HW A1

EMA 545
Mechanical Vibrations

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University of Wisconsin, Madison

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Contents

1 problem 1

A system has mass $M = 20kg$ and $\omega_n = 100$ rad/sec. It is observed that steady state response is $q = 20\cos(110t - 1.5)$ mm, where t is in seconds. Determine harmonic excitation causing this response for $\zeta = 0$ and $\zeta = 0.4$

Let the harmonic excitation be

$$F(t) = \operatorname{Re}\left\{\hat{F}e^{i\omega t}\right\}$$

where $\hat{F}$ is its complex amplitude. Also let

$$q = \operatorname{Re}\left\{\hat{Q}e^{i\omega t}\right\}$$

be the steady state response. We are given that $q = 20 \times 10^{-3}\cos(110t - 1.5)$, therefore

$$\begin{aligned} q &= \operatorname{Re}\left\{20 \times 10^{-3} e^{i(100t-1.5)}\right\} \\ &= \operatorname{Re}\left\{20 \times 10^{-3} e^{-1.5i} e^{i100t}\right\} \end{aligned}$$

Therefore

$$\hat{Q} = 20 \times 10^{-3} e^{-1.5i}$$

But the transfer function for second order system is

$$\hat{Q} = \frac{\hat{F}}{k}\frac{1}{\left(1 - r^2\right) + 2i\zeta r}$$

where $r = \frac{\omega}{\omega_n}$, hence we can now solve for $\hat{F}$ from the above.

$$\hat{F} = \hat{Q}\left(k\left(\left(1 - r^2\right) + 2i\zeta r\right)\right)$$

But $k = M\omega_n^2$ hence

$$\boxed{\hat{F} = \hat{Q}\left(M\omega_n^2\left(\left(1 - r^2\right) + 2i\zeta r\right)\right)}$$

When $\zeta = 0$ we find

$$\begin{aligned} \hat{F} &= 20 \times 10^{-3} e^{-1.5i}\left(20 \times 100^2\left(1 - \left(\frac{110}{100}\right)^2\right)\right) \\ &= 20 \times 10^{-3} e^{-1.5i}(-42000.0) \\ &= -42000.0 \times 20 \times 10^{-3} e^{-1.5i} \\ &= -840.0 e^{-1.5i} \end{aligned}$$

Hence

$$\begin{aligned} F(t) &= \operatorname{Re}\left\{\hat{F}e^{i\omega t}\right\} \\ &= \operatorname{Re}\left\{-840.0 e^{-1.5i} e^{i110t}\right\} \\ &= \operatorname{Re}\left\{-840.0 e^{i(100t-1.5)}\right\} \end{aligned}$$

Therefore

$$\boxed{F(t) = -840\cos(100t - 1.5)}$$

When $\zeta = 0.4$ we find

$$\begin{aligned}
\hat{F} &= \hat{Q}\left(M\omega_n^2\left(\left(1-r^2\right)+2i\zeta r\right)\right) \\
&= 20\times 10^{-3}e^{-1.5i}\left(20\times 100^2\left(\left(1-\left(\frac{110}{100}\right)^2\right)+i2(0.4)\left(\frac{110}{100}\right)\right)\right) \\
&= 20\times 10^{-3}e^{-1.5i}\left(20\times 100^2(-0.21+0.88i)\right) \\
&= 4000e^{-1.5i}(-0.21+0.88i) \\
&= 4000e^{-1.5i}\left(\sqrt{(0.21)^2+(0.88)^2}\,e^{i\tan^{-1}\left(\frac{.88}{-0.21}\right)}\right)
\end{aligned}$$

```
In[4]:= ArcTan[-0.21, 0.88]

Out[4]= 1.80505
```

Hence

$$\begin{aligned}
\hat{F} &= 4000e^{-1.5i}\left(0.90471e^{i1.80505}\right) \\
&= 3618.8e^{-1.5i+1.80505i} \\
&= 3618.8e^{0.30505i}
\end{aligned}$$

Therefore

$$\begin{aligned}
F(t) &= \operatorname{Re}\left\{\hat{F}e^{i\omega t}\right\} \\
&= \operatorname{Re}\left\{3618.8e^{0.30505i}e^{i110t}\right\} \\
&= \operatorname{Re}\left\{3618.8e^{i(100t+0.30505)}\right\}
\end{aligned}$$

Hence

$$\boxed{F(t) = 3618.8\cos(100t+0.30505)}$$

2 Problem 2

3.11 Measurement of the steady-state response of a one-degree-of-freedom system to a harmonic excitation $F\cos(\omega t)$ indicates that at a frequency of 100 Hz, the response is $x = 4\sin(\omega t)$. It also is observed that 105 Hz is a half-power point.

