Homework #A1

EMA 545, Spring 2013

Instructions:

- If you scored 23 or below on Problem #2 on Exam #1, do Problems 1, 2 and 3.
- If you scored 23 or below on Problem #4 on Exam #1, do Problems 4 and 5
- If neither of those criteria apply to you then you do not need to turn in this assignment.

When working out your solutions to the following problems, you must derive your answers starting from the following. You may not use any equations from the book without first deriving them from these basic principles:

The general solution to an underdamped SDOF system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^{\ 2}x = 0$$

is:

$$x(t) = \mathrm{Re}\left(Ae^{-\zeta\omega_n t}e^{\mathrm{i}\omega_d t}\right)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and A is a complex constant.

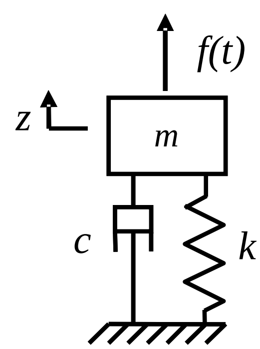
You are free to use Appendix B as needed and the fact that the forced response of a system is

$$x(t) = x_{IC} + x_F$$

Where x_{IC} and x_F are found in Appendix B for a variety of forcing functions.

$$f(t) = \mathrm{Re}\left(Fe^{\mathrm{i}\omega t}\right) \qquad \rightarrow \qquad x(t) = \mathrm{Re}\left(Xe^{\mathrm{i}\omega t}\right)$$

The half power points in a transfer function occur at frequencies $\omega \cong \omega_{peak} \pm \zeta\omega_n$



Problem 1: 3.1 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with q(t)).

Problem 2: 3.11 from Ginsberg. Begin by writing the equation of motion for the system (shown to the right after replacing z(t) with x(t)).

Problem 3: (3e2)

The equations of motion for the 2DOF system studied in class are given below. If the applied force is $f(t)=F\cos(\omega t)$, then the response of both coordinates x_1 and x_2 will also be harmonic. Use this fact to derive the transfer function between the force F and the response X_1.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

Problem 4: 2.52 from Ginsberg. Check your answer by comparing it to the solution from ode45 in Matlab.

Problem 5: 2.54 from Ginsberg. Choose values for the parameters and check your answer by plotting the response and comparing it to the solution from ode45 in Matlab.

Exercise 3.1

$M = 20\ kg,\ \omega_{nat} = 100\ rad/s \Rightarrow k = M\omega_{nat}^2 = 2(10^5) N/m$

$q = 0.02\cos(110t - 1.5)$ meter

$= Re[0.02\, e^{i(\omega t - 1.5)}] \quad \omega = 110\ rad/s$

$Q(t) = M\ddot{q} + C\dot{q} + Kq$

$= Re[(-M\omega^2 + Ci\omega + K)(0.02)e^{i(\omega t - 1.5)}]$

But $\frac{K}{M} = \omega_{nat}^2$ & $C/M = 2\zeta\omega_{nat}$, so

$Q(t) = M\, Re[(-\omega^2 + 2i\zeta\omega_{nat}\omega + \omega_{nat}^2)(0.02)$

$\times e^{i(\omega t - 1.5)}]$

For $\zeta = 0$ & $\omega = 110$:

$Q = 20\, Re[-42\ e^{i(110t - 1.5)}]$

$\Rightarrow \quad = -840\cos(110t - 1.5)$ newton

For $\zeta = 0.4$ & $\omega = 110$ rad/s:

$Q = 20\, Re[(-2100 + 8800i)(0.02)e^{i(110t - 1.5)}]$

$= Re[(-840 + 3520i)e^{i(110t - 1.5)}]$

$= Re[3619\, e^{i1.8051} e^{i(110t - 1.5)}$

$\Rightarrow \quad = 3619\cos(110t + 0.3051)$ newton

3.11 Solution

$m\ddot{x} + c\dot{x} + Kx = F\cos(\omega t)$

$f = 100\ Hz \longrightarrow x(t) = 4\sin(\omega t)$ ✓ ? meters? not given

$-90°$ out of phase, so $f_n = 100\ Hz$

$f = 105\ Hz \longrightarrow$ half power point

→ This part is tedious to re-derive, but we know from the book that:

