\frac{L^2}{3}m_1\theta_1'' = -kL^2\left(\sin 2\beta\right)\theta_2$. Assume $\theta_2 \geq 0$ and the system is now released to move. We should expect the top bar to accelerate down (negative), since the spring is stretched. Looking at the above, we see that $\theta_1'' \leq 0$. hence this is **correct**.

Now let $\theta_2 = 0$. Hence $\frac{L^2}{3}m_1\theta_1'' = -kL^2\left(\sin^2\beta\right)\theta_1$. Assume $\theta_1 \geq 0$ and the system is now released to move. We should expect the top bar to accelerate down (negative) since the spring was stretched. Looking at the above, we see that $\theta_1'' \leq 0$. This is **correct**.

Checking the second EOM

$$\frac{L^2}{3}m_2\theta_2'' + kL^2\left(\sin 2\beta\right)\theta_1 + kL^2\left(\cos^2\beta\right)\theta_2 = FL - m_2 g\frac{L}{2}\theta_2$$

Let $\theta_1 = 0$ and $F = 0$ then

$$\frac{L^2}{3}m_2\theta_2'' = -m_2 g\frac{L}{2}\theta_2 - L^2\left(\cos^2\beta\right)\theta_2$$

Assume $\theta_2 \geq 0$ and the system is now released to move. We would expect the right bar to accelerate back (negative) when released to move. From the equation we see that $\theta_2'' \leq 0$. This is **correct**.

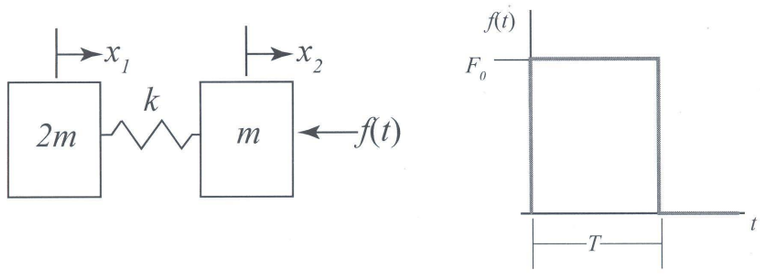
Now let $\theta_2 = 0$ and $F = 0$ then

$$\frac{L^2}{3}m_2\theta_2'' = -kL^2\left(\sin 2\beta\right)\theta_1$$

Assume $\theta_1 \leq 0$ and the system is now released to move. We would expect the bar to accelerate to the right (positive) since the spring was compressed. From the equation we see that $\theta_2'' > 0$. This is **correct**.

4.5.3 Problem 2

Problem #2 (20 pts)
The impact of a tennis ball with a racquet can be modeled using the two degree-of-freedom system shown below to represent the ball (the masses are only permitted to move in the horizontal direction). A ball is initially traveling to the right at speed v_0, (i.e., with $\dot{x}_1 = \dot{x}_2 = v_0$) when it strikes a racquet. Suppose that the impact force is known and is modeled as a square pulse whose duration is T. Damping is negligible.



The equations of motion of this system are:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix}$$

a.) (10 pts) Find the natural frequencies and mass-normalized mode shapes of the system.
b.) (10 pts) Find two uncoupled, second-order differential equations that could be solved to find the response of the tennis ball. Be sure to substitute all known quantities into each of the equations.
c.) (3 pts extra credit) Use the result from (b) to sketch the response of the first mass, $x_1(t)$, qualitatively for $t > T$, explaining any important features.

4.5.3.1 part(a)

$$\det\left([k] - \omega^2[m]\right) = 0$$

$$\det\left(\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - \omega^2 \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{m}{k} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

For normalization, let $t' = \omega t$ then $\frac{dt'}{dt} = \omega$ and using t' instead of t as the independent variable the above becomes

$$\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 - 2\omega^2 & -1 \\ -1 & 1 - \omega^2 \end{bmatrix}\right) = 0$$

$$\left(1 - 2\omega^2\right)\left(1 - \omega^2\right) - 1 = 0$$

The roots are $\omega = 0$ and $\omega = \sqrt{\frac{3}{2}}$. When $\omega = 0$ it is a rigid body motion, So any φ will do. Let $\varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. When $\omega = \sqrt{\frac{3}{2}}$ then

$$\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

let $\varphi_{12} = 1$ then $-2 - \varphi_{22} = 0$ or $\varphi_{22} = -2$ hence $\varphi_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$.

$$\mu_1 = \varphi_1^T [M] \varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 3$$
$$\mu_2 = \varphi_2^T [M] \varphi_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = 6$$

Hence

$$\Phi_1 = \frac{\varphi_1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.57735 \\ 0.57735 \end{Bmatrix}$$
$$\Phi_2 = \frac{\varphi_2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{6}} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = \begin{Bmatrix} 0.40825 \\ -0.81650 \end{Bmatrix}$$

4.5.3.2 part(b)

$$\Phi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

The EOM is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \Phi^T \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{3}\sqrt{3} f(t) \\ \frac{1}{3}\sqrt{6} f(t) \end{Bmatrix}$$

initial conditions are $\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and

$$\begin{Bmatrix} \eta_1'(0) \\ \eta_2'(0) \end{Bmatrix} = \Phi^T [M] \begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix} = \begin{Bmatrix} \sqrt{3} v_0 \\ 0 \end{Bmatrix}$$

Therefore, the first ODE is

$$\eta_1'' = -\frac{1}{3}\sqrt{3} F_0 (h(t) - h(t-T))$$

with IC $\eta_1(0) = 0$ and $\eta_1'(0) = \sqrt{3} v_0$. The second ODE is

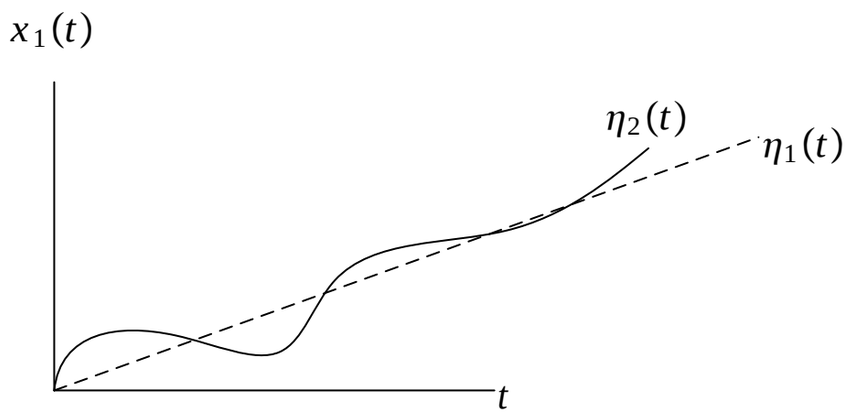
$$\eta_2'' + \frac{3}{2}\eta_2 = \frac{1}{3}\sqrt{6} F_0 (h(t) - h(t-T))$$

with IC $\eta_1(0) = 0$ and $\eta_1'(0) = 0$

4.5.3.3 part(c)

$$
\begin{aligned}
x_1(t) &= \Phi_{11}\eta_1(t) + \Phi_{12}\eta_2(t) \\
&= \frac{1}{\sqrt{3}}\eta_1(t) + \frac{1}{\sqrt{6}}\eta_2(t)
\end{aligned}
$$

Therefore, $x_1(t)$ solution has contribution from $\eta_1(t)$ and $\eta_2(t)$. But $\eta_1(t)$ is linear with positive slope of v_0 and $\eta_2(t)$ is a sinusoidal, with no damping. So adding both together, here is a sketch of possible solution

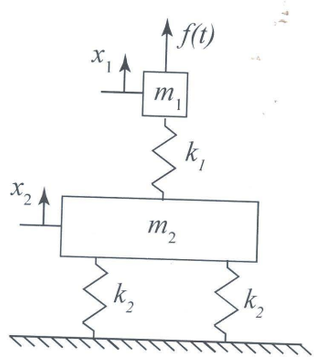


4.5.4 Problem 3

Name: _____

Problem #3 (30 pts)

The system below is a simplified model of an aircraft with an engine mounted on its tail.



The equations of motion for certain values of the k_1, m_1, etc…, are known except for the mass matrix,

$$
\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}
$$

so M_{11}, M_{12} and M_{22} are unknown constants. The mass normalized modes are also known and are:

$$
\{\Phi\}_1 = \begin{Bmatrix} 0.85 \\ 0.65 \end{Bmatrix} \quad \{\Phi\}_2 = \begin{Bmatrix} 1.1 \\ -0.5 \end{Bmatrix}
$$

The second natural frequency is $\omega_2 = 16.9$ rad/s. Suppose the system is initially at rest when the engine starts exerting a force $f(t) = A\cos(\omega t)h(t)$ where $h(t)$ is the unit step function.

a.) (10 pts) What is the first natural frequency ω_1?

b.) (10 pts) How long will it take for the system's response to settle to within approximately 1% of its steady state value? (Think carefully about what is being asked here and only answer the question that was asked.)

c.) (10 pts) Find an expression for the steady state response of the first mass $x_1(t)$ in terms of the forcing frequency ω.

4.5.4.1 part(a)

$$\Phi^T[K]\Phi^T = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 46.25 & -0.5 \\ -0.5 & 281.0 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

Hence $\omega_1^2 = 46.25$ or $\omega_1 = 6.8$ rad/sec

4.5.4.2 part(b)

Using the first natural frequency, since this has the longest time constant $\tau = \frac{1}{\zeta_1\omega_1}$and solving for the number of periods using logarithmic decrement method

$$\frac{1}{N}\ln\left(\frac{y_1}{y_N}\right) = 2\pi\zeta_1 \tag{1}$$

ζ_1 is not known but can be found by evaluating $\Phi^T[C]\Phi^T$

$$\Phi^T[K]\Phi^T = \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.05 & 0.021 \\ 0.021 & 0.061 \end{bmatrix}$$

and assuming small damping approximation, then $2\zeta_1\omega_1 = 0.05$. Hence $\zeta_1 = \frac{0.05}{2\omega_1} = \frac{0.05}{2(6.8)} = 0.0038$. Now that the critical damping ratio for the first mode is found, we can use the method of logarithmic decrement to find how many periods it takes to attenuate by 99%

Let $\frac{y_1}{y_N} = \frac{1}{0.01} = 100$ then Eq (1) becomes

$$\begin{aligned} \frac{1}{N}\ln(100) &= 2\pi(0.0038) \\ N &= \frac{(4.605)}{2\pi(0.0038)} = 192.87 \\ &= 193 \end{aligned}$$

Where N is the number or periods needed. But $T = \frac{2\pi}{\omega_1}$,hence the time needed is

$$t = NT = 192T = 192\frac{2\pi}{\omega_1} = 192\frac{2\pi}{6.8} = 177.41 \text{ sec}$$

So it takes 178 seconds for the first modal (decoupled) solution to attenuate in amplitude by 99%. Since this is the dominant time constant, we expect the physical solution to attenuate in approximately the same amount of time as well.

