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$$\xi = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{c}{2\omega_n m}$$

$$\frac{y_n}{y_{n+1}} = e^{\frac{\zeta\omega_n 2\pi}{\omega_D}}$$ small damping $\Rightarrow$ $e^{\frac{\zeta 2\pi}{\sqrt{1-\zeta^2}}} \Rightarrow e^{\zeta 2\pi}$

$$\ln\left(\frac{y_n}{y_{n+1}}\right) = \zeta 2\pi$$

$$\frac{1}{M}\ln\left(\frac{y_n}{y_{n+M}}\right) = \zeta 2\pi$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

$$r = \frac{\varpi}{\omega_n}$$

$$T_d = \frac{2\pi}{\omega_d}$$

$$\beta = \frac{1}{\sqrt{(1-r^2)^2+(2r\xi)^2}}$$

$$\beta_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

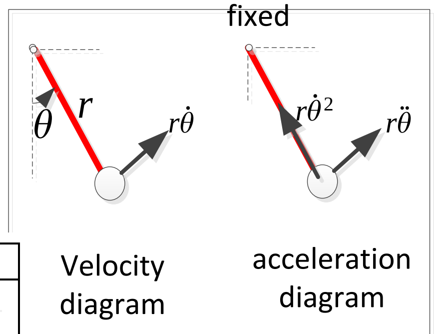
time constant $\tau = \frac{1}{\zeta\omega_n}$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2}$$
$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Springs: in parallel, ADD

Sin[x]	Cos[x]
$\frac{1}{2} i e^{-ix} - \frac{1}{2} i e^{ix}$	$\frac{e^{-ix}}{2} + \frac{e^{ix}}{2}$

Length of pendulum fixed



$$f = \varepsilon m\Omega^2\sin(\Omega t)$$

number of cycles needed for peak to decay by half	ξ (%)
1.	11.0318
2.	5.51589
3.	3.67726
4.	2.75795
5.	2.20636
6.	1.83863
7.	1.57597
8.	1.37897
9.	1.22575
10.	1.10318

$\partial_t t^3$	=	$3t^2$
∂_t Cos[3 t]	=	-3 Sin[3 t]
∂_t Sin[t]	=	Cos[t]
∂_t Cos[t]	=	-Sin[t]
∂_t Tan[t]	=	$\text{Sec}[t]^2$

$\beta_{\max}$ when $r = \sqrt{1-2\xi^2}$

$$u'' + 2\xi\omega u' + \omega^2 u = 0$$

	roots
$\xi < 1$	$\{-\xi\omega + j\omega_n\sqrt{1-\xi^2}, -\xi\omega - i\omega_n\sqrt{1-\xi^2}\}$
$\xi = 1$	$\{-\omega, -\omega\}$
$\xi > 1$	$\{-\omega_n\xi + \omega_n\sqrt{\xi^2-1}, -\omega_n\xi - \omega_n\sqrt{\xi^2-1}\}$

$$my'' + cy' + ky = \text{Re}\left(\hat{F}e^{i\varpi t}\right)$$
$$x = \text{Re}\left\{\hat{X}e^{i\varpi t}\right\}$$
$$\hat{X} = \frac{\hat{F}}{k}D(r,\zeta)$$
$$D(r,\zeta) = \frac{1}{(1-r^2)+2i\zeta r}$$
$$x = \text{Re}\left\{\frac{\hat{F}}{k}|D(r,\zeta)|e^{i(\varpi t-\theta)}\right\}$$
$$\theta = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

$\int t^3\,dt$	=	$\frac{t^4}{4}$
$\int$ Cos[3 t] dt	=	$\frac{1}{3}$ Sin[3 t]
$\int$ Sin[t] dt	=	-Cos[t]
$\int$ Cos[t] dt	=	Sin[t]
$\int e^{3t}\,dt$	=	$\frac{e^{3t}}{3}$

