my solution to second finals practice exam

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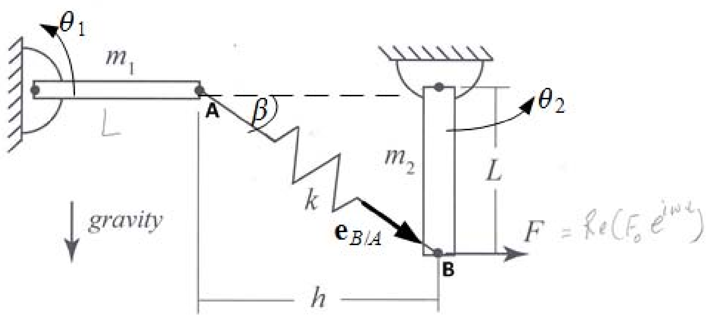
1 Problem 1

Problem #1 (20 pts)

Two rigid beams are pinned at their ends and arranged as shown below with a stiff spring connecting their tips. Gravity acts in the direction indicated. The position shown corresponds to the static equilibrium position. The masses of the two beams are m_1 and m_2 and they both have the same length, L. They are separated by a distance h. A dynamic force is applied to the tip of the right beam as shown. The moment of inertia of a bar is I_g=(1/12)mL^2 about its center and I_{end}= (1/3)mL^2 about its end.

Find the linearized equation(s) of motion for this system and check that your equation(s) are physically reasonable.



This is a 2 D.O.F. system. The degrees of freedom are θ_1 and θ_2 shown above in the positive sense. The method of power balance is used to obtain the EOM.

The system kinetic energy is $T = \frac{1}{2}m_1\frac{L^2}{3}\left(\theta_1'\right)^2 + \frac{1}{2}m_1\frac{L^2}{3}\left(\theta_2'\right)^2$, hence by comparing term to the quadratic form, the mass matrix part of the EOM is obtained

$$\frac{L^2}{3}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix}$$

To find spring stiffness, the spring deformation is found using stiff spring approximation.

$$\begin{aligned}\Delta' &= (V_B - V_A)\cdot e_{B/A}\\ &= \left(L\theta_2'\mathbf{i} - L\theta_1'\mathbf{j}\right)\cdot\left(\cos\beta\mathbf{i} - \sin\beta\mathbf{j}\right)\end{aligned}$$

Where $e_{B/A}$ is unit vector oriented to B from A and $\tan\beta = \frac{L}{h}$. The above becomes

$$\Delta' = L\theta_2'\cos\beta + L\theta_1'\sin\beta$$

Hence, integrating, squaring and collecting terms gives

$$\begin{aligned}\Delta &= L\theta_2\cos\beta + L\theta_1\sin\beta\\ \Delta^2 &= L^2\theta_2^2\cos^2\beta + L^2\theta_1^2\sin^2\beta + 2L^2\theta_1\theta_2\sin\beta\cos\beta\\ &= \theta_1^2\left(L^2\sin^2\beta\right) + \theta_2^2\left(L^2\cos^2\beta\right) + \theta_1\theta_2\left(2L^2\sin\beta\cos\beta\right)\end{aligned}$$

Using the quadratic form of the power balance method, the spring stiffness matrix part of the EOM is found from $V_{spring} = \frac{1}{2}k\left(\Delta^2\right)$ and by comparing quadratic terms, which leads to

$$V_{spring} = kL^2\begin{bmatrix} \sin^2\beta & 2\sin\beta\cos\beta \\ 2\sin\beta\cos\beta & \cos^2\beta \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

But $\sin\beta\cos\beta = \frac{1}{2}\left(\sin 2\beta\right)$ hence

$$V_{spring} = kL^2\begin{bmatrix} \sin^2\beta & \sin 2\beta \\ \sin 2\beta & \cos^2\beta \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