(a) Determine the phase lag of the response relative to the excitation at 105 Hz.

(b) Determine the amplitude and phase lag of the response at 110 Hz.

Let

$$P(t) = \operatorname{Re}\left\{\hat{F}e^{i\omega t}\right\}$$

where $\hat{F}$ is the complex amplitude of the excitation. Hence by comparing this to $P(t) = F\cos\omega t = \operatorname{Re}\left\{Fe^{i\omega t}\right\}$ we see that $\hat{F} = F$.

When $\omega = 2\pi 100$ then the response was $q = \operatorname{Re}\left\{\hat{Q}e^{i\omega t}\right\} = 4\sin(2\pi 100 t)$ hence $q = \operatorname{Re}\left\{\overbrace{4e^{-i\frac{\pi}{2}}}^{\hat{Q}}e^{i\omega t}\right\} = q = \operatorname{Re}\left\{4e^{i\left(\omega t-\frac{\pi}{2}\right)}\right\}$ therefore

$$\hat{Q} = 4e^{-i\frac{\pi}{2}}$$

But, from the transfer function of second order system we know that

$$\hat{Q} = \frac{\hat{F}}{k}\frac{1}{\left(1-r^2\right)+2i\zeta r}$$

Hence

$$\begin{aligned} 4e^{-i\frac{\pi}{2}} &= \frac{\hat{F}}{k}\frac{1}{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)+2i\zeta\left(\frac{2\pi 100}{\omega_n}\right)} \\ &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 100}{\omega_n}\right)\right)^2}}e^{-i\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)} \end{aligned} \tag{1}$$

By comparing sides we see that

$$\begin{aligned} \frac{\pi}{2} &= \frac{2\zeta r}{1-r^2} \\ &= \frac{2\zeta\frac{2\pi 100}{\omega_n}}{1-\left(\frac{2\pi 100}{\omega_n}\right)^2} \end{aligned} \tag{2}$$

When $\omega = 105Hz$ we are told it is half power point, which means the amplitude there is 0.707 of the maximum amplitude which occurs when $r = 1$. Hence

$$\begin{aligned} 0.707\frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-(1)^2\right)^2+(2\zeta(1))^2}} &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 105}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 105}{\omega_n}\right)\right)^2}} \\ 0.707\frac{1}{2\zeta} &= \frac{1}{\sqrt{\left(1-\left(\frac{2\pi 105}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{2\pi 105}{\omega_n}\right)\right)^2}} \end{aligned} \tag{3}$$

We now have 2 equations 2 and 3 to solve numerically for ζ and ω_n . Solving and keeping the positive solutions results in

$$\begin{aligned} \zeta &= 0.0309 \\ \omega_n &= 640.8 \text{ rad/sec} \\ &= 101.987 \text{ Hz} \end{aligned}$$

Hence at $\omega = 105$ hz the phase is

$$\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.0309)\frac{2\pi(105)}{640.8}}{1-\left(\frac{2\pi(105)}{640.8}\right)^2}\right) = 133.305^0$$

```
In[35]:= ArcTan[1 - ((2 Pi 105)/640.8)^2, 2 (0.0309) ((2 Pi 105)/640.8)]*180/Pi
Out[35]= 133.305
```

2.1 Part(b)

When $\omega = 100$ Hz we found from Eq 1 that

$$4 = \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 100}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2\pi 100}{\omega_n}\right)\right)^2}}$$

But we found that $\omega_n = 640.8$ rad/sec and $\zeta = 0.0309$, hence

$$\begin{aligned}\frac{\hat{F}}{k} &= 4\sqrt{\left(1-\left(\frac{2\pi 100}{640.8}\right)^2\right)^2 + \left(2(0.0309)\left(\frac{2\pi 100}{640.8}\right)\right)^2} \\ &= 0.28733\end{aligned}$$