4.5.4.3 part(c)

The EOM is, in modal coordinates

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \Phi^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \Phi \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \Phi^T \begin{Bmatrix} \mathrm{Re}\left(Ae^{i\omega t}\right) \\ 0 \end{Bmatrix}$$

But

$$\Phi^T\begin{bmatrix}0.04 & 0\\ 0 & 0.05\end{bmatrix}\Phi = \begin{bmatrix}0.85 & 1.1\\ 0.65 & -0.5\end{bmatrix}^T\begin{bmatrix}0.04 & 0\\ 0 & 0.05\end{bmatrix}\begin{bmatrix}0.85 & 1.1\\ 0.65 & -0.5\end{bmatrix} = \begin{bmatrix}0.05 & 0.021\\ 0.021 & 0.061\end{bmatrix}$$

Hence EOM in modal coordinates become

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\eta_1''\\ \eta_2''\end{Bmatrix} + \begin{bmatrix}0.05 & 0.021\\ 0.021 & 0.061\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\end{Bmatrix} + \begin{bmatrix}6.8^2 & 0\\ 0 & 16.9^2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\end{Bmatrix} = \begin{bmatrix}0.85 & 1.1\\ 0.65 & -0.5\end{bmatrix}^T\begin{Bmatrix}\mathrm{Re}\left(Ae^{i\omega t}\right)\\ 0\end{Bmatrix}$$

and using *small damping approximation*

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\eta_1''\\ \eta_2''\end{Bmatrix} + \begin{bmatrix}0.05 & 0\\ 0 & 0.061\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\end{Bmatrix} + \begin{bmatrix}6.8^2 & 0\\ 0 & 16.9^2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\end{Bmatrix} = \begin{Bmatrix}0.85\,\mathrm{Re}\left(Ae^{it\omega}\right)\\ 1.1\,\mathrm{Re}\left(Ae^{it\omega}\right)\end{Bmatrix}$$

Hence the 2 EOM's are

$$\begin{aligned}\eta_1'' + 0.05\eta_1' + 46.24\eta_1 &= \mathrm{Re}\left(0.85Ae^{i\omega t}\right)\\ \eta_2'' + 0.061\eta_2' + 285.61\eta_2 &= \mathrm{Re}\left(1.1Ae^{i\omega t}\right)\end{aligned}$$

Let $\eta_1 = \mathrm{Re}\left(X_1e^{it\omega}\right)$ then $X_1 = \frac{0.85A}{-\omega^2+i0.05\omega+46.24}$ and $\eta_2 = \mathrm{Re}\left(X_2e^{it\omega}\right)$ then $X_2 = \frac{1.1A}{-\omega^2+i0.0609\omega+285.61}$ then

$$\begin{aligned}\mathbf{x} &= \Phi_1\eta_1 + \Phi_2\eta_2\\ \begin{Bmatrix}x_1(t)\\ x_2(t)\end{Bmatrix} &= \begin{Bmatrix}0.85\\ 0.65\end{Bmatrix}\mathrm{Re}\left(X_1e^{i\omega t}\right) + \begin{Bmatrix}1.1\\ -0.5\end{Bmatrix}\mathrm{Re}\left(X_2e^{i\omega t}\right)\end{aligned}$$

Hence

$$\begin{aligned}x_1(t) &= 0.85\,\mathrm{Re}\left(X_1e^{i\omega t}\right) + 1.1\,\mathrm{Re}\left(X_2e^{i\omega t}\right)\\ x_2(t) &= 0.65\,\mathrm{Re}\left(X_1e^{i\omega t}\right) - 0.5\,\mathrm{Re}\left(X_2e^{i\omega t}\right)\end{aligned}$$

hence

$$\begin{aligned}x_1(t) &= 0.85\,\mathrm{Re}\left(\frac{0.85A}{-\omega^2 + i0.05\omega + 46.24}e^{i\omega t}\right) + 1.1\,\mathrm{Re}\left(\frac{1.1A}{-\omega^2 + i0.061\omega + 285.61}e^{i\omega t}\right)\\ x_2(t) &= 0.65\,\mathrm{Re}\left(\frac{0.85A}{-\omega^2 + i0.05\omega + 46.24}e^{i\omega t}\right) - 0.5\,\mathrm{Re}\left(\frac{1.1A}{-\omega^2 + i0.061\omega + 285.61}e^{i\omega t}\right)\end{aligned}$$

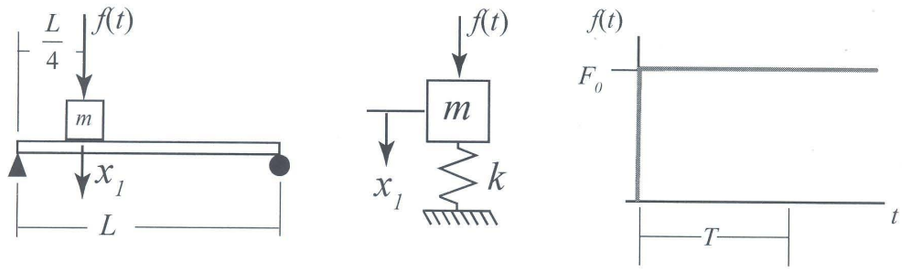
These can be combined to

$$\begin{aligned}x_1(t) &= \mathrm{Re}\left(\left[\frac{0.85^2A}{-\omega^2 + i0.05\omega + 46.24} + \frac{1.1^2A}{-\omega^2 + i0.0609\omega + 285.61}\right]e^{i\omega t}\right)\\ x_2(t) &= \mathrm{Re}\left(\left[\frac{(0.65)(0.85)A}{-\omega^2 + i0.05\omega + 46.24} - \frac{(0.5)(1.1)A}{-\omega^2 + i0.061\omega + 285.61}\right]e^{i\omega t}\right)\end{aligned}$$

4.5.5 Problem 4

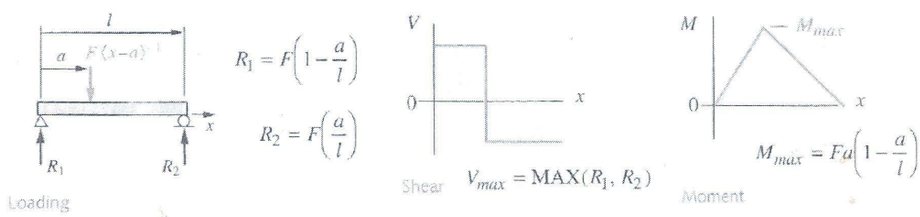
Problem #4 (10 pts)

The system shown consists of a beam with a large mass mounted one fourth of the distance from its end. This can be represented with the undamped spring-mass system shown to the right, with $k=85EI/L^3$. The system is initially in its static equilibrium position when a step force, $f(t)=F_0h(t)$, is applied to the mass.



The following information is available from a static analysis of the beam. When a static load, F, is applied to a beam, the maximum bending stress occurs in the outer fiber of the beam is given by $\sigma_{max}=-M_{max}c/I$, where M_{max} is the maximum bending moment in the beam, c is the (known) distance to the outer fiber and I is the area moment of inertia (also known). See the figure below for additional details regarding a static loading scenario.

(a) Simply supported beam with concentrated loading



What is the amplitude of the load, F_0, that causes the beam to exceed its yield stress, σ_y?

The transient response is given in appendix B as

$$x(t) = \frac{F_0}{k}(1 - \cos\omega_n t)h(t)$$

Hence maximum amplitude of the response is $u_{\max} = \frac{2F_0}{k}$. Compare this to static deflection which is $u_{static} = \frac{F_0}{k}$ then we can say that dynamic load is twice as large as the static load. Therefor using $2F_0$ in place of F in the expression for stress gives the result needed

$$\begin{aligned}\sigma_y &= \frac{-Mc}{I}\\ &= \frac{-(2F_0)\frac{L}{4}\left(\frac{3}{4}\right)}{I/c}\\ &= -\frac{3}{8}L\frac{c}{I}F_0\end{aligned}$$

Therefore

$$F_0 = -\frac{8I}{3Lc}\sigma_y$$

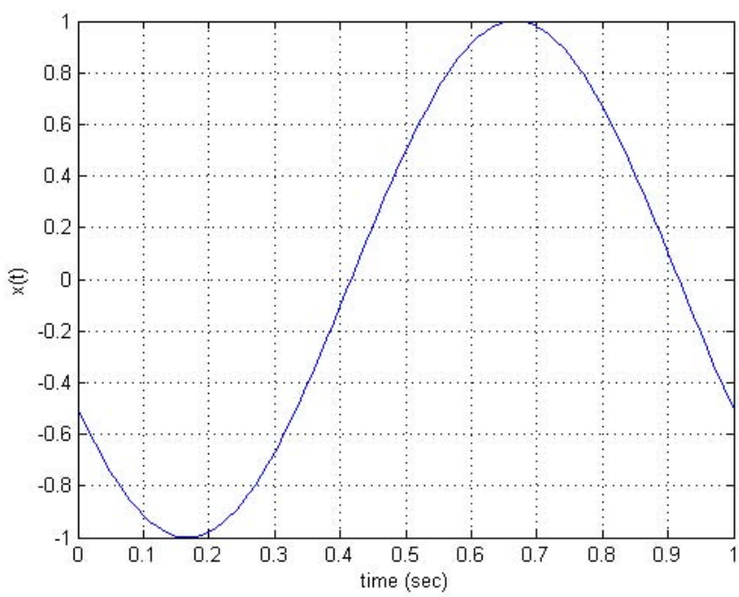
4.5.6 Problem 5

Problem #5 (10 pts)

A single degree-of-freedom system's response is given by $x(t) = \mathrm{Re}\left(Xe^{i\omega t}\right)$, with $X = e^{i2\pi/3}$. Sketch the complex amplitude, X, in the complex plane and sketch the corresponding time function $x(t)$ over at least one cycle.

