Homework #9
EMA 545, Spring 2013

For the following problems, you may have the equations of motion for some of these systems in your past homework assignments or in the solutions to those that were posted online, so you may use those if you wish.

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

1.) **Problem 4.3** in Ginsberg. <u>Sketch the deformation of the system when it moves in each of the modes</u>. (Notice that you can pull out factors such as *k* and *m* so that only numbers remain in the mass and stiffness matrices. Then it is possible to check your answers using Matlab.)

2.) **Problem 4.7** as given in the text. You may use the following equations of motion:

$$mL^2\begin{bmatrix} 4/3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + mgL\begin{bmatrix} 2\beta - 3/2 & -\beta \\ -\beta & \beta - 1/2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

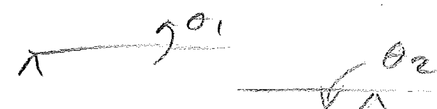
3.) **Problem 4.11** as given in the text. Hint: normal modes = mass normalized modes

4.) **Problem 4.29** as given in the text. Also, plot the motion of the automobile as a function of time. Is the response a pure-sinusoid? Why or why not? Note: The answer provided by the book is incorrect. The correct answer is:

$y_1(t) = 0.16\cos(1.5t') + 0.84\cos(2.0t')$
$y_2(t) = 0.45\cos(1.5t') - 0.45\cos(2.0t')$

– Solve by hand for w_r's and $\{\phi_r\}$'s
– Sketch deformation shape in each mode.

From HW#2 (Prob 1.30)



$$\begin{bmatrix} \frac{1}{3}m_1L^2 & 0 \\ 0 & \left(\frac{1}{12}+\frac{1}{16}\right)m_2L^2 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} \left(\frac{9}{16}k_1L^2+k_2L^2\right) & \left(\frac{9}{16}k_1L^2+\frac{1}{2}k_2L^2\right) \\ \text{sym} & \left(\frac{9}{16}k_1L^2+\frac{1}{4}k_2L^2\right)\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}0\\ \frac{FL}{2}\end{Bmatrix}$$

let $m_1 = m$, $m_2 = 2m$, $k_1 = k$, $k_2 = k/2$

$$\begin{bmatrix} \frac{1}{3}mL^2 & 0 \\ 0 & \frac{7}{24}mL^2 \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix} \left(\frac{9}{16}+\frac{1}{2}\right), & \left(\frac{9}{16}+\frac{1}{4}\right) \\ \text{sym} & \left(\frac{9}{16}+\frac{1}{8}\right)\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}\cdot kL^2 = \begin{Bmatrix}0\\ \frac{FL}{2}\end{Bmatrix}$$

$K_{11} = (51/48)kL^2 \quad K_{12} = (39/48)kL^2 \quad K_{22} = (33/48)kL^2$

let M + K be matrices containing only the numbers,

EVP: $\left([K]kL^2 - w^2\cdot mL^2[M]\right)\{\phi\} = 0$

let $\lambda = w^2\cdot\frac{mL^2}{kL^2} = w^2\frac{m}{k}$

multiply by 48 to clear fractions

$$\left(\begin{bmatrix}51 & 39\\ 39 & 33\end{bmatrix} - \lambda\begin{bmatrix}16 & 0\\ 0 & 14\end{bmatrix}\right)\{\phi\} = 0$$

$$\left(\begin{bmatrix}51-16\lambda & 39\\ 39 & 33-14\lambda\end{bmatrix}\right)\{\phi\} = 0$$

Char eq: $(51-16\lambda)(33-14\lambda) - 39^2 = 0$

$51\cdot 33 - (51\cdot 14 + 16\cdot 33) + 16\cdot 14\lambda^2 - 39^2 = 0$

polynomial solver on calculator or in Matlab:
roots([16·14, -(51·14+16·33), 51·33-39²])

$\lambda_1 = 0.1337 \qquad \lambda_2 = 5.411$

1st eq
$51-16\lambda\,\phi_1 + 39\phi_2 = 0 \longrightarrow \phi_2 = \frac{16\lambda-51}{39}\phi_1$ value in 2nd row, value in 1st row