Stiffness due to gravity V_g is now found. Let datum for zero potential energy be at the horizontal level of the top bar, hence $V_g = m_1 g\frac{L}{2}\sin\theta_1 - m_2 g\frac{L}{2}\cos\theta_2$. Since the derivatives are evaluated at static equilibrium $\theta_1 = 0$ and $\theta_2 = 0$, the only term that remains is $m_2 g\frac{L}{2}$ which is now added to the k_{22} term of the stiffness matrix. FL is the generalized force for θ_2 since work done by F in making virtual $\delta\theta_2$ is $FL\delta\theta_2$. Therefore, the EOM becomes

$$\frac{L^2}{3}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix} + kL^2\begin{bmatrix} \sin^2\beta & \sin 2\beta \\ \sin 2\beta & \cos^2\beta + m_2 g\frac{L}{2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ FL \end{Bmatrix}$$

To check units of the above EOM, looking at the first EOM from above

$$\frac{L^2}{3}m_1\theta_1'' + kL^2\left(\sin^2\beta\right)\theta_1 + kL^2\left(\sin 2\beta\right)\theta_2 = 0$$

Let $\theta_1 = 0$. Hence $\frac{L^2}{3}m_1\theta_1'' = -kL^2\left(\sin 2\beta\right)\theta_2$. Assume $\theta_2 \geq 0$ and the system is now released to move. We should expect the top bar to accelerate down (negative), since the spring is stretched. Looking at the above, we see that $\theta_1'' \leq 0$. hence this is **correct**.

Now let $\theta_2 = 0$. Hence $\frac{L^2}{3}m_1\theta_1'' = -kL^2\left(\sin^2\beta\right)\theta_1$. Assume $\theta_1 \geq 0$ and the system is now released to move. We should expect the top bar to accelerate down (negative) since the spring was stretched. Looking at the above, we see that $\theta_1'' \leq 0$. This is **correct**.

Checking the second EOM

$$\frac{L^2}{3}m_2\theta_2'' + kL^2\left(\sin 2\beta\right)\theta_1 + kL^2\left(\cos^2\beta\right)\theta_2 = FL - m_2 g\frac{L}{2}\theta_2$$

Let $\theta_1 = 0$ and $F = 0$ then

$$\frac{L^2}{3}m_2\theta_2'' = -m_2 g\frac{L}{2}\theta_2 - L^2\left(\cos^2\beta\right)\theta_2$$

Assume $\theta_2 \geq 0$ and the system is now released to move. We would expect the right bar to accelerate back (negative) when released to move. From the equation we see that $\theta_2'' \leq 0$. This is **correct**.

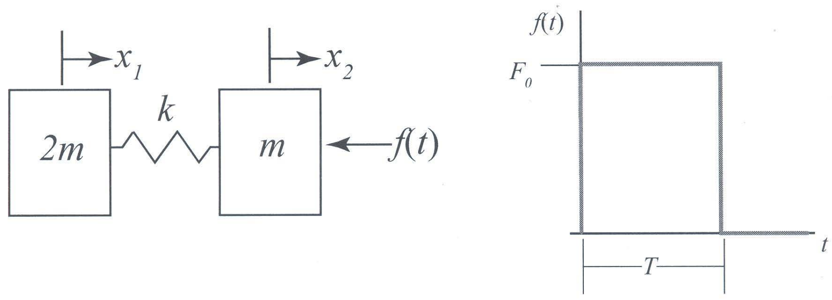
Now let $\theta_2 = 0$ and $F = 0$ then

$$\frac{L^2}{3}m_2\theta_2'' = -kL^2\left(\sin 2\beta\right)\theta_1$$

Assume $\theta_1 \leq 0$ and the system is now released to move. We would expect the bar to accelerate to the right (positive) since the spring was compressed. From the equation we see that $\theta_2'' > 0$. This is **correct**.