At $t = 0$ then $x(t) = \mathrm{Re}\left(e^{i\frac{2\pi}{3}}\right)$ which is $-\cos\left(60^0\right) = -\frac{1}{2}$. Using $\omega = 2\pi$ rad/sec then $x(t)$ can be traced. Here is a plot

```
I=sqrt(-1);
w=2*pi;
x=@(t) real(exp(I*2*pi/3)*exp(I*w*t))
t=0:.01:1;
plot(t,x(t))
grid
xlabel('time (sec)'); ylabel('x(t)');
```



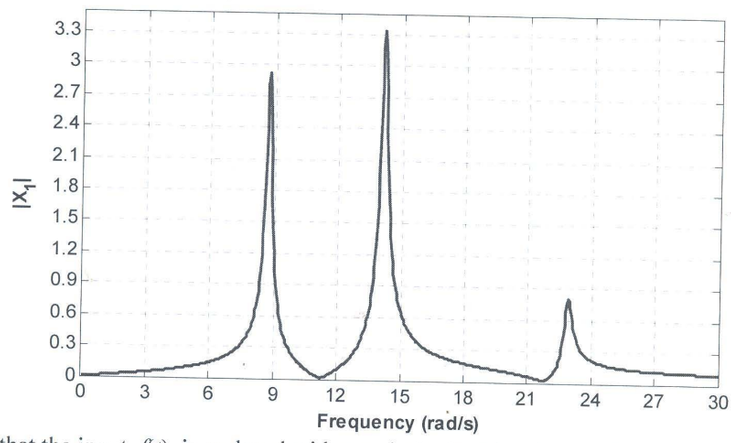
4.5.7 Problem 6

Problem #6 (10 pts)

A three degree-of-freedom system is excited by a sinusoidal force, $f(t)=\cos(\omega t)$.

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}f(t)$$

The frequency response was computed using $\{X\}=\left(-\omega^2[M]+i\omega[C]+[K]\right)^{-1}\{F\}$ and $|X_1|$ from that calculation is plotted below.



Suppose that the input, $f(t)$, is replaced with a periodic function that can be expressed as follows,

$$f(t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{100-n}{n}\right)e^{in\omega_1 t}$$

with $\omega_1 = 3.0$ rad/s. What frequencies would be present in the steady-state response $x_1(t)$? Which of those would be dominant (i.e. have the largest amplitude)?

Damped resonances are seen at $\omega = 8.5, 14$ and 23 rad/sec. This is where $r = \frac{\omega}{\omega_i}$ is close to unity, where ω is the forcing frequency and ω_i is the natural frequency. Since this is a 3 dof system, it will have 3 natural frequencies.

The response of each dof will take contributions from each mode of vibration. Each mode vibrates at different natural frequency. From the plot above it is seen that the response of $x_1(t)$ has the largest response when the forcing frequency is close to the $\omega_2 = 14$ rad/sec.

The new force now has the following set of discrete harmonics in it: ($n = 0$ is not counted, DC). $\frac{100}{1}e^{3t}, \frac{99}{2}e^{6t}, \frac{98}{3}e^{9t}, \frac{97}{4}e^{12t}, \frac{96}{5}e^{15t}, \frac{95}{6}e^{18t}, \frac{94}{7}e^{21t}, \frac{98}{8}e^{24t}, \cdots$ or

$$f(t) = 100e^{3t}, 49.5e^{6t}, 32.7e^{9t}, 24.3e^{12t}, 19.2e^{15t}, 15.8e^{18t}, 13.4e^{21t}, 12.3e^{24t}$$

So the input force has only discrete frequencies. Since linear sum, each $f_i(t)$ will cause the response $|X|$ at that specific forcing frequency as shown in the plot. Looking the plot it can be seen that when forcing frequency is 9 rad/sec, this will cause the largest $|X|$ among all these set of discrete frequencies. Hence the dominant harmonic is 9 rad/sec and will have amplitude around 2.4 from looking at the plot.

Chapter 5

Quizes

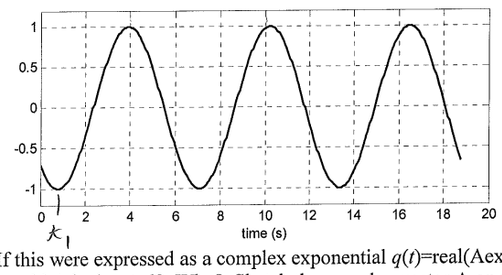
Local contents

5.1 Quiz 1

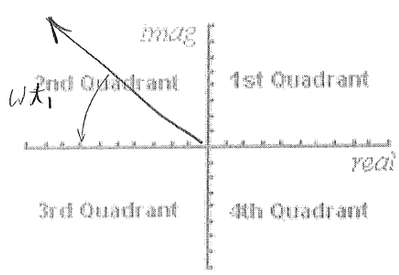
Name: Solution

EMA 545 – Pop Quiz #1 - Prof. M. S. Allen Spr. 2013
Closed book, no calculators

Problem #1 (5 pts each)
The figure below shows a harmonic function $q(t)$ whose frequency is ω=1.0 rad/s.

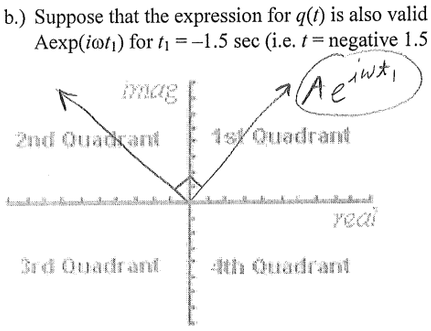


a.) If this were expressed as a complex exponential $q(t)$=real(Aexp($i\omega t$)), in which quadrant would A be located? Why? Sketch the complex vector A on the axes below.



2nd Quad
→ q is negative and increasing, so it would reach its neg. extreme soon
(neg peak at t_1)

b.) Suppose that the expression for $q(t)$ is also valid before time zero. Sketch the complex vector Aexp($i\omega t_1$) for t_1 = –1.5 sec (i.e. t = negative 1.5 seconds).



$\omega = \frac{2\pi}{6} \approx 1$

$Ae^{i(1)(-1.5)} \approx Ae^{-i1.5}$

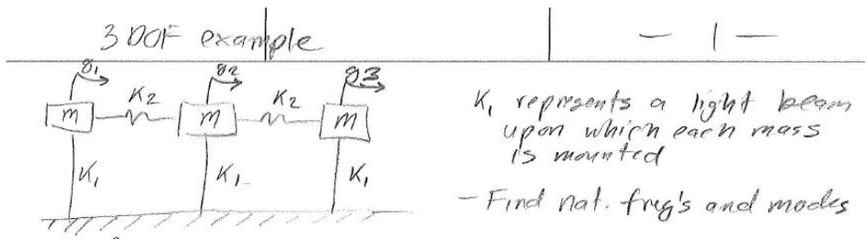
rotate backwards by 1.5 rad ~ 90°

Page 1 of 1

5.2 Quiz 2

5.2.1 Problem description

Consider this 3 DOF system



Suppose a harmonic force $f(t) = A\cos(\bar{\omega}t)$ is applied to the mass in the center. Use modal analysis to do the following:

1. Find the uncoupled modal equations of motion. Consider the steady state solution for each of these equations. Sketch the modal amplitude (X_j in the book on page 275) for each mode versus frequency. A hand sketch is sufficient.
2. Use that result to sketch the frequency response of each of the masses, in other words the complex amplitude Y_n versus ω

5.2.2 Answer part (1)

A summary of the steps needed for full modal analysis is first given. In these steps, a column vector is shown as bold letter $\mathbf{Y}$ and a matrix is shown as $[M]$. In this summary, the system is assumed to have n degree of freedom.

The steps are

1. Determine the system of equations of motion and set up $[M]\mathbf{Y}'' + [C]\mathbf{Y}' + [K]\mathbf{Y} = \mathbf{F}$ in matrix form.
2. Solve the eigenvalue problem $\det\big([K] - \omega^2[M]\big) = 0$ in order to determine the n natural frequencies.
3. For each natural frequency ω_j determine the corresponding j^{th} eigenvector $_j$ by solving $\big([K] - \omega_j^2[M]\big)_j = \mathbf{0}$. In this step, the first component of $_j$ is set to 1 and the other components are solved relative to it.
4. Obtain the normalized eigenvectors Φ_j for each $_j$ using $\Phi_j = \frac{_j}{\sqrt{u_j}}$ where $u_j = {}_j^T[M]_j$. Each u_j will be a scalar.
5. Set up the modal transformation matrix $[\Phi] = [\Phi_1\Phi_2 \cdots \Phi_n]$. This will be an $n \times n$ matrix.
6. The transformation from normal solution $y(t)$ to modal $\eta(t)$ will be $\mathbf{Y} = [\Phi]$ and $= [\Phi]^{-1}\mathbf{Y} = [\Phi]^T[M]\mathbf{Y}$
7. Apply the above transformation on the original equations of motions in matrix form to obtain the equations of motion in modal coordinates $[\Phi]^T[M]\ [\Phi]\mathbf{Y}'' + [\Phi]^T[C][\Phi]\mathbf{Y}' + [\Phi]^T[C][\Phi]\mathbf{Y} = [\Phi]^T\mathbf{F}$. This becomes $\mathbf{I}''(t) + \left[\tilde{C}\right]'(t) + \left[\tilde{K}\right](t) = [\Phi]^T\mathbf{F}$ where $\mathbf{I}$ is the identity matrix, $\left[\tilde{C}\right]$ is a diagonal damping matrix obtained using a method such as weak damping approximation and $\left[\tilde{K}\right]$ is diagonal matrix with diagonal that contains the natural frequencies squared ω_j^2 in each of entries.
8. For steady state solution in modal coordinates, the loading vector $[\Phi]^T\mathbf{F}$ is assumed to be $\mathbf{Q} = [\Phi]^T\mathbf{F} = \mathrm{Re}\big(\hat{\mathbf{Q}}e^{i\bar{\omega}t}\big)$ where $\hat{\mathbf{Q}}$ is the complex amplitude of the loading vector in modal coordinates. Therefore, the steady state solution is $_{ss}(t) = \mathrm{Re}\big(\hat{\mathbf{X}}e^{i\bar{\omega}t}\big)$

where $\hat{\mathbf{X}}$ is the complex amplitude of each modal response is $\hat{X}_j = \frac{{}_j^T\mathbf{F}}{-\varpi^2 + i2\zeta_j\omega_j\varpi + \omega_j^2}$. For a system with no damping this simplifies to $\hat{X}_j = \frac{{}_j^T\mathbf{F}}{-\varpi^2 + \omega_j^2}$. In here, ${}_j^T$ represents the transpose of the jth column of the modal transformation matrix $[\Phi]$, or the transpose of the jth mass normalized eigenvector, and ω_j is the jth natural frequency.