Let $y = \text{Re}\left(\hat{Y}e^{i\varpi t}\right)$ $\hat{Y} = \frac{\frac{\hat{F}}{m}}{(-\varpi^2+2\zeta\omega_n i\varpi+\omega_n^2)}$

$$\zeta = 0 \quad y = \text{Re}\left(\frac{\hat{F}}{k}\frac{1}{(1-r^2)}e^{i\varpi t}\right)$$

$$\zeta > 0 \quad y = \text{Re}\left(\frac{\hat{F}}{k}\frac{1}{(1-r^2)+i2\zeta r}e^{i\varpi t}\right)$$

{Cos[Pi / 2 + x], Cos[Pi / 2 - x], Sin[Pi / 2 - x], Sin[Pi / 2 + x]}

{-Sin[x], Sin[x], Cos[x], Cos[x]}

When y=0, then y in complex plan is pure imaginary. When force is max, then f in complex plan is all real

system	equation used to derive	transfer function
isolate base from force transmitted by machine	$f_{tr}(t) = f_{spring} + f_{damper}$	$\frac{\lvert f_{tr}(t)\rvert_{\max}}{\lvert\hat{F}\rvert} = \lvert D\rvert\sqrt{1+(2\zeta r)^2}$
isolate machine from motion of base	Use absolute mass position $my'' + cy' + ky = cz' + kz$	$\frac{\lvert Y\rvert_{\max}}{\lvert Z\rvert} = \lvert D\rvert\sqrt{1+(2\zeta r)^2}$
accelerameter: Measue base acc. using relative displacement	Use relative mass position $mu'' + cu' + ku = -mz''$	$U = \frac{-1}{(\omega_n^2-\omega^2)+i\omega 2\zeta\omega_n}Z_a \Rightarrow \lvert D(r,\zeta)\rvert = \frac{-1}{\sqrt{(\omega_n^2-\omega^2)^2+(2\omega\zeta\omega_n)^2}}$
seismometer: Measue base motion using relative displacement	Use relative mass position $mu'' + cu' + ku = -mz''$	$U = \frac{r^2}{(1-r^2)+i2\zeta r}Z \Rightarrow \lvert D(r,\zeta)\rvert = \frac{r^2}{\sqrt{(1-r^2)+i2\zeta r}}$

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_i'}\right) - \frac{\partial L}{\partial x_i}$$

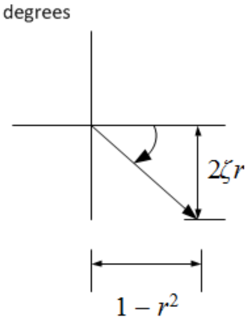
$$f_{tr}(t) = f_{spring} + f_{damper} = \text{Re}\left\{\left(\hat{F} + ci\varpi\frac{\hat{F}}{k}\right)D(r,\zeta)e^{i\varpi t}\right\}$$

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin(\varpi t - \theta)$$

$$A = u_0 + \frac{F}{K}\frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}}\sin\theta$$

$$B = \frac{v_0}{\omega_d} + \frac{u_0\xi\omega}{\omega_d} + \frac{F}{K}\frac{1}{\omega_d\sqrt{(1-r^2)^2+(2\xi r)^2}}(\xi\omega\sin\theta - \varpi\cos\theta)$$

Phase of response complex amplitude for underdamped and when r<1
Phase will be from 0 to -90 degrees



$$|f_{tr}(t)|_{\max} \qquad \zeta < 1$$

$$= \left|\hat{F}\right||D|\sqrt{1+(2\zeta r)^2}$$

Same for vibration isolation

$$u(t)\ \ \zeta = 0 \begin{cases} \varpi = \omega \rightarrow u(0)\cos\varpi t + \frac{u'(0)}{\varpi}\sin\varpi t - \frac{F}{K}\frac{\varpi t}{2}\cos(\varpi t) \\ \varpi \neq \omega \rightarrow u(0)\cos\omega t + \left(\frac{u'(0)}{\omega} - \frac{F}{K}\frac{r}{1-r^2}\right)\sin\omega t + \frac{F}{K}\frac{1}{1-r^2}\sin\varpi t \end{cases}$$