2 Problem 2

Problem #2 (20 pts)
The impact of a tennis ball with a racquet can be modeled using the two degree-of-freedom system shown below to represent the ball (the masses are only permitted to move in the horizontal direction). A ball is initially traveling to the right at speed v_0, (i.e. with $\dot{x}_1 = \dot{x}_2 = v_0$) when it strikes a racquet. Suppose that the impact force is known and is modeled as a square pulse whose duration is T. Damping is negligible.



The equations of motion of this system are:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix}$$

a.) (10 pts) Find the natural frequencies and mass-normalized mode shapes of the system.
b.) (10 pts) Find two uncoupled, second-order differential equations that could be solved to find the response of the tennis ball. Be sure to substitute all known quantities into each of the equations.
c.) (3 pts extra credit) Use the result from (b) to sketch the response of the first mass, $x_1(t)$, qualitatively for $t > T$, explaining any important features.

2.1 part(a)

$$\det\left([k] - \omega^2 [m]\right) = 0$$

$$\det\left(\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} - \omega^2 \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \frac{m}{k} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

For normalization, let $t' = \omega t$ then $\frac{dt'}{dt} = \omega$ and using t' instead of t as the independent variable the above becomes

$$\det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1 - 2\omega^2 & -1 \\ -1 & 1 - \omega^2 \end{bmatrix}\right) = 0$$

$$\left(1 - 2\omega^2\right)\left(1 - \omega^2\right) - 1 = 0$$

The roots are $\omega = 0$ and $\omega = \sqrt{\frac{3}{2}}$. When $\omega = 0$ it is a rigid body motion, So any φ will do. Let $\varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$. When $\omega = \sqrt{\frac{3}{2}}$ then

$$\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -2 & -1 \\ -1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

let $\varphi_{12} = 1$ then $-2 - \varphi_{22} = 0$ or $\varphi_{22} = -2$ hence $\varphi_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$.

$$\mu_1 = \varphi_1^T [M] \varphi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 3$$

$$\mu_2 = \varphi_2^T [M] \varphi_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = 6$$

Hence

$$\Phi_1 = \frac{\varphi_1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{3}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.57735 \\ 0.57735 \end{Bmatrix}$$

$$\Phi_2 = \frac{\varphi_2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{6}} \begin{Bmatrix} 1 \\ -2 \end{Bmatrix} = \begin{Bmatrix} 0.40825 \\ -0.81650 \end{Bmatrix}$$

2.2 part(b)

$$\Phi = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

The EOM is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \Phi^T \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T \begin{Bmatrix} 0 \\ -f(t) \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{3}\sqrt{3} f(t) \\ \frac{1}{3}\sqrt{6} f(t) \end{Bmatrix}$$

initial conditions are $\begin{Bmatrix} \eta_1(0) \\ \eta_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ and

$$\begin{Bmatrix} \eta_1'(0) \\ \eta_2'(0) \end{Bmatrix} = \Phi^T [M] \begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix}^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v_0 \\ v_0 \end{Bmatrix} = \begin{Bmatrix} \sqrt{3}\, v_0 \\ 0 \end{Bmatrix}$$