9. Now the steady state solution in modal coordinate is used to obtain the solution in normal coordinates since $\mathbf{Y} = [\Phi]$. Therefore $\mathbf{Y}_{ss} = \mathrm{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left([\Phi]\hat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left(\hat{\mathbf{Y}}e^{i\varpi t}\right)$. In component form $\mathbf{Y}_{ss} = \mathrm{Re}\left(\left(\sum_{j=1}^{n} {}_j\hat{X}_j\right)e^{i\varpi t}\right)$

The EOM are derived in the hand out given. The force $f(t)$ acting on the second mass is now added, resulting in the following equations of motion for the system

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} q_1'' \\ q_2'' \\ q_3'' \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_1 + 2k_2 & -k_2 \\ 0 & -k_2 & k_1 + k_2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ A\cos(\varpi t) \\ 0 \end{Bmatrix}$$

The first step is to obtain the natural frequencies of the system. This is done by solving the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$. The solutions are also given in handout. They are $\omega_1^2 = \frac{k_1}{m}, \omega_2^2 = \frac{k_1+k_2}{m}, \omega_3^2 = \frac{k_1+3k_2}{m}$. The non mass normalized eigenvectors associated with these eigenvalues are found as

$${}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, {}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, {}_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

The next step is to mass normalize the eigenvectors as follows

$$\mu_1 = {}_1^T[M]_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3m$$

$$\mu_2 = {}_2^T[M]_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2m$$

$$\mu_3 = {}_3^T[M]_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}^T \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6m$$

Hence the mass normalized eigenvectors are

$$
\begin{aligned}
{}_1 &= \frac{{}_1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{3m}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}\\
{}_2 &= \frac{{}_2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{2m}}\begin{Bmatrix}1\\0\\-1\end{Bmatrix}\\
{}_3 &= \frac{{}_3}{\sqrt{\mu_3}} = \frac{1}{\sqrt{6m}}\begin{Bmatrix}1\\-2\\1\end{Bmatrix}
\end{aligned}
$$

Hence the modal transformation matrix $[\Phi]$ is

$$
[\Phi] = \begin{bmatrix}{}_{123}\end{bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}}\end{bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix}0.577 & 0.707 & 0.408\\ 0.577 & 0 & -0.816\\ 0.577 & -0.707 & 0.408\end{bmatrix}
$$

The modal EOM's are now found using the modal transformation matrix $[\Phi]$

$$
[\Phi]^T[M][\Phi]\left\{\eta''\right\} + [\Phi]^T[K][\Phi]\left\{\eta\right\} = [\Phi]^T\mathbf{Q}
$$

$$
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\end{Bmatrix} + \begin{bmatrix}\omega_1^2&0&0\\0&\omega_2^2&0\\0&0&\omega_3^2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{Bmatrix} = [\Phi]^T\begin{Bmatrix}0\\ A\cos(\varpi t)\\ 0\end{Bmatrix}
$$

$$
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\end{Bmatrix} + \frac{1}{m}\begin{bmatrix}k_1&0&0\\0&k_1+k_2&0\\0&0&k_1+3k_2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{bmatrix}0.577&0.707&0.408\\0.577&0&-0.816\\0.577&-0.707&0.408\end{bmatrix}^T\begin{Bmatrix}0\\ A\cos(\varpi t)\\ 0\end{Bmatrix}
$$

$$
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\end{Bmatrix} + \frac{1}{m}\begin{bmatrix}k_1&0&0\\0&k_1+k_2&0\\0&0&k_1+3k_2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\end{Bmatrix} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.577\,A\cos(\varpi t)\\ 0\\ -0.816\,A\cos(\varpi t)\end{Bmatrix}
$$

Therefore, the 3 uncoupled modal EOM's are

$$
\begin{aligned}
\eta_1''(t) + \frac{k_1}{m}\eta_1(t) &= \frac{0.577\,A}{\sqrt{m}}\cos(\varpi t)\\
\eta_2''(t) + \frac{k_1+k_2}{m}\eta_2(t) &= 0\\
\eta_3''(t) + \frac{k_1+3k_2}{m}\eta_3(t) &= -\frac{0.816\,A}{\sqrt{m}}\cos(\varpi t)
\end{aligned}
$$

To complete the solution, the above EOM are written as follows by using complex form

for the loading vector

$$\begin{aligned}
\eta_1''(t) + \frac{k_1}{m}\eta_1(t) &= \mathrm{Re}\left(\frac{0.577\,A}{\sqrt{m}}e^{i\varpi t}\right)\\
\eta_2''(t) + \frac{k_1+k_2}{m}\eta_2(t) &= 0\\
\eta_3''(t) + \frac{k_1+3k_2}{m}\eta_3(t) &= \mathrm{Re}\left(\frac{-0.816\,A}{\sqrt{m}}e^{i\varpi t}\right)
\end{aligned}$$

Assuming the steady state solution is

$$= \mathrm{Re}\left(\hat{\mathbf{X}}e^{i\varpi t}\right)$$

or in expanded form

$$\begin{aligned}
\eta_1(t) &= \mathrm{Re}\left(\hat{X}_1 e^{i\varpi t}\right)\\
\eta_2(t) &= \mathrm{Re}\left(\hat{X}_2 e^{i\varpi t}\right)\\
\eta_3(t) &= \mathrm{Re}\left(\hat{X}_3 e^{i\varpi t}\right)
\end{aligned}$$

Where

$$\begin{aligned}
\hat{X}_1 &= \frac{\frac{0.577\,A}{\sqrt{m}}}{\omega_1^2 + 2i\zeta_1\omega_1\varpi - \varpi^2}\\
\hat{X}_2 &= 0\\
\hat{X}_3 &= \frac{\frac{-0.816\,A}{\sqrt{m}}}{\omega_3^2 + 2i\zeta_3\omega_3\varpi - \varpi^2}
\end{aligned}$$

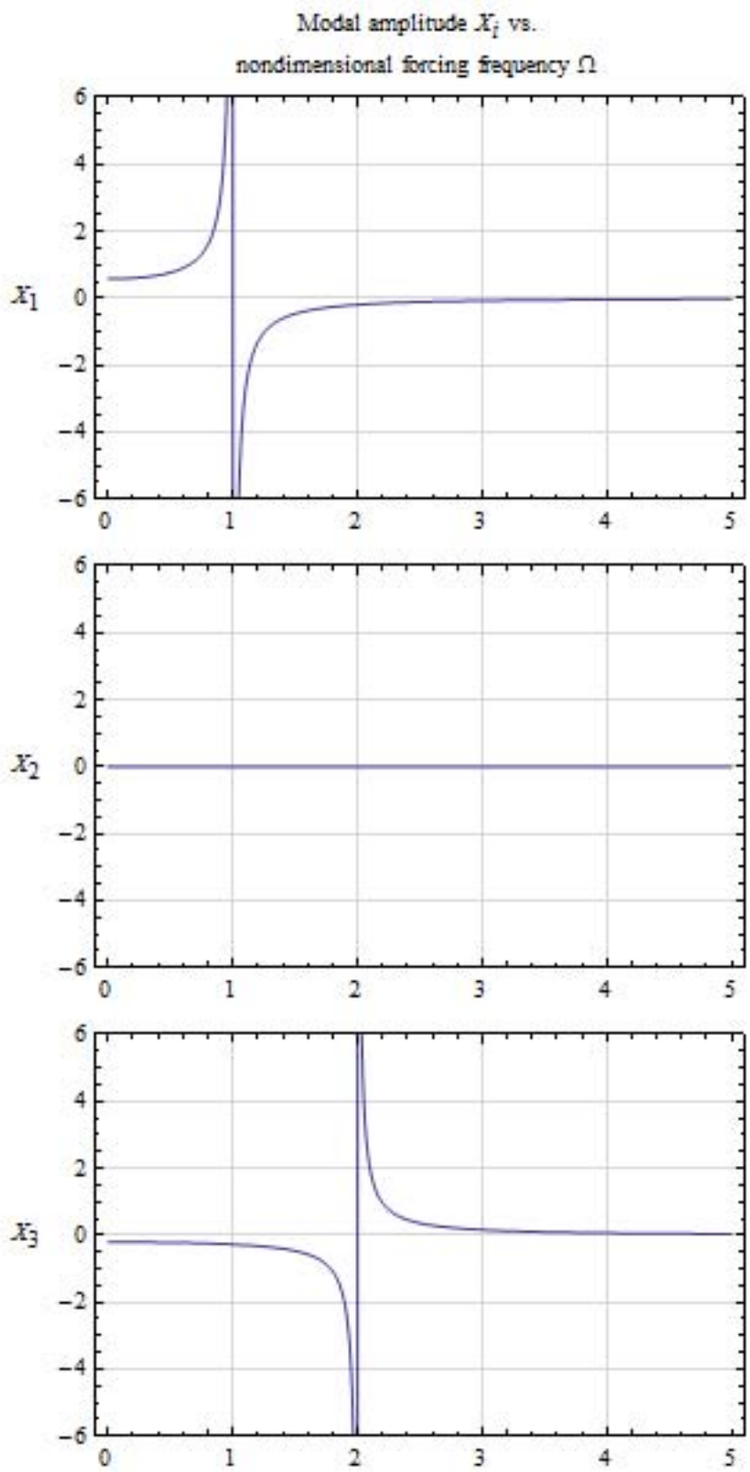
Dividing the numerator and the denominator by ω_i^2 where $i = 1, 2, 3$ and using $r_i = \frac{\varpi}{\omega_i}$ and letting $\zeta = 0$ since no damping exists, results in

$$\begin{aligned}
\hat{X}_1 &= \frac{A\sqrt{m}}{k_1}\left(\frac{0.577}{1 - m\frac{\varpi^2}{k_1}}\right)\\
\hat{X}_2 &= 0\\
\hat{X}_3 &= \frac{A\sqrt{m}}{k_1+3k_2}\left(\frac{-0.816}{1 - m\frac{\varpi^2}{k_1+3k_2}}\right)
\end{aligned}$$