$\omega_{peak} \approx \omega_n \qquad \omega_{HP} = \omega_{peak} \pm \zeta\omega_n$

so

$105 - 100 = 5\ Hz \cdot 2\pi = \zeta\omega_n$

$\longrightarrow \zeta = \frac{10\pi}{105 \cdot 2\pi} = 0.0476$

a.) $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = (F/m)\cos(\omega t) \longrightarrow x(t) = Re(X e^{i\omega t})$

$\longrightarrow \quad X = \frac{(F/m)}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$

$X = (F/m) \frac{1}{\cdots}$ $\qquad \omega = 105\ Hz$, $\omega_n = 100\ Hz$, $\zeta = 0.0476$

$X = (F/m) \cdot (-1.26 - 1.24)\times 10^{-5}$

$X = (F/m) \cdot 1.77\, e^{-2.37}$

$\longrightarrow \boxed{\text{phase lag} = 2.37\ \text{rad} = 135.7°}$

b.) Amplitude: at resonance:

$|X| = \left| \frac{(F/m)}{i\omega_n(2\zeta\omega_n)} \right| \longrightarrow (F/m) = 4 \cdot 2\zeta\omega_n^2 = 1.5\times 10^5\ N/kg$

at $\omega = 110\ Hz$

$X = \frac{F/m}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n} \longrightarrow$

$\boxed{|X| = 1.62 \qquad \angle X = -153°}$

Units not given – m? (Very large!)

3e2 Solution

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$

$$f(t) = F\cos(\omega t)$$

assume: $x_j(t) = Re(X_j e^{i\omega t})$

$\ddot{x}_j(t) = Re(-\omega^2 X_j e^{i\omega t})$

$$-\omega^2 m X_1 + 2K X_1 - K X_2 = 0$$

$$-\omega^2 m X_2 - K X_1 + K X_2 = F$$

$X_1 = \dfrac{K X_2}{(2K - \omega^2 m)}$, sub into

$$-\omega^2 m X_2 - \frac{K^2 X_2}{(2K - \omega^2 m)} + K X_2 = F$$

$$X_2 = \left(-\omega^2 m - \frac{K^2}{(2K - \omega^2 m)} + K\right)^{-1} F$$

$$\boxed{X_1 = \frac{K}{(2K - \omega^2 m)}\left(-\omega^2 m - \frac{K^2}{(2K - \omega^2 m)} + K\right)^{-1} F}$$

Or → easier method → solve linear equations above

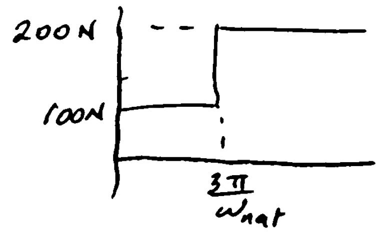
$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{bmatrix} -\omega^2 m + 2K, & -K \\ -K, & -\omega^2 m + K \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$= \frac{\begin{bmatrix} -\omega^2 m + K & K \\ K & -\omega^2 m + 2K \end{bmatrix}\begin{Bmatrix} 0 \\ F \end{Bmatrix}}{(-\omega^2 m + 2K)(-\omega^2 m + K) - K^2}$$