At $\omega = 110$ Hz

$$\begin{aligned}\left|\hat{Q}\right| &= \frac{\hat{F}}{k}\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 110}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{2\pi 110}{\omega_n}\right)\right)^2}} \\ &= 0.28733\frac{1}{\sqrt{\left(1-\left(\frac{2\pi 110}{640.8}\right)^2\right)^2 + \left(2(0.0309)\left(\frac{2\pi 110}{640.8}\right)\right)^2}} \\ &= \boxed{1.6288}\end{aligned}$$

The phase is

$$\tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.0309)\left(\frac{2\pi 110}{640.8}\right)}{1-\left(\frac{2\pi 110}{640.8}\right)^2}\right) = \boxed{157.798^0}$$

```
In[37]:= ArcTan[1 - ((2 Pi 110)/640.8)^2, 2 (0.0309) ((2 Pi 110)/640.8)]*180/Pi
Out[37]= 157.798
```

3 Problem 3

Problem 3: (3e2)
The equations of motion for the 2DOF system studied in class are given below. If the applied force is $f(t)$=Fcos(ωt), then the response of both coordinates x_1 and x_2 will also be harmonic. Use this fact to derive the transfer function between the force F and the response X_1.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f(t) \end{Bmatrix}$$

The two equations are

$$\begin{aligned} mx_1'' + 2kx_1 - kx_2 &= 0 \\ mx_2'' - kx_1 + kx_2 &= f(t) \end{aligned}$$

Since the responses are harmonic and the input is harmonic, then we can write

$$\begin{aligned} x_1(t) &= \operatorname{Re}\left\{\hat{X}_1 e^{i\omega t}\right\} \\ x_1(t) &= \operatorname{Re}\left\{\hat{X}_2 e^{i\omega t}\right\} \end{aligned}$$

Therefore the two equations can be written in terms of the complex amplitudes as

$$-m\omega^2\hat{X}_1 + 2k\hat{X}_1 - k\hat{X}_2 = 0 \tag{4}$$
$$-m\omega^2\hat{X}_2 - k\hat{X}_1 + k\hat{X}_2 = F \tag{5}$$

From Eq 4

$$\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 = \hat{X}_2$$

Substitute the above into Eq 5 gives

$$-m\omega^2\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 - k\hat{X}_1 + k\frac{\left(-m\omega^2 + 2k\right)}{k}\hat{X}_1 = F$$
$$\left(\frac{\left(-m^2\omega^4 - m\omega^2 2k\right)}{k} + k - m\omega^2\right)\hat{X}_1 = F$$
$$\hat{X}_1 = kF\frac{1}{\left(-m^2\omega^4 - m\omega^2 2k + k^2 - km\omega^2\right)}$$

Dividing the numerator and denominator of the RHS by k^2, and using $k^2 = \omega_n^4 m^2$ and using $r = \frac{\omega}{\omega_n}$

$$\hat{X}_1 = \frac{F}{k}\frac{1}{\left(\frac{-m^2\omega^4}{\omega_n^4 m^2} - \frac{m\omega^2 2}{\omega_n^2 m} + 1 - \frac{m\omega^2}{\omega_n^2 m}\right)}$$
$$\hat{X}_1 = \frac{F}{k}\frac{1}{\left(-r^4 - 2r^2 + 1 - r^2\right)}$$

Hence the transfer function is

$$\hat{X}_1 = \frac{F}{k}\frac{1}{\left(1 - r^4 - 3r^2\right)}$$

4 Problem 4

2.52 An undamped system is subjected to a sequence of step excitations, $Q(t) = 100$ N if $0 < t < 3\pi/\omega_{nat}$, $Q(t) = 200$ N if $t > 3\pi/\omega_{nat}$. The system mass is 5 kg and $\omega_{nat} = 50$ rad/s. Determine and graph the response.

Summary of method of solution: There are 2 ways to solve these problem. We will solve it using both methods. The first method is using known standard solution for step input, the solution $y(t)$ is found for the period of $0 < t < \frac{3\pi}{\omega_n}$ using zero initial conditions. Next, the solution $y(t)$ and $y'(t)$ is evaluated again at $t = \frac{3\pi}{\omega_n}$. These values are now used as the initial conditions for the solution for $t > \frac{3\pi}{\omega_n}$. The solution for $t > \frac{3\pi}{\omega_n}$ will have the same form, but the step input now is $200N$ instead of 100N.

The second method as follows: Let $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ or $F(t) = 100h(t) + 100h\left(\tilde{t}\right)$ where $\tilde{t} = t - \frac{3\pi}{\omega_n}$, then assuming the transient solution to $h(t)$ is $s(t)$ then the solution to $F(t)$ is $100s(t) + 100s\left(\tilde{t}\right)$. The second method is simplet than the first method.