Therefore, the first ODE is

$$\eta_1'' = -\frac{1}{3}\sqrt{3}\, F_0 (h(t) - h(t - T))$$

with IC $\eta_1(0) = 0$ and $\eta_1'(0) = \sqrt{3}\,v_0$. The second ODE is

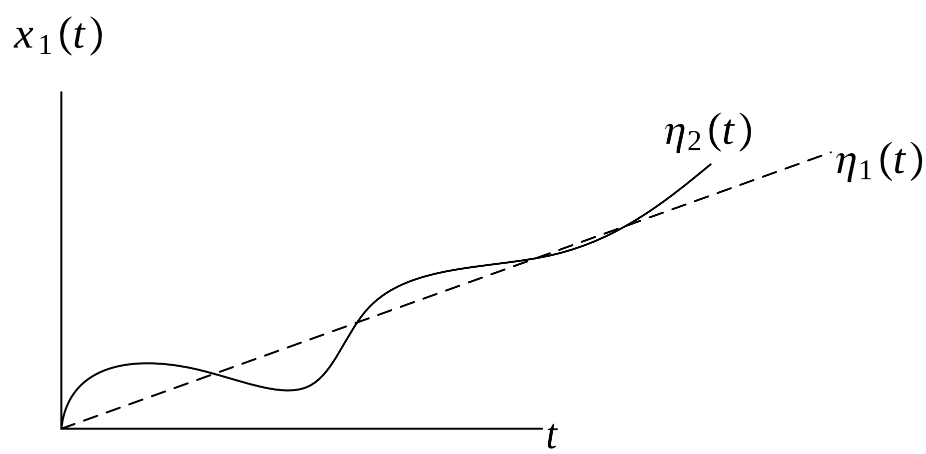
$$\eta_2'' + \frac{3}{2}\eta_2 = \frac{1}{3}\sqrt{6}\,F_0(h(t) - h(t - T))$$

with IC $\eta_1(0) = 0$ and $\eta_1'(0) = 0$

2.3 part(c)

$$\begin{aligned} x_1(t) &= \Phi_{11}\eta_1(t) + \Phi_{12}\eta_2(t) \\ &= \frac{1}{\sqrt{3}}\eta_1(t) + \frac{1}{\sqrt{6}}\eta_2(t) \end{aligned}$$

Therefore, $x_1(t)$ solution has contribution from $\eta_1(t)$ and $\eta_2(t)$. But $\eta_1(t)$ is linear with positive slope of v_0 and $\eta_2(t)$ is a sinusoidal, with no damping. So adding both together, here is a sketch of possible solution

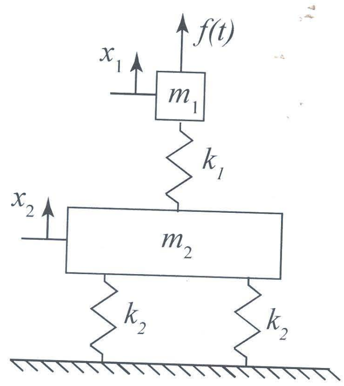


3 Problem 3

Name: ____________________

Problem #3 (30 pts)

The system below is a simplified model of an aircraft with an engine mounted on its tail.



The equations of motion for certain values of the k_1, m_1, etc…, are known except for the mass matrix,

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

so M_{11}, M_{12} and M_{22} are unknown constants. The mass normalized modes are also known and are:

$$\{\Phi\}_1 = \begin{Bmatrix} 0.85 \\ 0.65 \end{Bmatrix} \quad \{\Phi\}_2 = \begin{Bmatrix} 1.1 \\ -0.5 \end{Bmatrix}$$

The second natural frequency is ω_2= 16.9 rad/s. Suppose the system is initially at rest when the engine starts exerting a force $f(t) = A\cos(\omega t)h(t)$ where $h(t)$ is the unit step function.

a.) (10 pts) What is the first natural frequency ω_1?

b.) (10 pts) How long will it take for the system's response to settle to within approximately 1% of its steady state value? (Think carefully about what is being asked here and only answer the question that was asked.)

c.) (10 pts) Find an expression for the steady state response of the first mass $x_1(t)$ in terms of the forcing frequency ω.