To sketch these amplitudes, the equations are normalized. This is in effect the same as setting $m = 1, k_1 = k_2 = 1, A = 1$ resulting in

$$\hat{\mathbf{X}} = \begin{Bmatrix}\hat{X}_1\\ \hat{X}_2\\ \hat{X}_3\end{Bmatrix} = \begin{Bmatrix}\frac{0.577}{1-\varpi^2}\\ 0\\ \frac{1}{4}\left(\frac{-0.816}{1-\frac{\varpi^2}{4}}\right)\end{Bmatrix}$$

Here is a plot of each X_i vs ϖ. The x-axis is the nondimensional forcing frequency Ω



Since there is no damping, resonance will occur at $\Omega = 1$ in first mode and at $\Omega = 2$ for mode 3.

5.2.3 Answer part (2)

The transformation from modal coordinates to normal coordinates is

$$\mathbf{q} = [\Phi]$$

In expanded form

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \Phi_1^T \{\eta\} \\ \Phi_2^T \{\eta\} \\ \Phi_3^T \{\eta\} \end{Bmatrix}$$

But $[\Phi] = \frac{1}{\sqrt{m}}\begin{bmatrix} 0.577 & 0.707 & 0.408 \\ 0.577 & 0 & -0.816 \\ 0.577 & -0.707 & 0.408 \end{bmatrix}$ and $= \text{Re}\left(\hat{\mathbf{X}}e^{i\omega t}\right)$ hence the above becomes

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} \begin{Bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{Bmatrix}^T \begin{Bmatrix} \text{Re}\left(\hat{X}_1 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_2 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_3 e^{i\omega t}\right) \end{Bmatrix} \\ \begin{Bmatrix} 0.707 \\ 0 \\ -0.707 \end{Bmatrix}^T \begin{Bmatrix} \text{Re}\left(\hat{X}_1 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_2 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_3 e^{i\omega t}\right) \end{Bmatrix} \\ \begin{Bmatrix} 0.408 \\ -0.816 \\ 0.408 \end{Bmatrix}^T \begin{Bmatrix} \text{Re}\left(\hat{X}_1 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_2 e^{i\omega t}\right) \\ \text{Re}\left(\hat{X}_3 e^{i\omega t}\right) \end{Bmatrix} \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.577\ \text{Re}\left(X_1 e^{it\omega}\right) + 0.577\ \text{Re}\left(X_2 e^{it\omega}\right) + 0.577\ \text{Re}\left(X_3 e^{it\omega}\right) \\ 0.707\ \text{Re}\left(X_1 e^{it\omega}\right) - 0.707\ \text{Re}\left(X_3 e^{it\omega}\right) \\ 0.408\ \text{Re}\left(X_1 e^{it\omega}\right) - 0.816\ \text{Re}\left(X_2 e^{it\omega}\right) + 0.408\ \text{Re}\left(X_3 e^{it\omega}\right) \end{Bmatrix}$$

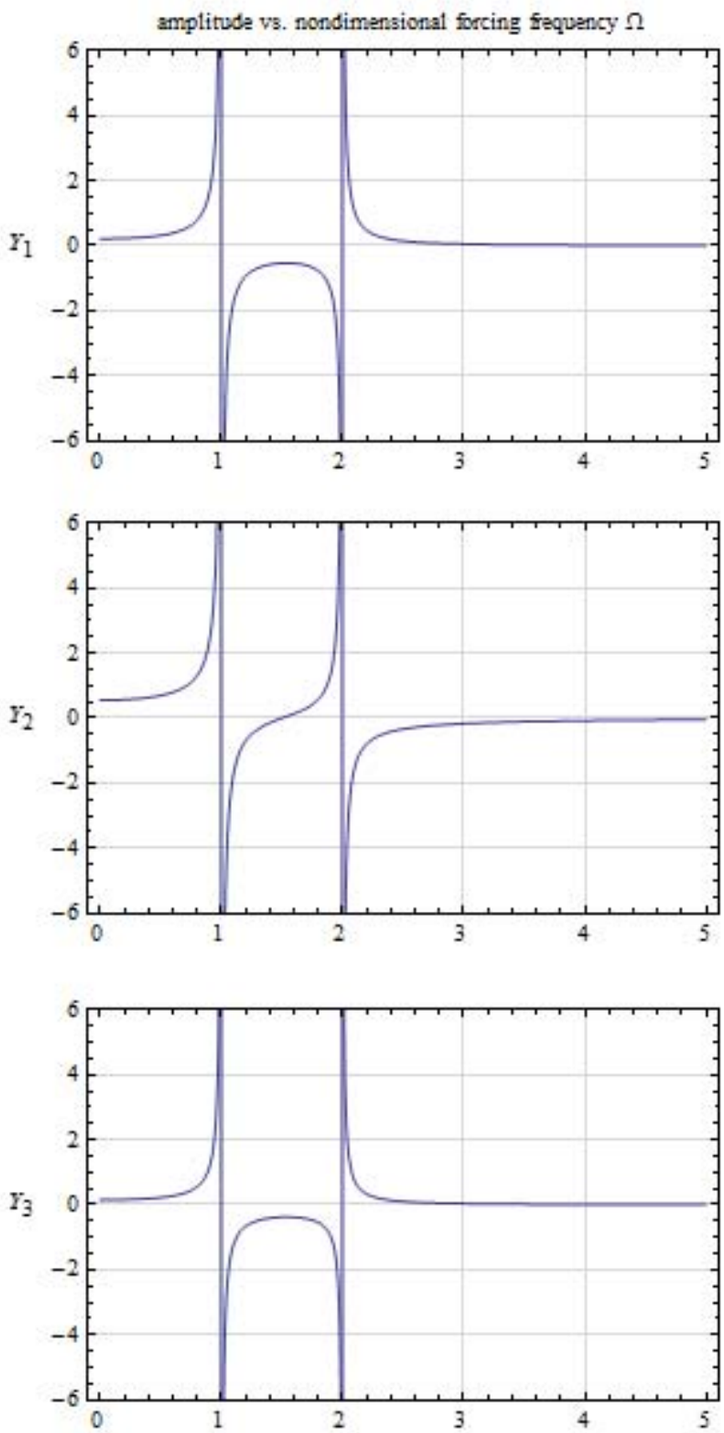
$$= \text{Re}\left(\begin{Bmatrix} 0.577\, X_1 + 0.577\, X_2 + 0.577\, X_3 \\ 0.707\, X_1 - 0.707\, X_3 \\ 0.408\, X_1 - 0.816\, X_2 + 0.408\, X_3 \end{Bmatrix} e^{i\omega t}\right)$$

Comparing the above to $\mathbf{q}_{ss} = \text{Re}\left(\mathbf{Y}e^{i\omega t}\right)$ shows that

$$\mathbf{Y} = \begin{Bmatrix} 0.577\, X_1 + 0.577\, X_2 + 0.577\, X_3 \\ 0.707\, X_1 - 0.707\, X_3 \\ 0.408\, X_1 - 0.816\, X_2 + 0.408\, X_3 \end{Bmatrix}$$

To plot each Y_i, let $m = 1, k_1 = 1, k_2 = 1, A = 1$, and letting $X_2 = 0$ as found earlier, results in

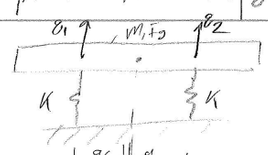
$$\mathbf{Y} = \begin{Bmatrix} 0.577\, \frac{0.577}{1-\omega^2} + \frac{0.577}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \\ 0.707 \frac{0.577}{1-\omega^2} - \frac{0.707}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \\ 0.408\, \frac{0.577}{1-\omega^2} + \frac{0.408}{4}\left(\frac{-0.816}{1-\frac{\omega^2}{4}}\right) \end{Bmatrix}$$



The above shows that when the nondimensional frequency Ω is not close to a one of the nondimensional natural frequencies, then the Y values have comparable magnitudes. For nondimensional frequency Ω larger than 3 all amplitude are zero, which means the whole system does not oscillate any more in steady state.

5.2.4 Key solution

Repeated Nat. Freq's — 2 —



$J_g = m r^2$

$[M]\{\ddot{q}\} + [K]\{q\} = 0$

$$M = \begin{bmatrix} \left(\frac{ma}{4} + \frac{J_g}{a}\right) & \left(\frac{ma}{4} - \frac{J_g}{a}\right) \\ \left(\frac{ma}{4} - \frac{J_g}{a}\right) & \left(\frac{ma}{4} + \frac{J_g}{a}\right) \end{bmatrix}$$

$$M = ma \begin{bmatrix} \frac{1}{4} + \left(\frac{r}{a}\right)^2 & \frac{1}{4} - \left(\frac{r}{a}\right)^2 \\ \frac{1}{4} - \left(\frac{r}{a}\right)^2 & \frac{1}{4} + \left(\frac{r}{a}\right)^2 \end{bmatrix} \qquad K = Ka \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\omega_1^2 = \frac{2K}{m} \quad \phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \qquad \omega_2^2 = \frac{2K}{m}\left(\frac{a}{2r}\right)^2 \quad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

if $(r/a) = 1/2$, $\omega_1 = \omega_2$

$$M = \frac{ma}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ ~ already diagonal (more complicated in general)

$[K - \omega^2 M]\phi =$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\{\phi\} = 0$$ ~ anything could be an eigenvector!