Solution using first method:

The system is

$$my''(t) + ky(t) = F(t)$$

When $F(t)$ is a fixed input, such as a step input of magnitude F then the response is given by

$$y(t) = \left(y_0 - \frac{F}{k}\right)\cos\omega_n t + \frac{y_0'}{\omega_n}\sin\omega_n t + \frac{F}{k}$$

Where in the above, y_0 and y_0' are the initial position and initial velocity. For $0 < t < 1.5T_n$ the solution is

$$\begin{aligned} y(t) &= -\frac{F}{k}\cos\omega_n t + \frac{F}{k} \\ &= \frac{F}{k}(1 - \cos\omega_n t) \end{aligned}$$

Let $F = Q_1 = 100\text{N}$, and since $k = m\omega_n^2$ then the above becomes

$$y(t) = \frac{Q_1}{m\omega_n^2}(1 - \cos\omega_n t) \qquad 0 < t < \frac{3\pi}{\omega_n}$$

Now we need first to evaluate $y\left(t = \frac{3\pi}{\omega_n}\right)$ and $y'\left(t = \frac{3\pi}{\omega_n}\right)$. From the above

$$y'(t) = \frac{Q_1}{m\omega_n}\sin\omega_n t \qquad 0 < t < \frac{3\pi}{\omega_n}$$

Hence

$$y\left(t = \frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n^2}\left(1 - \cos\omega_n\frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n^2}(1 - \cos 3\pi) = \frac{Q_1}{m\omega_n^2}(1 - (-1)) = \frac{2Q_1}{m\omega_n^2}$$

and

$$y'\left(t = \frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n}\sin\left(\omega_n\frac{3\pi}{\omega_n}\right) = \frac{Q_1}{m\omega_n}\sin(3\pi) = 0$$

Now let $\tilde{t} = t - \frac{3\pi}{\omega_n}$Hence the solution for $\tilde{t} > 0$ is

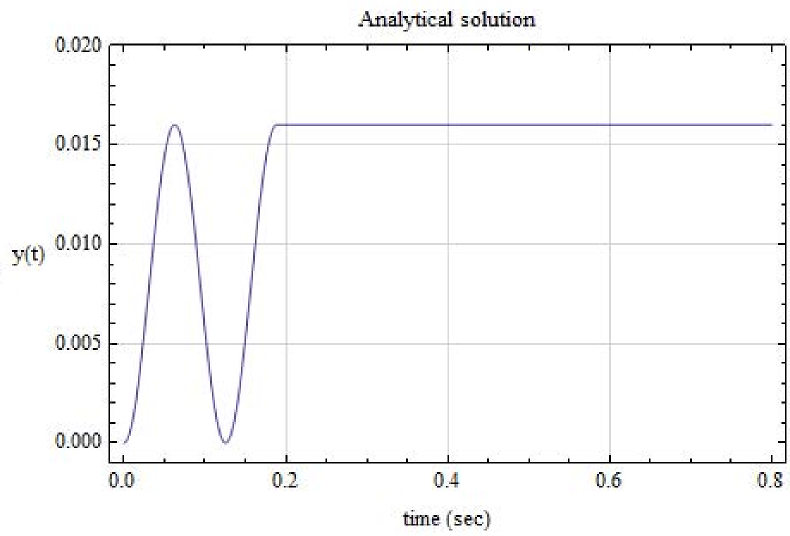
$$\begin{aligned} y(\tilde{t}) &= \left(y(\tilde{t} = 0) - \frac{Q_2}{m\omega_n^2}\right)\cos\omega_n\tilde{t} + \frac{y'(\tilde{t} = 0)}{\omega_n}\sin\omega_n\tilde{t} + \frac{Q_2}{k} \\ &= \left(\frac{2Q_1}{m\omega_n^2} - \frac{Q_2}{m\omega_n^2}\right)\cos\omega_n\tilde{t} + \frac{Q_2}{k} \end{aligned}$$