3.1 part(a)

$$\Phi^T[K]\Phi^T = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix}\begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

$$\begin{bmatrix} 46.25 & -0.5 \\ -0.5 & 281.0 \end{bmatrix} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

Hence $\omega_1^2 = 46.25$ or $\omega_1 = 6.8$ rad/sec

3.2 part(b)

Using the first natural frequency, since this has the longest time constant $\tau = \frac{1}{\zeta_1\omega_1}$ and solving for the number of periods using logarithmic decrement method

$$\frac{1}{N}\ln\left(\frac{y_1}{y_N}\right) = 2\pi\zeta_1 \tag{1}$$

ζ_1 is not known but can be found by evaluating $\Phi^T[C]\Phi^T$

$$\Phi^T[K]\Phi^T = \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.05 & 0.021 \\ 0.021 & 0.061 \end{bmatrix}$$

and assuming small damping approximation, then $2\zeta_1\omega_1 = 0.05$. Hence $\zeta_1 = \frac{0.05}{2\omega_1} = \frac{0.05}{2(6.8)} = 0.0038$. Now that the critical damping ratio for the first mode is found, we can use the method of logarithmic decrement to find how many periods it takes to attenuate by 99%

Let $\frac{y_1}{y_N} = \frac{1}{0.01} = 100$ then Eq (1) becomes

$$\begin{aligned} \frac{1}{N}\ln(100) &= 2\pi(0.0038) \\ N &= \frac{(4.605)}{2\pi(0.0038)} = 192.87 \\ &= 193 \end{aligned}$$

Where N is the number or periods needed. But $T = \frac{2\pi}{\omega_1}$,hence the time needed is

$$t = NT = 192T = 192\frac{2\pi}{\omega_1} = 192\frac{2\pi}{6.8} = 177.41 \text{ sec}$$

So it takes 178 seconds for the first modal (decoupled) solution to attenuate in amplitude by 99%. Since this is the dominant time constant, we expect the physical solution to attenuate in approximately the same amount of time as well.

3.3 part(c)

The EOM is, in modal coordinates

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \Phi^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \Phi \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \Phi^T \begin{Bmatrix} \operatorname{Re}\left(Ae^{i\varpi t}\right) \\ 0 \end{Bmatrix}$$

But

$$\Phi^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \Phi = \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{bmatrix} 0.04 & 0 \\ 0 & 0.05 \end{bmatrix} \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix} = \begin{bmatrix} 0.05 & 0.021 \\ 0.021 & 0.061 \end{bmatrix}$$

Hence EOM in modal coordinates become

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 0.05 & 0.021 \\ 0.021 & 0.061 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 6.8^2 & 0 \\ 0 & 16.9^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 0.85 & 1.1 \\ 0.65 & -0.5 \end{bmatrix}^T \begin{Bmatrix} \operatorname{Re}\left(Ae^{i\varpi t}\right) \\ 0 \end{Bmatrix}$$

and using *small damping approximation*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 0.05 & 0 \\ 0 & 0.061 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 6.8^2 & 0 \\ 0 & 16.9^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{Bmatrix} 0.85 \operatorname{Re}\left(Ae^{it\varpi}\right) \\ 1.1 \operatorname{Re}\left(Ae^{it\varpi}\right) \end{Bmatrix}$$

Hence the 2 EOM's are

$$\begin{aligned} \eta_1'' + 0.05\eta_1' + 46.24\eta_1 &= \operatorname{Re}\left(0.85Ae^{i\varpi t}\right) \\ \eta_2'' + 0.061\eta_2' + 285.61\eta_2 &= \operatorname{Re}\left(1.1Ae^{i\varpi t}\right) \end{aligned}$$

Let $\eta_1 = \operatorname{Re}\left(X_1 e^{it\varpi}\right)$ then $X_1 = \frac{0.85A}{-\varpi^2+i0.05\varpi+46.24}$ and $\eta_2 = \operatorname{Re}\left(X_2 e^{it\varpi}\right)$ then $X_2 = \frac{1.1A}{-\varpi^2+i0.0609\varpi+285.61}$ then

$$\mathbf{x} = \Phi_1\eta_1 + \Phi_2\eta_2$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 0.85 \\ 0.65 \end{Bmatrix} \operatorname{Re}\left(X_1 e^{i\varpi t}\right) + \begin{Bmatrix} 1.1 \\ -0.5 \end{Bmatrix} \operatorname{Re}\left(X_2 e^{i\varpi t}\right)$$