$\{\phi_1\} = \begin{Bmatrix}1\\ -1.2529\end{Bmatrix} \quad \phi_2 = \begin{Bmatrix}1\\ 0.9122\end{Bmatrix}$

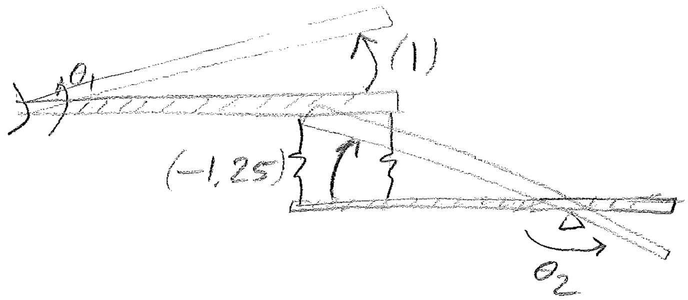
$w_1 = 0.3656\sqrt{\frac{k}{m}} \quad w_2 = 2.3262\sqrt{\frac{k}{m}}$

Matlab gives the exact same answer

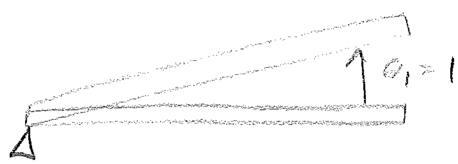
Sketch

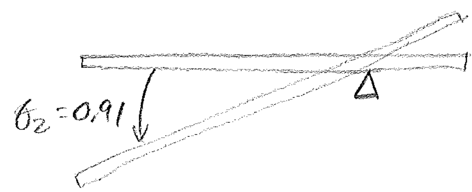
Mode 1 : $\{\theta\} = \mathrm{Re}(B\{\phi_1\}e^{i\omega_1 t})$, so

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ -1.25\end{Bmatrix}$$



Mode 2 : $\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = \begin{Bmatrix}1\\ 0.91\end{Bmatrix}$

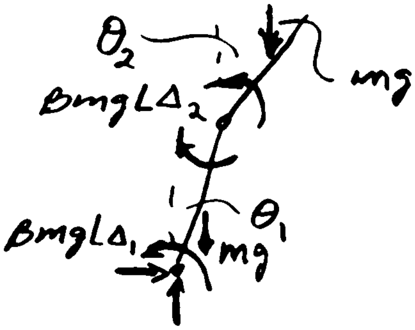


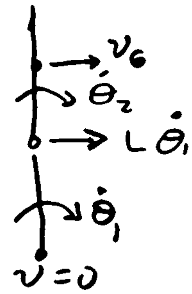


→ Second mode involves more strain in springs, and hence a higher frequency.

→ The effective mass of the two bars is similar since $m_2 = 2m$ (even though its pivot is closer to the center of mass.) $m_{11} = (1/3\, mL^2)$; $m_{22} = 0.29\, mL^2$ Because this is the case, both bars rotate by similar ammounts in each mode.

Exercise 4.7





$$\bar{v}_G = L\dot{\theta}_1 \bar{i} + \frac{L}{2}\dot{\theta}_2 \bar{i}$$

$$T = \frac{1}{2}\left(\frac{1}{3}mL^2\right)\dot{\theta}_1^2 + \frac{1}{2}m\left(L\dot{\theta}_1 + \frac{L}{2}\dot{\theta}_2\right)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\dot{\theta}_2^2 = \frac{1}{2}mL^2\left(\frac{4}{3}\dot{\theta}_1^2 + \frac{1}{3}\dot{\theta}_2^2 + \dot{\theta}_1\dot{\theta}_2\right)$$

$$M_{11} = \frac{4}{3}mL^2,\quad M_{22} = \frac{1}{3}mL^2,\quad M_{12} = M_{21} = \frac{1}{2}mL^2$$

$$V_{sp} = \frac{1}{2}(\beta mgL)\theta_1^2 + \frac{1}{2}(\beta mgL)(\theta_2-\theta_1)^2 = \frac{1}{2}\beta mgL\left[2\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2\right]$$