Therefore, we have obtain the complete solution, which is

time	solution
$0 < t < \frac{3\pi}{\omega_n}$	$\frac{Q_1}{m\omega_n^2}(1 - \cos\omega_n t) = \frac{100}{5(50)^2}(1 - \cos 50t) = 0.008(1 - \cos 50t)$
$\tilde{t} = t - \frac{3\pi}{\omega_n}$	$\left(\frac{2Q_1}{m\omega_n^2} - \frac{Q_2}{k}\right)\cos\omega_n\tilde{t} + \frac{Q_2}{k} = \left(\frac{2(100)}{5(50)^2} - \frac{200}{5(50)^2}\right)\cos\omega_n\tilde{t} + \frac{200}{5(50)^2} = 0.016$

This is a plot of the solution. Then a numerical ODE solver is used to verify the result

```
y[t_] := Piecewise[{{0.008 (1 - Cos[50 t]), 0 ≤ t < 3 π/50}, {0.016, True}}];
Plot[y[t], {t, 0, 0.8}, Exclusions → None,
 PlotRange → {Automatic, {-0.001, 0.02}}, AxesOrigin → {0, 0},
 Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "Analytical solution"}},
 RotateLabel → False]
```



Now a numerical ODE solver was used to verify. Here is the result

```
force[t_?NumericQ] :=
  Piecewise[{{100, 0 ≤ t < 3. π/50}, {200, True}, {0, t < 0}}];
ode = y''[t] + 50^2 y[t] == force[t]/5;
sol = NDSolve[{ode, y[0] == 0, y'[0] == 0}, y, {t, 0, 0.8}];
```

```
Plot[Evaluate[y[t] /. sol], {t, 0, 0.8}, Exclusions → None,
 PlotRange → {Automatic, {-0.001, 0.02}}, AxesOrigin → {0, 0},
 Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
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```
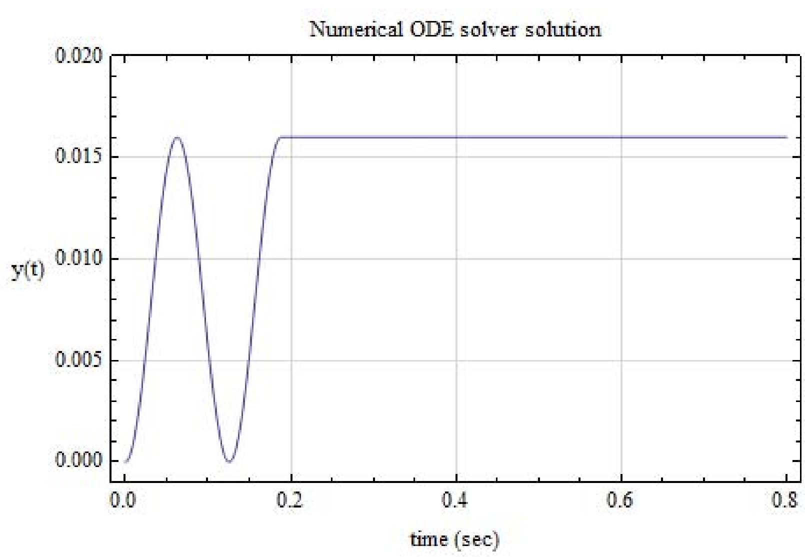



We can see the solutions match very well.

Solution using second method:

Let $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ then assuming the transient solution to $h(t)$ is $s(t)$ then the solution to $F(t)$ is $100s(t) + 100s\left(t - \frac{3\pi}{\omega_n}\right)h\left(t - \frac{3\pi}{\omega_n}\right)$. From appendix B, the solution to $h(t)$ is given by

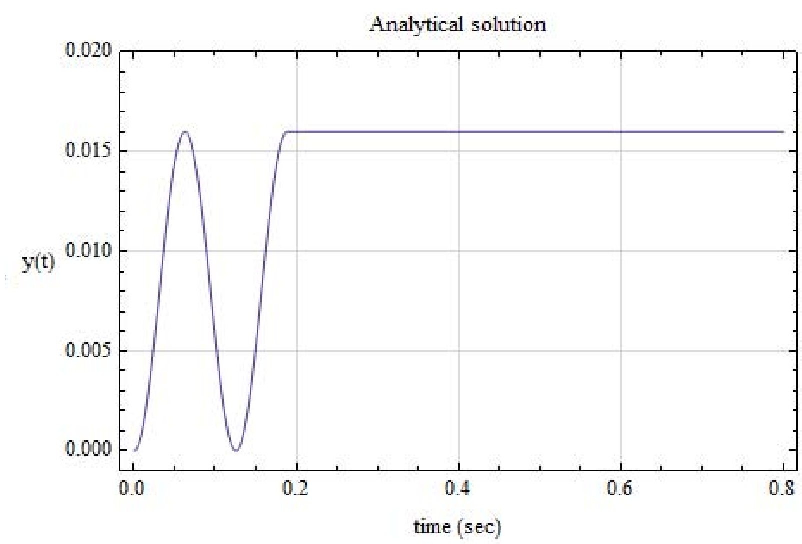
$$s(t) = \frac{1}{m\omega_n^2}(1 - \cos \omega_n t)$$

hence the solution to $F(t) = 100h(t) + 100h\left(t - \frac{3\pi}{\omega_n}\right)$ is

$$\begin{aligned} y(t) &= 100s(t) + 100s\left(t - \frac{3\pi}{\omega_n}\right)h\left(t - \frac{3\pi}{\omega_n}\right) \\ &= \frac{100}{m\omega_n^2}(1 - \cos \omega_n t) + \frac{100}{m\omega_n^2}\left(1 - \cos \omega_n\left(t - \frac{3\pi}{\omega_n}\right)\right)h\left(t - \frac{3\pi}{\omega_n}\right) \end{aligned}$$

To verify, this is a plot of the above solution. We see it is the same as the first analytical solution, and it is the same solution as the one using numerical ODE solver as well.