Hence

$$x_1(t) = 0.85\operatorname{Re}\left(X_1 e^{i\varpi t}\right) + 1.1\operatorname{Re}\left(X_2 e^{i\varpi t}\right)$$
$$x_2(t) = 0.65\operatorname{Re}\left(X_1 e^{i\varpi t}\right) - 0.5\operatorname{Re}\left(X_2 e^{i\varpi t}\right)$$

hence

$$x_1(t) = 0.85\operatorname{Re}\left(\frac{0.85A}{-\varpi^2 + i0.05\varpi + 46.24} e^{i\varpi t}\right) + 1.1\operatorname{Re}\left(\frac{1.1A}{-\varpi^2 + i0.061\varpi + 285.61} e^{i\varpi t}\right)$$
$$x_2(t) = 0.65\operatorname{Re}\left(\frac{0.85A}{-\varpi^2 + i0.05\varpi + 46.24} e^{i\varpi t}\right) - 0.5\operatorname{Re}\left(\frac{1.1A}{-\varpi^2 + i0.061\varpi + 285.61} e^{i\varpi t}\right)$$

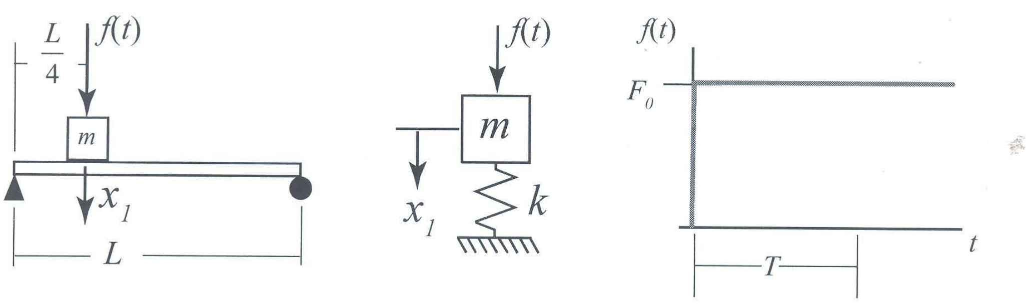
These can be combined to

$$x_1(t) = \operatorname{Re}\left(\left[\frac{0.85^2A}{-\varpi^2 + i0.05\varpi + 46.24} + \frac{1.1^2A}{-\varpi^2 + i0.0609\varpi + 285.61}\right] e^{i\varpi t}\right)$$
$$x_2(t) = \operatorname{Re}\left(\left[\frac{(0.65)(0.85)A}{-\varpi^2 + i0.05\varpi + 46.24} - \frac{(0.5)(1.1)A}{-\varpi^2 + i0.061\varpi + 285.61}\right] e^{i\varpi t}\right)$$

4 Problem 4

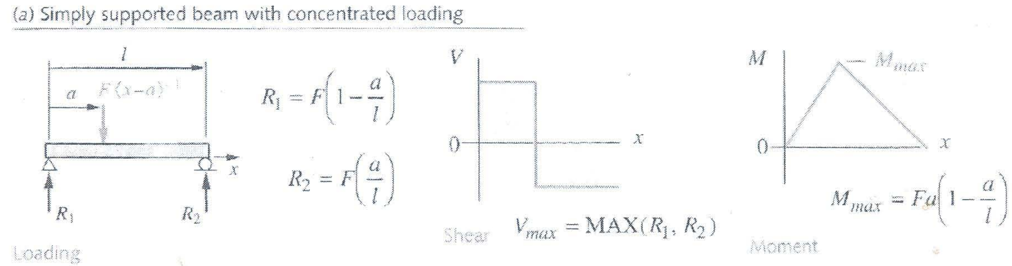
Problem #4 (10 pts)

The system shown consists of a beam with a large mass mounted one fourth of the distance from its end. This can be represented with the undamped spring-mass system shown to the right, with $k=85EI/L^3$. The system is initially in its static equilibrium position when a step force, $f(t)=F_0h(t)$, is applied to the mass.