1. Determine the system of equations of motion and set up $[M]\mathbf{Y}'' + [C]\mathbf{Y}' + [K]\mathbf{Y} = \mathbf{F}$ in matrix form.
2. Solve the eigenvalue problem $\det([K] - \omega^2[M]) = 0$ in order to determine the n natural frequencies.
3. For each natural frequency ω_j determine the corresponding j^{th} eigenvector φ_j by solving $([K] - \omega_j^2[M])\varphi_j = \mathbf{0}$. In this step first component of φ_j is set to 1 and the other components are solved relative to it.
4. Obtain the normalized eigenvectors Φ_j for each φ_j using $\Phi_j = \frac{\varphi_j}{\sqrt{u_j}}$ where $u_j = \varphi_j^T[M]\varphi_j$. Each u_j will be a scalar.
5. Set up the modal transformation matrix $[\Phi] = [\Phi_1 \Phi_2 \cdots \Phi_n]$. This will be an $n \times n$ matrix.
6. The transformation from normal solution $y(t)$ to modal $\eta(t)$ will be $\mathbf{Y} = [\Phi]\eta$ and $\eta = [\Phi]^{-1}\mathbf{Y} = [\Phi]^T[M]\mathbf{Y}$
7. Apply the above transformation on the original equations of motions in matrix form to obtain the equations of motion in modal coordinates $[\Phi]^T[M]\,[\Phi]\mathbf{Y}'' + [\Phi]^T[C][\Phi]\mathbf{Y}' + [\Phi]^T[C][\Phi]\mathbf{Y} = [\Phi]^T\mathbf{F}$. This becomes $\mathbf{I}\eta''(t) + [\widetilde{C}]\eta'(t) + [\widetilde{K}]\eta(t) = [\Phi]^T\mathbf{F}$ where $\mathbf{I}$ is the identity matrix, $[\widetilde{C}]$ is a diagonal damping matrix obtained using a method such as weak damping approximation and $[\widetilde{K}]$ is diagonal matrix with diagonal that contains the natural frequencies squared ω_j^2 in each of entries.
8. For steady state solution in modal coordinates, the loading vector $[\Phi]^T\mathbf{F}$ is assumed to be $\mathbf{Q} = [\Phi]^T\mathbf{F} = \mathrm{Re}(\widehat{\mathbf{Q}}e^{i\varpi t})$ where $\widehat{\mathbf{Q}}$ is the complex amplitude of the loading vector in modal coordinates. Therefore, the steady state solution is $\eta_{ss}(t) = \mathrm{Re}(\widehat{\mathbf{X}}e^{i\varpi t})$ where $\widehat{\mathbf{X}}$ is the complex amplitude of each modal response is $\widehat{X}_j = \frac{\Phi_j^T\mathbf{F}}{-\varpi^2 + i2\zeta_j\omega_j\varpi + \omega_j^2}$. For a system with no damping this simplifies to $\widehat{X}_j = \frac{\Phi_j^T\mathbf{F}}{-\varpi^2 + \omega_j^2}$. In here, Φ_j^T represents the transpose of the j^{th} column of the modal transformation matrix $[\Phi]$, or the transpose of the j^{th} mass normalized eigenvector, and ω_j is the j^{th} natural frequency.
9. Now the steady state solution in modal coordinate is used to obtain the solution in normal coordinates since $\mathbf{Y} = [\Phi]\eta$. Therefore $\mathbf{Y}_{ss} = \mathrm{Re}(\widehat{\mathbf{X}}e^{i\varpi t}) = \mathrm{Re}([\Phi]\widehat{\mathbf{X}}e^{i\varpi t}) = \mathrm{Re}(\hat{\mathbf{Y}}e^{i\varpi t})$. In component form $\mathbf{Y}_{ss} = \mathrm{Re}\left(\left(\sum_{j=1}^{n}\Phi_j\widehat{X}_j\right)e^{i\varpi t}\right)$