```
y[t_] := 100/(m wn^2) (1 - Cos[wn t]) +
  100/(m wn^2) (1 - Cos[wn (t - 3 π/50)]) UnitStep[t - 3 π/50]
Plot[Evaluate[y[t] /. {m → 5, wn → 50}], {t, 0, 0.8},
 PlotRange → {Automatic, {-0.001, 0.02}}, Exclusions → None,
 AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "Analytical solution"}},
 RotateLabel → False]
```



5 Problem 5

2.54 An undamped system is subjected to a constant force for an interval T, after which it begins to roll off exponentially. Specifically, $Q(t) = F_0$ if $0 < t < T$, $Q(t) = F_0 \exp[-\beta(t - T)]$ if $t > T$. Derive an expression for the response.

The input can be written as $F_0 h(t) - F_0 h(t - T) + F_0 e^{-\beta(t-T)} h(t - T)$ or, by letting $\tilde{t} = t - T$, the input becomes

$$F_0 h(t) - F_0 h(\tilde{t}) + F_0 e^{-\beta\tilde{t}} h(\tilde{t})$$

If the response to $h(t)$ is $s(t)$ and the response to $e^{-\beta\tilde{t}}$ is $s_1(\tilde{t})$ then the response to the above becomes

$$F_0 s(t) - F_0 s(\tilde{t}) h(\tilde{t}) + F_0 s_1(\tilde{t}) h(\tilde{t})$$

From appendix B, we see that

$$s(t) = \frac{1}{m\omega_n^2}(1 - \cos\omega_n t)$$

and

$$s_1(\tilde{t}) = \frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta\tilde{t}} - \left(\cos(\omega_n\tilde{t}) + \frac{-\beta}{\omega_n}\sin\omega_n\tilde{t}\right)\right)h(\tilde{t})$$

Therefore the the final response is

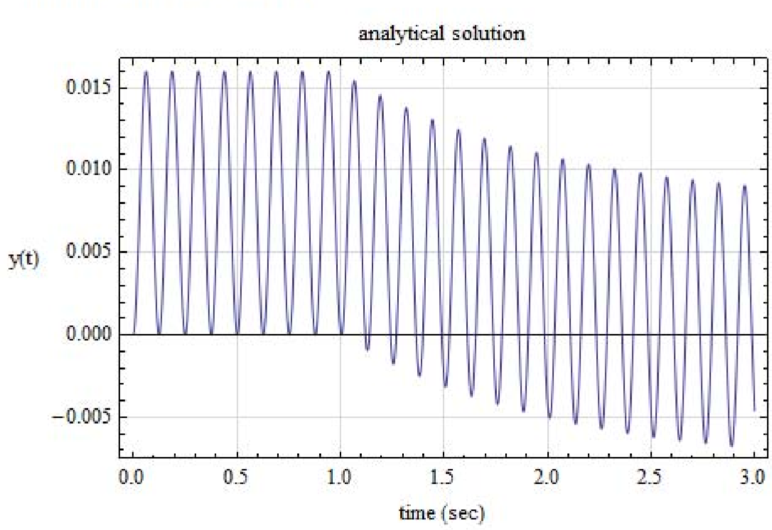
$$\begin{aligned} y(t) &= F_0 h(t) - F_0 h(\tilde{t}) + F_0 e^{-\beta\tilde{t}} h(\tilde{t}) \\ &= F_0 \frac{1}{m\omega_n^2}(1 - \cos\omega_n t)h(t) - F_0 \frac{1}{m\omega_n^2}\left(1 - \cos\omega_n\tilde{t}\right)h(\tilde{t}) + \\ F_0 &\frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta\tilde{t}} - \left(\cos(\omega_n\tilde{t}) + \frac{-\beta}{\omega_n}\sin\omega_n\tilde{t}\right)\right)h(\tilde{t}) \end{aligned}$$

$$\begin{aligned} &= F_0 \frac{1}{m\omega_n^2}(1 - \cos\omega_n t) - F_0 \frac{1}{m\omega_n^2}(1 - \cos(\omega_n(t - T)))h(t - T) + \\ F_0 &\frac{1}{m(\omega_n^2 + \beta^2)}\left(e^{-\beta(t-T)} - \left(\cos(\omega_n(t - T)) + \frac{-\beta}{\omega_n}\sin\omega_n(t - T)\right)\right)h(t - T) \end{aligned}$$

To plot this, we need to choose values for parameters. Let $F_0 = 100, \omega_n = 50 rad/\sec, m = 5kg, \beta = 1, T = 1$, then a plot of the above is below, followed by solution from numerical ODE solver.

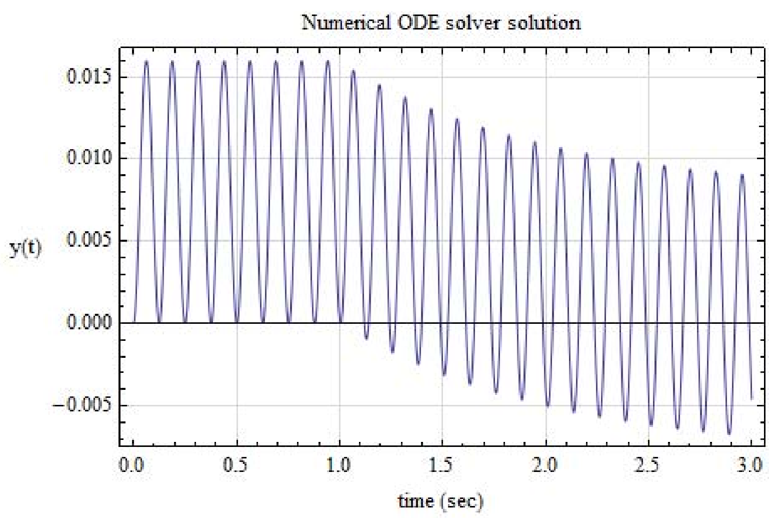
Plot of the analytical solution