$$(K_{11})_{sp} = 2\beta mgL,\ (K_{22})_{sp} = \beta mgL,\ (K_{12})_{sp} = (K_{21})_{sp} = -\beta mgL$$

$$V_{gr} = mg\frac{L}{2}\cos(\theta_1) + mg\left[L\cos(\theta_1) + \frac{L}{2}\cos(\theta_2)\right]$$

$$(K_{11})_{gr} = \left.\frac{\partial^2 V_{gr}}{\partial\theta_1^2}\right|_{\theta_1=\theta_2=0} = -\frac{3}{2}mgL$$

$$(K_{22})_{gr} = \left.\frac{\partial^2 V_{gr}}{\partial\theta_2^2}\right|_{\theta_1=\theta_2=0} = -\frac{1}{2}mgL$$

$$(K_{12})_{gr} = \left.\frac{\partial^2 V}{\partial\theta_1\partial\theta_2}\right|_{\theta_1=\theta_2=0} = 0$$

$$\left[[K] - \omega^2[M]\right]\{\phi\} = \left[mgL\begin{bmatrix}(2\beta-\frac{3}{2}) & -\beta \\ -\beta & (\beta-\frac{1}{2})\end{bmatrix} - mL^2\omega^2\begin{bmatrix}\frac{4}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3}\end{bmatrix}\right]\{\phi\} = \{0\}$$

Let $\lambda = \dfrac{mL^2\omega^2}{mgL} \Rightarrow \omega^2 = \dfrac{g}{L}\lambda$, Then

$$\begin{bmatrix}((2\beta-\frac{3}{2}) - \frac{4}{3}\lambda) & -(\beta+\frac{1}{2}\lambda) \\ -(\beta+\frac{1}{2}\lambda) & ((\beta-\frac{1}{2}) - \frac{1}{3}\lambda)\end{bmatrix}\{\phi\} = \{0\}$$

When $\beta = 4$:
$$\begin{bmatrix}(\frac{13}{2} - \frac{4}{3}\lambda) & -(4+\frac{1}{2}\lambda) \\ -(4+\frac{1}{2}\lambda) & (\frac{7}{2} - \frac{1}{3}\lambda)\end{bmatrix}\begin{Bmatrix}\phi_1 \\ \phi_2\end{Bmatrix} = \begin{Bmatrix}0 \\ 0\end{Bmatrix}$$

Char eq $\dfrac{7}{36}\lambda^2 - \dfrac{65}{6}\lambda + \dfrac{27}{4} = 0 \Rightarrow \lambda = 0.6302, 55.08$

$$\begin{bmatrix}\frac{13}{2} - \frac{4}{3}\lambda_j & -(4+\frac{1}{2}\lambda_j) \\ \times & \times\end{bmatrix}\begin{Bmatrix}1 \\ \phi_{21}\end{Bmatrix} = \begin{Bmatrix}0 \\ \times\end{Bmatrix}$$

$$\phi_{2j} = \frac{13/2 - (4/3)\lambda_j}{4 + \lambda_j/2}$$

$$\lambda_1 = 0.6302 \Rightarrow \omega_1 = 0.794\left(\frac{g}{L}\right)^{1/2}, \{\phi_1\} = \begin{Bmatrix} 1 \\ 1.312 \end{Bmatrix} \quad \Longleftarrow$$

$$\lambda_2 = 55.08 \Rightarrow \omega_2 = 7.422\left(\frac{g}{L}\right)^{1/2}; \{\phi_2\} = \begin{Bmatrix} 1 \\ -2.122 \end{Bmatrix}$$

When $\beta = 2$

$$\begin{bmatrix} \frac{5}{2} - \frac{4}{3}\lambda & -\left(2 + \frac{\lambda}{2}\right) \\ -\left(2 + \frac{\lambda}{2}\right) & \frac{3}{2} - \frac{1}{3}\lambda \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Char eq: $\frac{7}{36}\lambda^2 - \frac{29}{6}\lambda - \frac{1}{4} = 0 \Rightarrow \lambda_1 = -0.053, 24.91 \quad \Longleftarrow$

