① Practice Midterm
4.34

1:35

find $\alpha \rightarrow \phi_1^T M \phi_2 = 0$ Modes must be $\perp$

$$[1 \quad 0.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \end{bmatrix} =$$

$$= [1 \quad 0.414]\begin{bmatrix} 3+\alpha \\ 1+\alpha \end{bmatrix} = 3+\alpha+0.414+0.414\alpha$$

$$= 3.414 + 1.414\alpha$$

$$\boxed{\alpha = -\frac{3.414}{1.414}} = -2.414$$

Mass Normalize: (Wouldn't have all This Arithmetic on final)

$$[1 \quad 0.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0.414 \end{bmatrix} = [1 \quad 0.414]\begin{bmatrix} 3.414 \\ 1.414 \end{bmatrix}$$

$$= 3.414 + 1.414 \cdot 0.414 = M_1$$

$$[1 \quad -2.414]\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2.414 \end{bmatrix} = [1 \quad -2.414]\begin{bmatrix} 3-2.414 \\ 1-2.414 \end{bmatrix} = M_2$$

$$\{\Phi_1\} = \{\phi_1\}\frac{1}{\sqrt{M_1}} \quad \text{etc...}$$

$$\left.\begin{aligned} \ddot{n}_1 + \omega_n^2 n_1 &= \{\Phi_1\}^T \begin{Bmatrix} 0 \\ 20t \end{Bmatrix} = \frac{0.414 \cdot 20t}{\sqrt{M_1}} = c_1 t \\ \ddot{n}_2 + \omega_n^2 n_2 &= \{\Phi_2\}^T \begin{Bmatrix} 0 \\ 20t \end{Bmatrix} = c_2 t \end{aligned}\right\} \text{Ramp}$$

$$n_1(t) = \frac{c_1}{\omega_1^3}\left[\omega_1 t - \sin(\omega_1 t)\right] h(t) \quad - \quad \text{App. B for } \zeta = 0$$

$$n_2(t) = \frac{c_2}{\omega_2^3}\left[\omega_2 t - \sin(\omega_2 t)\right] h(t)$$

2:47

$$q = [\Phi]\begin{Bmatrix} n_1(t) \\ n_2(t) \end{Bmatrix} \quad \ldots$$

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2:47

Nondimensionalize $\rightarrow$ $t' = w_n t$, $w_n = \sqrt{\dfrac{\cancel{m} g \frac{L}{2}}{\frac{1}{3}\cancel{m} L^{\cancel{2}}}} = \sqrt{\dfrac{3g}{2L}}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1'' \\ \theta_2'' \end{Bmatrix} + \begin{bmatrix} \alpha+0.5 & -\alpha \\ -\alpha & \alpha+0.5 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$

EVP

$$\begin{bmatrix} \alpha+0.5-w^2 & -\alpha \\ -\alpha & \alpha+0.5-w^2 \end{bmatrix}\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = 0$$

nontrivial if matrix is singular — det = 0

$$(\alpha+0.5-w^2)(\alpha+0.5-w^2) - \alpha^2 = 0$$

$$\cancel{\alpha^2} + 0.5\alpha - \underline{\alpha w^2} + 0.5\alpha + 0.25 - \underline{0.5 w^2} - \underline{\alpha w^2} - \underline{0.5 w^2} + w^4 - \cancel{\alpha^2} = 0$$

$$w^4 - 2(\alpha+0.5)w^2 + 0.25 + \alpha = 0$$

$$w^2 = \frac{2(\alpha+0.5) \pm \sqrt{4(\alpha+0.5)^2 - 4(1)(0.25+\alpha)}}{2}$$

$$w^2 = (\alpha+0.5) \pm \sqrt{(\alpha+0.5)^2 - (\alpha+0.25)}$$

$$= (\alpha+0.5) \pm \sqrt{\alpha^2 + \cancel{\alpha} + \cancel{0.25} - \cancel{\alpha} - \cancel{0.25}}$$

$$= (\alpha+0.5) \pm \alpha =$$

$$\boxed{w_1^2 = 0.5 \quad , w_2^2 = 0.5 + 2\alpha}$$

for small α — $w_1 \approx w_2$

Modes $\rightarrow$

$$\begin{bmatrix} \alpha & -\alpha \\ -\alpha & \alpha \end{bmatrix}\phi_1 = 0 \qquad \phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\begin{bmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{bmatrix}\phi_2 = 0 \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Given an initial condition,

$$\left.\begin{aligned} \eta_1(t) &= Re(N_1 e^{iw_1 t}) \\ \eta_2(t) &= Re(N_2 e^{iw_2 t}) \end{aligned}\right\}$$ No decay $\rightarrow$ damping neglected

$$\{\theta\} = [\Phi]\begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} \rightarrow$$

(2-cont.)

$$q_1 = \frac{1}{\sqrt{M_1}} \eta_1(t) + \frac{1}{\sqrt{M_2}} \eta_2(t)$$

$$q_2 = \frac{1}{\sqrt{M_1}} \eta_1(t) + \frac{-1}{\sqrt{M_2}} \eta_2(t)$$

$q_1(t)$ is the sum of 2 harmonics with nearly equal amplitudes and nearly the same frequency, so beating occurs