```
y[t_] := f0/(m wn^2) (1 - Cos[wn t]) - f0/(m wn^2) (1 - Cos[wn (t - T)]) UnitStep[t - T] +
   f0/(m (wn^2 β^2)) (Exp[-β (t - T)] - (Cos[wn (t - T)] - β/wn Sin[wn (t - T)]))
    UnitStep[t - T]
parms = {T → 1, β → 1, wn → 50, m → 5, f0 → 100};
Plot[Evaluate[y[t] /. parms], {t, 0, 3}, Exclusions → None,
 PlotRange → All, AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel → {{"y(t)", None}, {"time (sec)", "analytical solution"}},
 RotateLabel → False]
```



To verify, this is the result from numerical ODE solver

```
Clear[t, y]
T = 1; β = 1; wn = 50; m = 5; f0 = 100;
force[t_?NumericQ] :=
  Piecewise[{{f0, 0 ≤ t < T}, {f0 Exp[-β (t - T)] UnitStep[t - T], True}}];
ode = y''[t] + wn^2 y[t] == force[t]/m;
sol = NDSolve[{ode, y[0] == 0, y'[0] == 0}, y, {t, 0, 3}];
Plot[Evaluate[y[t] /. sol], {t, 0, 3}, Exclusions → None,
 PlotRange → All, AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
 GridLinesStyle → LightGray,
 FrameLabel →
  {{"y(t)", None}, {"time (sec)", "Numerical ODE solver solution"}},
 RotateLabel → False]
```



We can see that the solutions agree.