Choose arbitrary

$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \phi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\{\phi_1\}^T [M] \{\phi_2\} = 0$

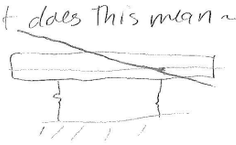
$$\{1 \ 0\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \neq 0$$ ~ we've lost orthogonality

Procedure: choose ϕ_1, find ϕ_2 to be orth (Graham Schmidt) $(\phi_1^T M \phi_2) = 0$

$\phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \{1 \ 0\}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \rightarrow a = 0$, b - anything choose 1

→ Mass normalize

What does this mean ~

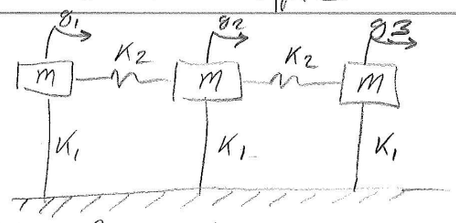


Start the system in any initial pattern, that pattern persists for all time at $\omega_1 = \omega_2$!

$q = \{\phi_1\}\eta_1 + \{\phi_2\}\eta_2$

same frequency

3 DOF example — 1 —



K_1 represents a light beam upon which each mass is mounted

– Find nat. freq's and modes

EOM → Power Balance →

$$T = \frac{1}{2}\left(m\dot{q}_1^2 + m\dot{q}_2^2 + m\dot{q}_3^2\right)$$

$$V = \frac{1}{2}K_1\left(q_1^2 + q_2^2 + q_3^2\right) + \frac{1}{2}K_2(q_2 - q_1)^2 + \frac{1}{2}K_2(q_3 - q_1)^2$$

all connected to ground

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}\{\ddot{q}\} + \begin{bmatrix} (K_1+K_2) & -K_2 & 0 \\ -K_2 & (K_1+2K_2) & -K_2 \\ 0 & -K_2 & (K_1+K_2) \end{bmatrix}\{q\} = 0$$

– No stiffness coupling between q_1 + q_3 ✓

Solve EVP — $[K - \omega^2[M]]\{\phi\} = 0$ → (Matlab)

$$\left|\begin{bmatrix} (K_1+K_2-\omega^2 m) & \boxed{-K_2} & 0 \\ -K_2 & (K_1+2K_2-\omega^2 m) & -K_2 \\ 0 & -K_2 & (K_1+K_2-\omega^2 m) \end{bmatrix}\right| = 0$$

Rule for determinant

det|[ ]| = 0 take ⬭ • det( ) + ▭ • det(...

$$(K_1+K_2-\omega^2 m)\left[(K_1+2K_2-\omega^2 m)(K_1+K_2-\omega^2 m) - K_2^2\right] +$$

$$-(-K_2)\left[(-K_2)(K_1+K_2-\omega^2 m) - 0\right] + 0[\quad]$$

Sign!

factor out leaves only K_2^2

$$(K_1+K_2-\omega^2 m)\left[(K_2+2K_2-\omega^2 m)(K_1+K_2-\omega^2 m) - K_2^2 - K_2^2\right] = 0$$

This gives one eigenvalue:

$$(K_1+K_2-\omega^2 m) = 0 \rightarrow \omega^2 = \frac{K_1+K_2}{m}$$

solving part in [ ] gives ω_1, ω_3 →

$$\omega_1^2 = \frac{K_1}{m} \qquad \omega_2^2 = \frac{K_1+K_2}{m} \qquad \omega_3^2 = \frac{K_1+3K_2}{m}$$

4/7 3 DOF Example - Continued — 2 —

for ϕ_2 $[\phi_1 \phi_2 \phi_3]$

$$\begin{bmatrix} 0 & -K_2 & 0 \\ -K_2 & K_2 & -K_2 \\ 0 & -K_2 & 0 \end{bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = 0 \rightarrow -K_2 \phi_{22} = 0$$

$$\rightarrow -K_2 \phi_{12} - K_2 \phi_{32} = 0$$

$$\phi_{12} = -\phi_{32}$$

$$\rightarrow \phi_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

ϕ_1

$$\begin{bmatrix} K_2 & -K_2 & 0 \\ -K_2 & 2K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = 0 \rightarrow K_2\phi_{11} - K_2\phi_{21} = 0 \rightarrow \phi_{11} = \phi_{21}$$

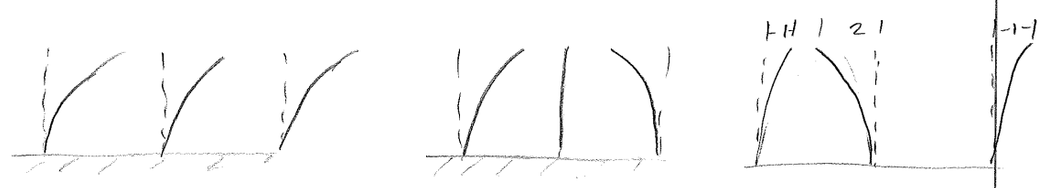
$$\rightarrow \phi_{21} = \phi_{31} \rightarrow \phi_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

ϕ_3 (skip + give answer)

$$\begin{bmatrix} -2K_2 & -K_2 & 0 \\ -K_2 & -K_2 & -K_2 \\ 0 & -K_2 & -2K_2 \end{bmatrix} \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \rightarrow 2\phi_{13} = -\phi_{23} \rightarrow \phi_3 = \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

$$\rightarrow \phi_{23} = -2\phi_{33}$$

Mode 1 Mode 2 Mode 3



→ Mass Normalize

In matlab, → Generalized eigenvalue prob – $[A]\{x\} = \lambda[B]\{x\}$

→ [phi, λ] = eig(K, M)

```
m = 1;   K1 = 1;   K2 = 2;

M = [m, 0, 0; 0, m, 0; 0, 0, m];
K = [K1+K2, -K2, 0; -K2, K1+2K2, -K2; 0, -K2, K1+K2];
[phi, lam] = eig(K, M);
wnsq = diag(lam);
wn = (wnsq).^(1/2)
[wnsort, Isort] = sort(wn);
phisort = phi(:, Isort);
```

4/7 3DOF Example — 3 —

mass normalizing:

```
for k = 1:length(wn)
    mu = phi(:,k).'*M*phi(:,k);
    PHI(:,k) = phi(:,k)/sqrt(mu);
end
```

or phisort

or — a slick way to do it in one line

```
PHI = phi*diag(sqrt(diag(phi.'*M*phi)).^-1);
```

$$\begin{bmatrix} \phi_1 & \phi_2 & \cdots \end{bmatrix} \begin{bmatrix} \mu_1^{-1/2} & & \\ & \mu_2^{-1/2} & \\ & & \ddots \end{bmatrix} \quad \checkmark$$

Chapter 6

appendix

Local contents

6.1 cheat sheet

By Nasser M. Abbasi

$$\xi = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

$$r = \frac{\varpi}{\omega_n}$$

$$T_d = \frac{2\pi}{\omega_d}$$

$$\beta = \frac{1}{\sqrt{\left(1-r^2\right)^2+(2r\xi)^2}}$$

$$\beta_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\frac{y_n}{y_{n+1}} = e^{\frac{\zeta\omega_n 2\pi}{\omega_D}} \quad \text{small damping} \Rightarrow e^{\frac{\zeta 2\pi}{\sqrt{1-\zeta^2}}} \Rightarrow e^{\zeta 2\pi}$$

$$\ln\left(\frac{y_n}{y_{n+1}}\right) = \zeta 2\pi$$

$$\frac{1}{M}\ln\left(\frac{y_n}{y_{n+M}}\right) = \zeta 2\pi$$

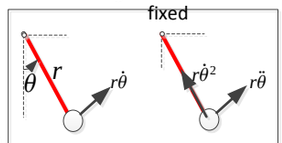
time constant $\tau = \frac{1}{\zeta\omega_n}$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Springs: in parallel, ADD

Sin[x]	Cos[x]
$\frac{1}{2} i e^{-ix} - \frac{1}{2} i e^{ix}$	$\frac{e^{-ix}}{2} + \frac{e^{ix}}{2}$

Length of pendulum fixed



Velocity diagram

acceleration diagram

$$f = \varepsilon m\Omega^2\sin(\Omega t)$$

number of cycles needed for peak to decay by half	ξ (%)
1.	11.0318
2.	5.51589
3.	3.67726
4.	2.75795
5.	2.20636
6.	1.83863
7.	1.57597
8.	1.37897
9.	1.22575
10.	1.10318

$\beta_{\max}$ when $r = \sqrt{1-2\xi^2}$

$$u'' + 2\xi\omega u' + \omega^2 u = 0$$

	roots
$\xi < 1$	$\left\{-\xi\omega + j\omega_n\sqrt{1-\xi^2}, -\xi\omega - i\omega_n\sqrt{1-\xi^2}\right\}$
$\xi = 1$	$\{-\omega, -\omega\}$
$\xi > 1$	$\left\{-\omega_n\xi + \omega_n\sqrt{\xi^2-1}, -\omega_n\xi - \omega_n\sqrt{\xi^2-1}\right\}$

$$my'' + cy' + ky = \mathrm{Re}\left(\hat{F}e^{i\varpi t}\right)$$
$$x = \mathrm{Re}\left\{\hat{X}e^{i\varpi t}\right\}$$
$$\hat{X} = \frac{\hat{F}}{k}D(r,\zeta)$$
$$D(r,\zeta) = \frac{1}{(1-r^2)+2i\zeta r}$$
$$x = \mathrm{Re}\left\{\frac{\hat{F}}{k}|D(r,\zeta)|e^{i(\varpi t-\theta)}\right\}$$
$$\theta = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

$\partial_t t^3$	=	$3t^2$
$\partial_t \mathrm{Cos}[3t]$	=	$-3\,\mathrm{Sin}[3t]$
$\partial_t \mathrm{Sin}[t]$	=	$\mathrm{Cos}[t]$
$\partial_t \mathrm{Cos}[t]$	=	$-\mathrm{Sin}[t]$
$\partial_t \mathrm{Tan}[t]$	=	$\mathrm{Sec}[t]^2$