In light daming, the off-diagonal entries are set to zero and then $c_{11} = 2\zeta_1\omega_1$

$$\frac{1}{i} = -i = e^{-\frac{\pi}{2}}$$

½ power point, means the max amplitude at that frequency is 0.707 of the maximum possible amplitude (which happens at resonance)

$\Phi_i^T[M]\Phi_i = [I]$ only if $i = j$, else 0

For modal solution, $\{Y\} = \sum_j \Phi_j X_j$

or

$$Y_1 = \Phi_{11}X_1 + \Phi_{12}X_2 + \cdots$$
$$Y_2 = \Phi_{21}X_1 + \Phi_{22}X_2 + \cdots$$

or

$$Y_1 = \{\Phi\}_{\text{row1}}\{X\}$$
$$Y_2 = \{\Phi^T\}_{\text{row2}}\{X\}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$
$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right)$$
$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$$
$$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$$
$$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u \sin v = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$$

If using power method

$$V_{g_{11}} = \left(\frac{\partial V_g^2}{\partial\theta_1^2}\right)_{\substack{\theta_1=0\\\theta_2=0}} = \left(-m_1 g\frac{L}{2}\sin\theta_1\right)_{\theta_1=0} = 0$$
$$V_{g_{22}} = \left(\frac{\partial V_g^2}{\partial\theta_2^2}\right)_{\substack{\theta_1=0\\\theta_2=0}} = \left(-m_1 g\frac{L}{2}\sin\theta_2\right)_{\theta_2=0} = 0$$
$$V_{g_{12}} = \left(\frac{\partial V_g^2}{\partial\theta_1\partial\theta_2}\right)_{\substack{\theta_1=0\\\theta_2=0}} = 0$$

$$T = \frac{1}{2}m\dot{y}_g^2 + \frac{1}{2}I_{cg}\dot{\theta}^2$$
$$= \frac{1}{2}m\left(\frac{\dot{y}_1+\dot{y}_2}{2}\right)^2 + \frac{1}{2}(mr_G^2)\left(\frac{\dot{y}_2-\dot{y}_1}{L}\right)^2$$

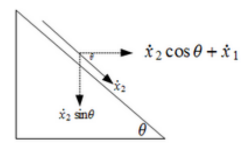
$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2v^2$$

:k. To find this v it is easier to resolve components on the x

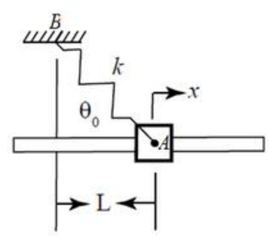
$$\vec{v} = \dot{x}_2\sin\theta j + (\dot{x}_2\cos\theta + \dot{x}_1)i$$

velocity

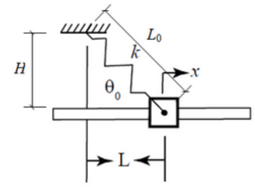


$$|\vec{v}|^2 = (\dot{x}_2\sin\theta)^2 + (\dot{x}_2\cos\theta + \dot{x}_1)^2$$
$$= (\dot{x}_2^2\sin^2\theta) + (\dot{x}_2^2\cos^2\theta + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta)$$
$$= \dot{x}_2^2(\sin^2\theta + \cos^2\theta) + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta$$
$$= \dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta$$

Stiff spring



$$\dot{\Delta} = (\dot{u}_A - \dot{u}_B)e_{A/B}$$
$$= (\dot{x}\hat{\imath} - 0\hat{\jmath})\cdot(\cos\theta_0\hat{\imath} - \sin\theta_0\hat{\jmath})$$
$$= \dot{x}\cos\theta_0$$



Physically, the constant represents the time it takes the system's step response to reach 63.2 % of its final (asymptotic) value $\tau = \frac{1}{\zeta_1\omega_1}$

Hence from the above diagram we see that $L_0 = \sqrt{H^2+L^2}$ and $L_{cur} = \sqrt{H^2+(L+x)^2}$, therefore

$$\Delta = \sqrt{H^2+(L+x)^2} - \sqrt{H^2+L^2}$$

Where $e_{B/A}$ is unit vector oriented to B from A