$$= \frac{\begin{Bmatrix} KF \\ (2K - \omega^2 m)F \end{Bmatrix}}{\cdots}$$

$$\boxed{X_1 = \frac{K}{(2K - \omega^2 m)(K - \omega^2 m) - K^2} F}$$

Exercise 2.52



$$Q(t) = 100\,h(t) + 100\,h\left(t - \frac{3\pi}{\omega_{nat}}\right)$$

$$m\ddot{q} + kq = Q.$$

Assume $q(0) = \dot{q}(0) = 0$

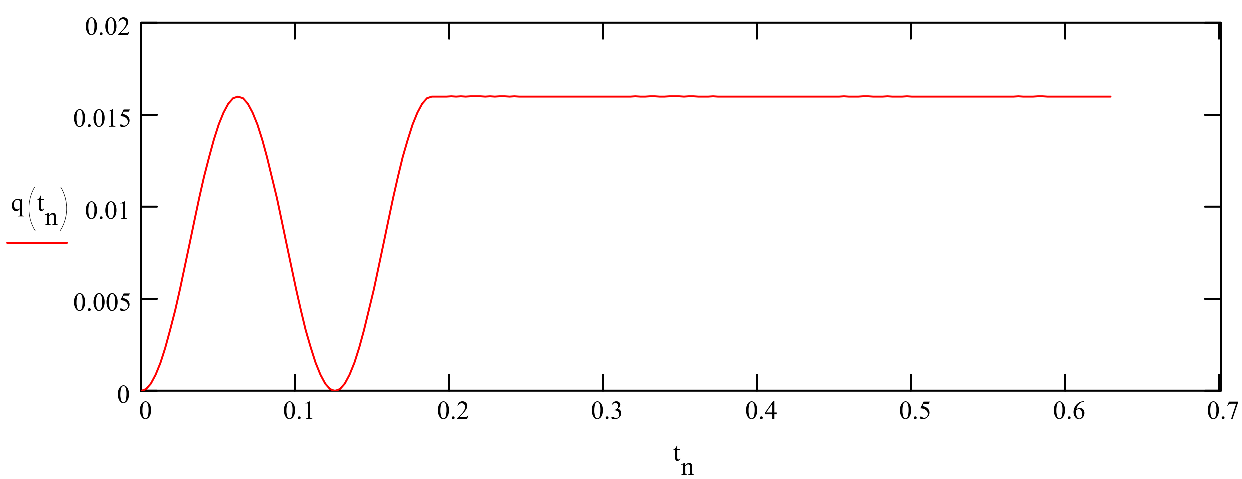
Let $u(t)$ be the unit step response.

$$q = 100\left[u(t) + u\left(t - \frac{3\pi}{\omega_{nat}}\right)\right]$$

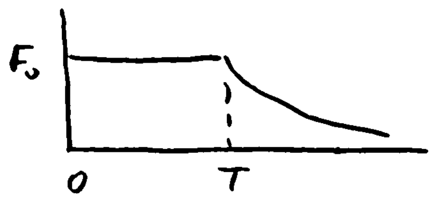
$M := 5$ $\qquad \omega_{nat} := 50$

$$u(t) := \frac{1}{M\cdot\omega_{nat}^{\ 2}}\cdot\left(1 - \cos\left(\omega_{nat}\cdot t\right)\right)\cdot(t>0) \qquad q(t) := 100\cdot\left(u(t) + u\left(t - \frac{3\cdot\pi}{\omega_{nat}}\right)\right)$$

$N := 201$ $\qquad n := 1\,..\,N$ $\qquad t_n := \frac{n-1}{N-1}\cdot\frac{10\cdot\pi}{\omega_{nat}}$



Exercise 2.54



$$Q = F_0\left[h(t) - h(t-T)\right] + F_0 \exp\left[-\beta(t-T)\right] h(t-T)$$

$m\ddot{q} + kq = Q$, Assume $q(0) = \dot{q}(0) = 0$

Let u be the unit step response, and x be the unit exponential response. Then

$$q = F_0\, u(t) - F_0\, u(t-T) + F_0\, x(t-T)$$

where

$$u(t) = \frac{1}{M\omega_{nat}^2}\left[1 - \cos(\omega_{nat} t)\right] h(t)$$

$$x(t) = \frac{1}{M(\omega_{nat}^2 - 2\beta\omega_{nat} + \beta^2)}\left\{\exp(-\beta t) - \left[\cos(\omega_{nat} t) - \frac{\beta}{\omega_{nat}} \sin(\omega_{nat} t)\right]\right\} h(t)$$