The following information is available from a static analysis of the beam. When a static load, F, is applied to a beam, the maximum bending stress occurs in the outer fiber of the beam is given by $\sigma_{max}=-M_{max}c/I$, where M_{max} is the maximum bending moment in the beam, c is the (known) distance to the outer fiber and I is the area moment of inertia (also known). See the figure below for additional details regarding a static loading scenario.

(a) Simply supported beam with concentrated loading



<u>What is the amplitude of the load, F_0, that causes the beam to exceed its yield stress, σ_y?</u>

The transient response is given in appendix B as

$$x(t) = \frac{F_0}{k}(1 - \cos\omega_n t)h(t)$$

Hence maximum amplitude of the response is $u_{\max} = \frac{2F_0}{k}$. Compare this to static deflection which is $u_{static} = \frac{F_0}{k}$ then we can say that dynamic load is twice as large as the static load. Therefor using $2F_0$ in place of F in the expression for stress gives the result needed

$$\begin{aligned}\sigma_y &= \frac{-Mc}{I} \\ &= \frac{-(2F_0)\frac{L}{4}\left(\frac{3}{4}\right)}{I/c} \\ &= -\frac{3}{8}L\frac{c}{I}F_0\end{aligned}$$

Therefore

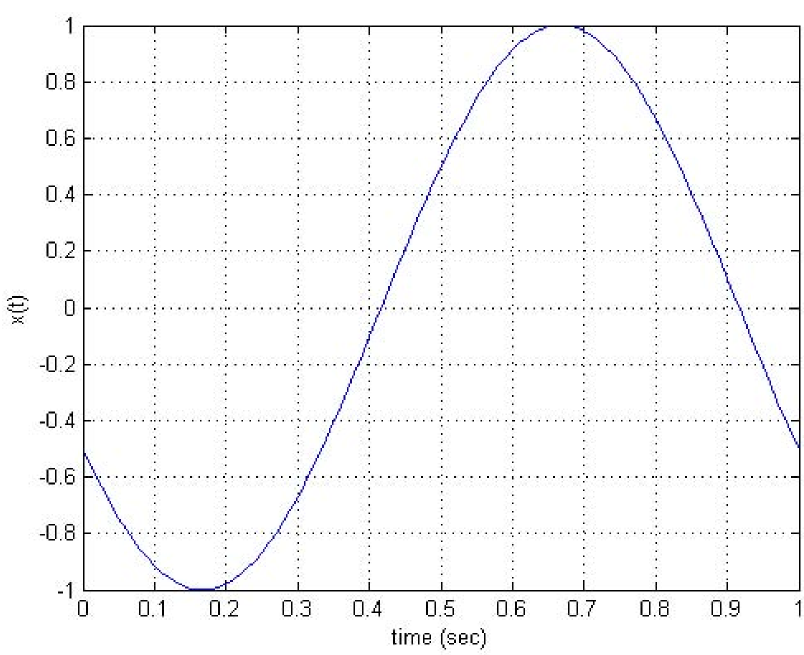
$$F_0 = -\frac{8I}{3Lc}\sigma_y$$

5 Problem 5

Problem #5 (10 pts)
A single degree-of-freedom system's response is given by $x(t) = \mathrm{Re}\left(Xe^{i\omega t}\right)$, with $X = e^{i2\pi/3}$.
Sketch the complex amplitude, X, in the complex plane and sketch the corresponding time function $x(t)$ over at least one cycle.