$\int t^3\,dt$	=	$\frac{t^4}{4}$
$\int \mathrm{Cos}[3t]\,dt$	=	$\frac{1}{3}\mathrm{Sin}[3t]$
$\int \mathrm{Sin}[t]\,dt$	=	$-\mathrm{Cos}[t]$
$\int \mathrm{Cos}[t]\,dt$	=	$\mathrm{Sin}[t]$
$\int e^{3t}\,dt$	=	$\frac{e^{3t}}{3}$

Let $y = \mathrm{Re}\left(\hat{Y}e^{i\varpi t}\right)$ $\hat{Y} = \frac{\frac{\hat{F}}{m}}{\left(-\varpi^2+2\zeta\omega_n i\varpi+\omega_n^2\right)}$

$$\zeta = 0 \quad y = \mathrm{Re}\left(\frac{\hat{F}}{k}\frac{1}{(1-r^2)}e^{i\varpi t}\right)$$

$$\zeta > 0 \quad y = \mathrm{Re}\left(\frac{F}{k}\frac{1}{(1-r^2)+i2\zeta r}e^{i\varpi t}\right)$$

```
{Cos[Pi/2+x], Cos[Pi/2-x], Sin[Pi/2-x], Sin[Pi/2+x]}
{-Sin[x], Sin[x], Cos[x], Cos[x]}
```

When y=0, then y in complex plan is pure imaginary. When force is max, then f in complex plan is all real

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right) - \frac{\partial L}{\partial x_i}$$

system	equation used to derive	transfer function
isolate base from force transmitted by machine	$f_{tr}(t) = f_{spring} + f_{damper}$	$\frac{\lvert f_{tr}(t)\rvert_{\max}}{\lvert\hat{F}\rvert} = \lvert D\rvert\sqrt{1+(2\zeta r)^2}$
isolate machine from motion of base	Use absolute mass position $my'' + cy' + ky = cz' + kz$	$\frac{\lvert Y\rvert_{\max}}{\lvert Z\rvert} = \lvert D\rvert\sqrt{1+(2\zeta r)^2}$
accelerameter: Measue base acc. using relative displacement	Use relative mass position $mu'' + cu' + ku = -mz''$	$U = \frac{-1}{\left(\omega_n^2-\omega^2\right)+i\omega 2\zeta\omega_n}Z_a \Rightarrow \lvert D(r,\zeta)\rvert = \frac{-1}{\sqrt{\left(\omega_n^2-\omega^2\right)^2+(2\omega\zeta\omega_n)^2}}$
seismometer: Measue base motion using relative displacement	Use relative mass position $mu'' + cu' + ku = -mz''$	$U = \frac{r^2}{(1-r^2)+i2\zeta r}Z \Rightarrow \lvert D(r,\zeta)\rvert = \frac{r^2}{\sqrt{(1-r^2)+i2\zeta r}}$

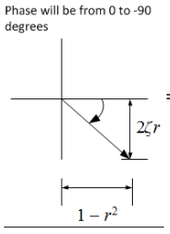
$$f_{tr}(t) = f_{spring} + f_{damper}$$
$$= \mathrm{Re}\left\{\left(\hat{F} + ci\varpi\frac{\hat{F}}{k}\right)D(r,\zeta)e^{i\varpi t}\right\}$$

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{K}\frac{1}{\sqrt{\left(1-r^2\right)^2+(2\xi r)^2}}\sin(\varpi t - \theta)$$

$$A = u_0 + \frac{F}{K}\frac{1}{\sqrt{\left(1-r^2\right)^2+(2\xi r)^2}}\sin\theta$$

$$B = \frac{v_0}{\omega_d} + \frac{u_0\xi\omega}{\omega_d} + \frac{F}{K}\frac{1}{\omega_d\sqrt{\left(1-r^2\right)^2+(2\xi r)^2}}(\xi\omega\sin\theta - \varpi\cos\theta)$$

Phase of response complex amplitude for underdamped and when r<1
Phase will be from 0 to -90 degrees



$$|f_{tr}(t)|_{\max} \qquad \zeta < 1$$

$$= \left|\hat{F}\right||D|\sqrt{1+(2\zeta r)^2}$$

Same for vibration isolation

$$u(t) \quad \zeta = 0 \begin{cases} \varpi = \omega \rightarrow u(0)\cos\varpi t + \frac{u'(0)}{\varpi}\sin\varpi t - \frac{F}{K}\frac{\varpi t}{2}\cos(\varpi t) \\ \varpi \neq \omega \rightarrow u(0)\cos\omega t + \left(\frac{u'(0)}{\omega} - \frac{F}{K}\frac{r}{1-r^2}\right)\sin\omega t + \frac{F}{K}\frac{1}{1-r^2}\sin\varpi t \end{cases}$$

1. Determine the system of equations of motion and set up $[M]\mathbf{Y}'' + [C]\mathbf{Y}' + [K]\mathbf{Y} = \mathbf{F}$ in matrix form.
2. Solve the eigenvalue problem $\det([K] - \omega^2[M]) = 0$ in order to determine the n natural frequencies.
3. For each natural frequency ω_j determine the corresponding j^{th} eigenvector $\boldsymbol{\varphi}_j$ by solving $\left([K] - \omega_j^2[M]\right)\boldsymbol{\varphi}_j = \mathbf{0}$. In this step first component of $\boldsymbol{\varphi}_j$ is set to $\mathbf{1}$ and the other components are solved relative to it.
4. Obtain the normalized eigenvectors $\boldsymbol{\Phi}_j$ for each $\boldsymbol{\varphi}_j$ using $\boldsymbol{\Phi}_j = \frac{\boldsymbol{\varphi}_j}{\sqrt{u_j}}$ where $u_j = \boldsymbol{\varphi}_j^T[M]\boldsymbol{\varphi}_j$. Each u_j will be a scalar.
5. Set up the modal transformation matrix $[\boldsymbol{\Phi}] = [\boldsymbol{\Phi}_1\boldsymbol{\Phi}_2 \cdots \boldsymbol{\Phi}_n]$. This will be an $n \times n$ matrix.
6. The transformation from normal solution $\boldsymbol{y}(t)$ to modal $\boldsymbol{\eta}(t)$ will be $\mathbf{Y} = [\boldsymbol{\Phi}]\boldsymbol{\eta}$ and $\boldsymbol{\eta} = [\boldsymbol{\Phi}]^{-1}\mathbf{Y} = [\boldsymbol{\Phi}]^T[M]\mathbf{Y}$
7. Apply the above transformation on the original equations of motions in matrix form to obtain the equations of motion in modal coordinates $[\boldsymbol{\Phi}]^T[M]\,[\boldsymbol{\Phi}]\mathbf{Y}'' + [\boldsymbol{\Phi}]^T[C][\boldsymbol{\Phi}]\mathbf{Y}' + [\boldsymbol{\Phi}]^T[C][\boldsymbol{\Phi}]\mathbf{Y} = [\boldsymbol{\Phi}]^T\mathbf{F}$. This becomes $\mathbf{I}\boldsymbol{\eta}''(t) + \left[\widetilde{C}\right]\boldsymbol{\eta}'(t) + \left[\widetilde{K}\right]\boldsymbol{\eta}(t) = [\boldsymbol{\Phi}]^T\mathbf{F}$ where $\mathbf{I}$ is the identity matrix, $\left[\widetilde{C}\right]$ is a diagonal damping matrix obtained using a method such as weak damping approximation and $\left[\widetilde{K}\right]$ is diagonal matrix with diagonal that contains the natural frequencies squared ω_j^2 in each of entries.
8. For steady state solution in modal coordinates, the loading vector $[\boldsymbol{\Phi}]^T\mathbf{F}$ is assumed to be $\mathbf{Q} = [\boldsymbol{\Phi}]^T\mathbf{F} = \mathrm{Re}\left(\widehat{\mathbf{Q}}e^{i\varpi t}\right)$ where $\widehat{\mathbf{Q}}$ is the complex amplitude of the loading vector in modal coordinates. Therefore, the steady state solution is $\boldsymbol{\eta}_{ss}(t) = \mathrm{Re}\left(\widehat{\mathbf{X}}e^{i\varpi t}\right)$ where $\widehat{\mathbf{X}}$ is the complex amplitude of each modal response is $\widehat{X}_j = \frac{\boldsymbol{\Phi}_j^T\mathbf{F}}{-\varpi^2 + i2\zeta_j\omega_j\varpi + \omega_j^2}$. For a system with no damping this simplifies to $\widehat{X}_j = \frac{\boldsymbol{\Phi}_j^T\mathbf{F}}{-\varpi^2 + \omega_j^2}$. In here, $\boldsymbol{\Phi}_j^T$ represents the transpose of the jth column of the modal transformation matrix $[\boldsymbol{\Phi}]$, or the transpose of the jth mass normalized eigenvector, and ω_j is the jth natural frequency.
9. Now the steady state solution in modal coordinate is used to obtain the solution in normal coordinates since $\mathbf{Y} = [\boldsymbol{\Phi}]\boldsymbol{\eta}$. Therefore $\mathbf{Y}_{ss} = \mathrm{Re}\left(\widehat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left([\boldsymbol{\Phi}]\widehat{\mathbf{X}}e^{i\varpi t}\right) = \mathrm{Re}\left(\hat{\mathbf{Y}}e^{i\varpi t}\right)$. In component form $\mathbf{Y}_{ss} = \mathrm{Re}\left(\left(\sum_{j=1}^{n}\boldsymbol{\Phi}_j\widehat{X}_j\right)e^{i\varpi t}\right)$

In light daming, the off-diagonal entries are set to zero and then $c_{11} = 2\zeta_1\omega_1$

$$\frac{1}{i} = -i = e^{-\frac{\pi}{2}}$$

½ power point, means the max amplitude at that frequency is 0.707 of the maximum possible amplitude (which happens at resonance)

$\Phi_i^T[M]\Phi_i = [I]$ only if $i = j$, else 0

For modal solution, $\{Y\} = \sum_j \Phi_j X_j$

or

$$Y_1 = \Phi_{11}X_1 + \Phi_{12}X_2 + \cdots$$
$$Y_2 = \Phi_{21}X_1 + \Phi_{22}X_2 + \cdots$$

or

$$Y_1 = \{\Phi\}_{\text{row}1}\{X\}$$
$$Y_2 = \{\Phi^T\}_{\text{row}2}\{X\}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$
$$\cos u \cos v = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$
$$\sin u \cos v = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$
$$\cos u \sin v = \frac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