3:05

$$q_1 \approx Re(A e^{i\Delta\omega t})\, Re(A e^{i\omega_{ave} t}) \quad \text{(see below)}$$

$$\omega_{ave} = \left(\sqrt{0.5} + \sqrt{0.5 + 2\alpha}\right)\frac{1}{2} \qquad (\alpha \text{ has little effect here.})$$

$$\Delta\omega = \left(\sqrt{0.5 + 2\alpha} - \sqrt{0.5}\right)\frac{1}{2}$$

depends on α

could solve this on calculator iteratively to find α for known $\Delta\omega$

This part is little intricate for an exam

for small α ~ Taylor series

$$\frac{\partial \Delta\omega}{\partial\alpha} = (0.5 + 2\alpha)^{-1/2}\left(\frac{1}{2}\right)(2)\Big|_{\alpha=0} = \frac{1}{\sqrt{0.5}}$$

$$\Delta\omega = 0 + \frac{\partial\Delta\omega}{\partial\alpha}\alpha = \sqrt{2}\,\alpha$$

$$\Delta\omega = \frac{2\pi}{T} = \frac{\cancel{2}\pi}{(63.\cancel{2})} \sim \text{2 beats per period} \quad ✓$$

$$\Delta\omega = \frac{\pi}{63} = \sqrt{2}\,\alpha$$

$$\boxed{\alpha \approx \frac{\pi}{63\sqrt{2}} \approx 0.035} \qquad (\text{True } \alpha = 0.036605)$$

3:15

(23 min)

Or, if you need to derive the beating equation above:

$$q_1 = Re(c_1 e^{i\omega_1 t}) + Re(c_2 e^{i\omega_2 t}) \qquad \text{for } c_1 = c_2$$

$$q_1 = c_1\left[Re(e^{i\omega_{ave} t} e^{-i\Delta\omega t} + e^{i\omega_{ave} t} e^{i\Delta\omega t})\right]$$

$$= \frac{1}{2}c_1\left[e^{i\omega_{ave} t} e^{-i\Delta\omega t} + e^{-i\omega_{ave} t} e^{i\Delta\omega t} + e^{i\omega_{ave} t} e^{i\Delta\omega t} + e^{-i\omega_{ave} t} e^{-i\Delta\omega t}\right]$$

$$q_1 = \frac{1}{2}c_1\left[e^{i\omega_{ave} t}\left(e^{i\Delta\omega t} + e^{-i\Delta\omega t}\right) + e^{-i\omega_{ave} t}\left(e^{i\Delta\omega t} + e^{-i\Delta\omega t}\right)\right]$$

→ equation for beating

→ $2\cos(\Delta\omega t)$ – Factor out

8:26

③

a) $\rightarrow \{x(t)\} = Re(\{\bar{X}\} e^{i\omega t}) \longrightarrow \quad y(t) = Re(\bar{Y} e^{i\omega t}) \longrightarrow$

$$\{\bar{X}\} = [K + i\omega C - \omega^2 M]^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

$$= \begin{bmatrix} K + i\omega c - \omega^2 m & -i\omega c \\ -i\omega c & 2K + i\omega c - \omega^2 m \end{bmatrix}^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

use $\omega = \omega_1 \rightarrow$ find ω_1

$$\left| [K - \omega^2 M] \right| = \left| \begin{bmatrix} K - m\omega^2 & \\ & 2K - m\omega^2 \end{bmatrix} \right|$$

by inspection (diagonal system) $\rightarrow \omega_1 = \sqrt{K/m}, \quad \omega_2 = \sqrt{\frac{2K}{m}}$

$$= \begin{bmatrix} \cancel{K} + i\omega c \cancel{-(K/m) \cdot m} & -i\omega c \\ -i\omega c & K + i\omega c \end{bmatrix}^{-1} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

$$= \begin{bmatrix} K + i\omega c & i\omega c \\ i\omega c & i\omega c \end{bmatrix} \frac{1}{(K + i\omega c) i\omega c - (i\omega c)^2} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

$$= \begin{bmatrix} (K + i\omega c) K \\ K(i\omega c) \end{bmatrix} \frac{1}{K(i\omega c)} \bar{Y}$$

$$= \begin{bmatrix} \frac{K^2 + i\omega c K}{i\omega c K} \\ 1 \end{bmatrix} \bar{Y}$$

b) General case:

$$\{\bar{X}\} = \begin{bmatrix} 2K + i\omega c - \omega^2 m & i\omega c \\ i\omega c & K + i\omega c - \omega^2 m \end{bmatrix} \frac{1}{\det(\ )} \begin{Bmatrix} K \\ 0 \end{Bmatrix} \bar{Y}$$

$$= \begin{Bmatrix} 2K + i\omega c - \omega^2 m \\ i\omega c \end{Bmatrix} \frac{K}{\det(\ )} \bar{Y}$$

x_2 cannot $= 0$ unless $c = 0$

$x_1 \rightarrow \quad 2K + i\omega c - \omega^2 m = 0$

$$\omega = \frac{-ic \pm \sqrt{-c^2 - 4(-m)(2K)}}{-2m}$$

a

solution for ω is imaginary unless $c = 0$

$\boxed{\rightarrow \text{so there is no real } \omega \text{ for which } x_1 = 0}$

8:37

3-cont.