At $t = 0$ then $x(t) = \mathrm{Re}\left(e^{i\frac{2\pi}{3}}\right)$ which is $-\cos\left(60^0\right) = -\frac{1}{2}$. Using $\omega = 2\pi$ rad/sec then $x(t)$ can be traced. Here is a plot

```
I=sqrt(-1);
w=2*pi;
x=@(t) real(exp(I*2*pi/3)*exp(I*w*t))
t=0:.01:1;
plot(t,x(t))
grid
xlabel('time (sec)'); ylabel('x(t)');
```



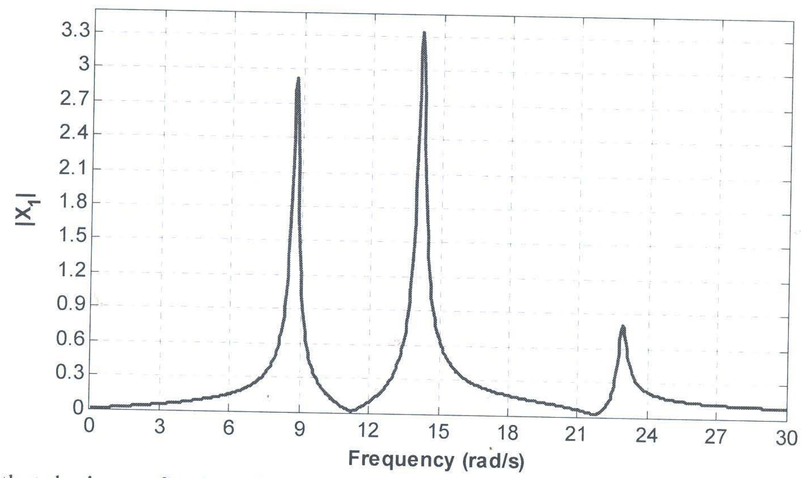
6 Problem 6

Problem #6 (10 pts)

A three degree-of-freedom system is excited by a sinusoidal force, $f(t)=\cos(\omega t)$.

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}\,f(t)$$

The frequency response was computed using $\{X\}=\left(-\omega^2[M]+i\omega[C]+[K]\right)^{-1}\{F\}$ and $|X_1|$ from that calculation is plotted below.



Suppose that the input, $f(t)$, is replaced with a periodic function that can be expressed as follows,

$$f(t)=\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{100-n}{n}\right)e^{in\omega_1 t}$$

with $\omega_1 = 3.0$ rad/s. What frequencies would be present in the steady-state response $x_1(t)$? Which of those would be dominant (i.e. have the largest amplitude)?

Damped resonances are seen at $\omega = 8.5, 14$ and 23 rad/sec. This is where $r = \frac{\varpi}{\omega_i}$ is close to unity, where ϖ is the forcing frequency and ω_i is the natural frequency. Since this is a 3 dof system, it will have 3 natural frequencies.

The response of each dof will take contributions from each mode of vibration. Each mode vibrates at different natural frequency. From the plot above it is seen that the response of $x_1(t)$ has the largest response when the forcing frequency is close to the $\omega_2 = 14$ rad/sec.

The new force now has the following set of discrete harmonics in it: ($n = 0$ is not counted, DC). $\frac{100}{1}e^{3t}, \frac{99}{2}e^{6t}, \frac{98}{3}e^{9t}, \frac{97}{4}e^{12t}, \frac{96}{5}e^{15t}, \frac{95}{6}e^{18t}, \frac{94}{7}e^{21t}, \frac{98}{8}e^{24t}, \cdots$ or

$$f(t) = 100e^{3t}, 49.5e^{6t}, 32.7e^{9t}, 24.3e^{12t}, 19.2e^{15t}, 15.8e^{18t}, 13.4e^{21t}, 12.3e^{24t}$$

So the input force has only discrete frequencies. Since linear sum, each $f_i(t)$ will cause the response $|X|$ at that specific forcing frequency as shown in the plot. Looking the plot it can be seen that when forcing frequency is 9 rad/sec, this will cause the largest $|X|$ among all these set of discrete frequencies. Hence the dominant harmonic is 9 rad/sec and will have amplitude around 2.4 from looking at the plot.