If using power method

$$V_{g_{11}} = \left(\frac{\partial V_g^2}{\partial\theta_1^2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = \left(-m_1 g\frac{L}{2}\sin\theta_1\right)_{\theta_1=0} = 0$$
$$V_{g_{22}} = \left(\frac{\partial V_g^2}{\partial\theta_2^2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = \left(-m_1 g\frac{L}{2}\sin\theta_2\right)_{\theta_2=0} = 0$$
$$V_{g_{12}} = \left(\frac{\partial V_g^2}{\partial\theta_1\partial\theta_2}\right)_{\substack{\theta_1=0\\ \theta_2=0}} = 0$$

$$T = \frac{1}{2}m\dot{y}_g^2 + \frac{1}{2}I_{cg}\dot{\theta}^2$$
$$= \frac{1}{2}m\left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2 + \frac{1}{2}(mr_G^2)\left(\frac{\dot{y}_2 - \dot{y}_1}{L}\right)^2$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Physically, the constant represents the time it takes the system's step response to reach 63.2 % of its final (asymptotic) value $\tau = \frac{1}{\zeta_1\omega_1}$

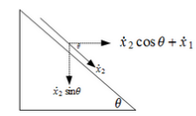
Where $e_{B/A}$ is unit vector oriented to B from A

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2v^2$$

:k. To find this v it is easier to resolve components on the x

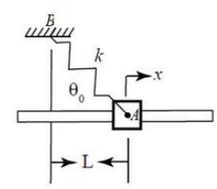
$$\vec{v} = \dot{x}_2\sin\theta j + (\dot{x}_2\cos\theta + \dot{x}_1)i$$

velocity

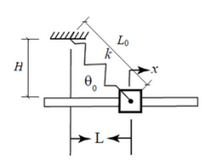


$$\begin{aligned}|\vec{v}|^2 &= (\dot{x}_2\sin\theta)^2 + (\dot{x}_2\cos\theta + \dot{x}_1)^2\\ &= \left(\dot{x}_2^2\sin^2\theta\right) + \left(\dot{x}_2^2\cos^2\theta + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta\right)\\ &= \dot{x}_2^2\left(\sin^2\theta + \cos^2\theta\right) + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta\\ &= \dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta\end{aligned}$$

Stiff spring



$$\begin{aligned}\dot{\Delta} &= (\dot{u}_A - \dot{u}_B)e_{A/B}\\ &= (\dot{x}\hat{\imath} - 0\hat{\jmath})\cdot(\cos\theta_0\hat{\imath} - \sin\theta_0\hat{\jmath})\\ &= \dot{x}\cos\theta_0\end{aligned}$$



Hence from the above diagram we see that $L_0 = \sqrt{H^2 + L^2}$ and $L_{cur} = \sqrt{H^2 + (L+x)^2}$, therefore

$$\Delta = \sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}$$

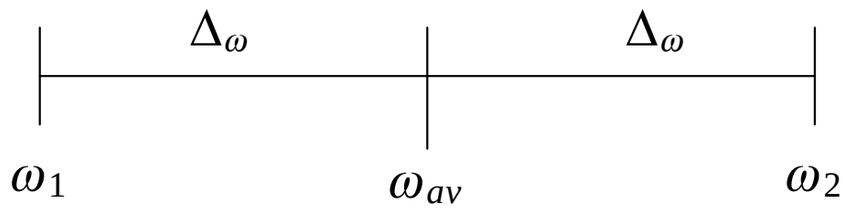
6.2 study notes

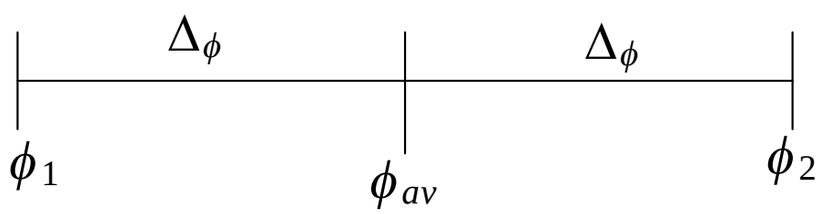
6.2.1 trig identities

$$
\begin{aligned}
\sin\omega t &= \mathrm{Re}\left[\frac{1}{i}e^{i\omega t}\right] = \mathrm{Re}\left[e^{i\left(\omega t-\frac{\pi}{2}\right)}\right]\\
\cos\omega t &= \mathrm{Re}\left[e^{i\omega t}\right]\\
\cos\omega t &= \frac{1}{2}\left(e^{i\omega t}+e^{-i\omega t}\right)\\
\sin\omega t &= \frac{1}{2i}\left(e^{i\omega t}-e^{-i\omega t}\right)
\end{aligned}
$$

When 2 harmonics have same amplitude, we can write then as envolope of one in another

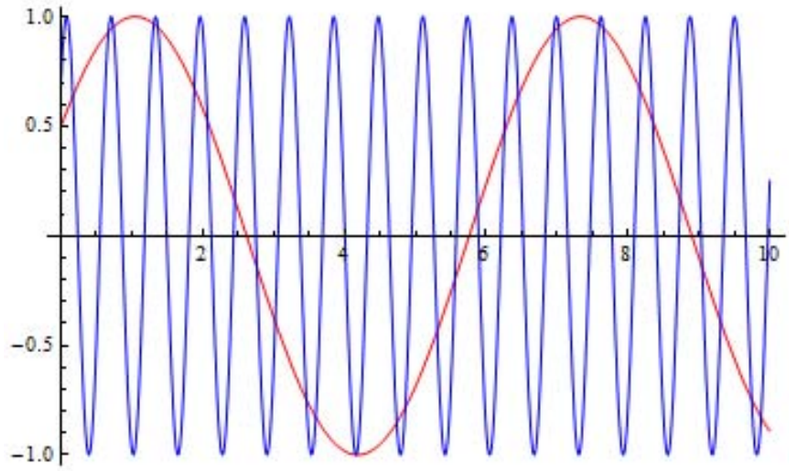
$$A\cos\left(\omega_1 t-\phi_1\right)+A\cos\left(\omega_2 t-\phi_2\right)=2A\cos\left(\Delta_\omega t-\Delta_\phi\right)\cos\left(\omega_{av}t-\phi_{av}\right)$$





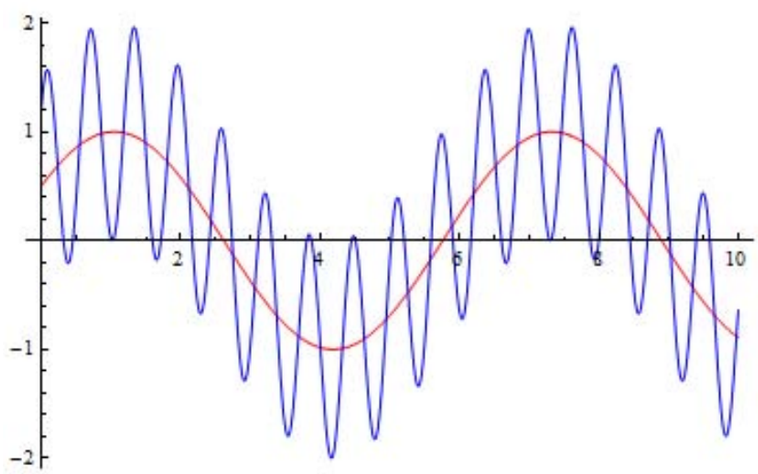
Here is an example of the above. We first draw the two signals on their own, then plot the additions of them

```
f1 = a Cos[w1 t - p1];
f2 = a Cos[w2 t - p2];
parms = {a -> 1, w1 -> 1, p1 -> Pi/3, w2 -> 10, p2 -> Pi/4};
Plot[Evaluate[{f1, f2} /. parms], {t, 0, 10},
 PlotStyle -> {Red, Blue}]
```



Now we add them to see the envelope effect

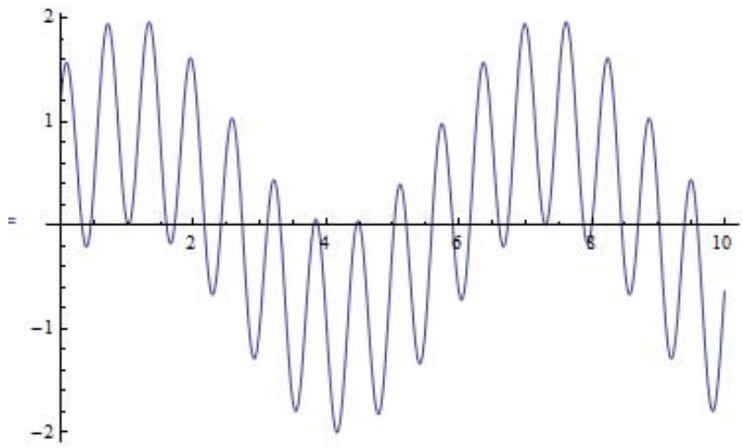
```
Plot[Evaluate[{f1, f1 + f2} /. parms], {t, 0, 10},
 PlotStyle -> {Red, Blue}]
```



Now we plot the same signal addition, but using the form after converting to use the mean and delta notation as shown above just to confirm it is the same signal

```
avW = Mean[{w1, w2}];
avP = Mean[{p1, p2}];
delW = w2 - avW;
delp = p2 - avP;

g = 2 a Cos[delW t - delp] Cos[avW t - avP];
Plot[g /. parms, {t, 0, 10}]
```



The beat period is $\frac{\pi}{\Delta\omega}$(this is the time between each beat to the next beat). The whole signal will be periodic only when $\frac{\omega_1}{\omega_2}$ is rational.

Beat shows up when we have 2 harmonics added, that has same amplitude. The beat signal itself will be period when the ratio between the frequencies of the two harmonics is rational. In the context of response of a system, we can think of the steady state response as one signal and the transient response as another singnal. The response will then show a beating signal when the amplitude of the steady state and transient singnals is the same. Here is an example of that from one of my demos