8:37 c) from previous

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} i\omega c & i\omega c \\ i\omega c & i\omega c \end{bmatrix} \frac{1}{\underline{\underline{0}}} \begin{Bmatrix} K \\ 0 \end{Bmatrix} Y$$

$$X_1, X_2 \rightarrow \infty$$

Explanation $\rightarrow$ equal nat freq's

$$[[K] - \omega^2 [M]]\{\phi\} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\{\phi\} = 0 \rightarrow \phi \text{ is arbitrary!}$$

if $\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ Then $\{\phi_1\}^T [C] \{\phi_1\} =$

$$[1 \;\; 1]\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = [1 \;\; 1]\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

so, $2\zeta_1 \omega_1 = 0$, $\zeta_1 = 0$
So this mode is undamped and it's response approaches ∞ when it is forced at it's nat. freq.

d) given $x_1 = Re(Ae^{i\omega t}) \rightarrow$ use 2nd EOM

x_2 must also be harmonic $\rightarrow x_2 = Re(X_2 e^{i\omega t})$

$$m\ddot{x}_2 + c\dot{x}_2 - c\dot{x}_1 + K_2 x_2 = 0$$

$$X_2 = \frac{i\omega c X_1}{K + i\omega c - \omega^2 m}$$

$$\boxed{X_2 = \frac{i\omega c A}{K + i\omega c - \omega^2 m}} \qquad x_2(t) = Re(X_2 e^{i\omega t})$$

E.C preload ~

$\downarrow mg$ $\quad K_2\Delta_0 - mg = 0$

K_2 $\quad \Delta_0 = \dfrac{mg}{K_2}$

$\uparrow K_2\Delta_0$

looses contact if $\Delta > \Delta_0$

$$\Delta = x_1 - y$$

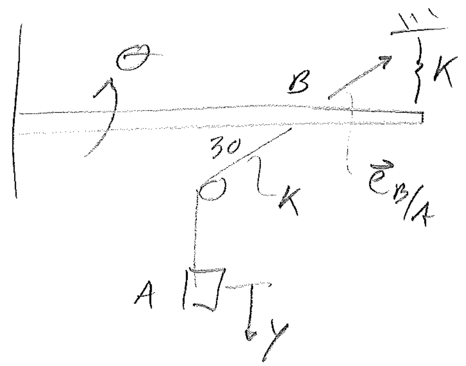
$$\boxed{x_1 - y > \frac{mg}{K}}$$

3:48

(4)

9:50

Find EOM $\quad q_1 = \theta \quad q_2 = y$



$$T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}^2 + \frac{1}{2}3m\dot{y}$$

I about pin

→ ok for bar in pure rotation, otherwise use $\frac{1}{2}mV_g^2$ and $\frac{1}{2}I_g\dot{\theta}^2$

$$M = \begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & 3m \end{bmatrix} \text{ for } \begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix}$$

$$V = \frac{1}{2}K\Delta_1^2 + \frac{1}{2}K\Delta_2^2$$

$$\Delta_1 = L\theta$$

$$\Delta_2 = U_a - U_B \quad \text{— stiff spring approx / Linearization}$$

$$= y - (\cos 30 i + \sin 30 j)\cdot\left(\frac{L}{2}\theta\hat{j}\right)$$

$$= y - \frac{L}{2}\sin 30\,\theta$$

$$V = \frac{1}{2}KL^2\theta^2 + \frac{1}{2}K\left(y^2 + 2\cdot\frac{L}{2}\sin 30\,\theta y + \frac{L^2}{4}\sin^2 30\,\theta^2\right)$$

$$K = \begin{bmatrix} KL^2 & -\frac{KL}{2}\sin 30 \\ -\frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}$$

$$\checkmark \sim I_o\ddot{\theta} + KL^2\theta - \frac{KL}{2}\sin 30\, y = 0$$

$\theta = 0,\ y > 0$ ~ oops – sign error on y — not consistent w $\vec{e}_{B/A}$

$$K = \begin{bmatrix} KL^2 & +\frac{KL}{2}\sin 30 \\ +\frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}$$

$$P_{in} = \dot{y}\cdot F$$

$$Q = \begin{Bmatrix} 0 \\ F \end{Bmatrix} \Rightarrow \boxed{\begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & 3m \end{bmatrix}\begin{Bmatrix} \ddot{\theta} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} KL^2 & \frac{KL}{2}\sin 30 \\ \frac{KL}{2}\sin 30 & \frac{KL^2}{4}\sin^2 30 \end{bmatrix}\begin{Bmatrix} \theta \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}}$$

0:00