Negative $\lambda_1 \Rightarrow$ imaginary $\omega_1 \Rightarrow$ divergence instability

Exercise 4.11

$$\left|[K]-\omega^2[M]\right| = \begin{vmatrix}(3(10^5)-4\omega^2) & -\omega^2 \\ -\omega^2 & (2(10^5)-3\omega^2)\end{vmatrix}$$

$$= 11\omega^4 - 17(10^5)\omega^2 + 6(10^{10}) = 0$$

$$\omega^2 = 5.455(10^4),\ 10(10^4)$$

$$\begin{bmatrix}(3(10^5)-4\omega_j^2) & -\omega_j^2 \\ x & x\end{bmatrix}\begin{Bmatrix}1 \\ \phi_{2j}\end{Bmatrix} = \begin{Bmatrix}0 \\ x\end{Bmatrix}$$

$$\phi_{2j} = \frac{3(10^5)-4\omega_j^2}{\omega_j^2}$$

$$\omega_1 = 233.5 \text{ rad/s} \Rightarrow \{\phi_1\} = \begin{Bmatrix}1 \\ 1.5\end{Bmatrix}$$ $\Leftarrow$

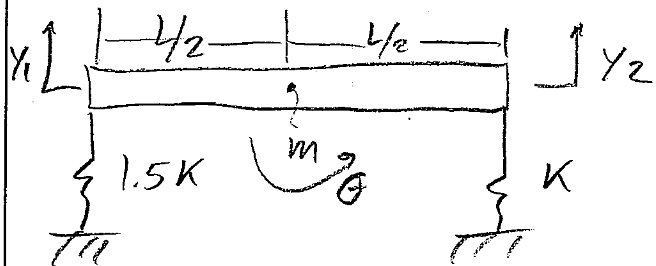
$$\omega_2 = 316.2 \text{ rad/s} \Rightarrow \{\phi_2\} = \begin{Bmatrix}1 \\ -1\end{Bmatrix}$$

$$\mu_j = \{\phi_j\}^T[M]\{\phi_j\} \Rightarrow \mu_1 = 13.75,\ \mu_2 = 5$$

$$\{\Phi_1\} = \frac{1}{\sqrt{13.75}}\begin{Bmatrix}1 \\ 1.5\end{Bmatrix} = \begin{Bmatrix}0.2697 \\ 0.4045\end{Bmatrix}$$ $\Leftarrow$

$$\{\Phi_2\} = \frac{1}{\sqrt{5}}\begin{Bmatrix}1 \\ -1\end{Bmatrix} = \begin{Bmatrix}0.4472 \\ -0.4472\end{Bmatrix}$$

4.29 Solution



$K_g = 0.4L \rightarrow I_g = mK_g^2$

$y_1(0) = \frac{mg}{K} \quad y_2(0) = 0$

find $\{y(t)\}$ for $t' = \sqrt{K/m}\, t$

$$T = \tfrac{1}{2} m\, v_g \cdot v_g + \tfrac{1}{2} I_g \dot{\theta}^2$$

Use velocity equations to relate $\dot{y}_1, \dot{y}_2$ to y_g, θ

$$\dot{y}_2 \hat{j} = \dot{y}_1 \hat{j} + L\dot{\theta}\hat{j} \longrightarrow L\dot{\theta} = \dot{y}_2 - \dot{y}_1$$

$$\dot{y}_g \hat{j} = \dot{y}_1 \hat{j} + \tfrac{L}{2}\dot{\theta} \qquad \dot{y}_g = \dot{y}_1 + \tfrac{L}{2}\left(\frac{\dot{y}_2 - \dot{y}_1}{L}\right) = \frac{\dot{y}_1 + \dot{y}_2}{2}$$

$$T = \tfrac{1}{2} m \left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2 + \tfrac{1}{2} m (0.4\cancel{L})^2 \cdot \left(\frac{\dot{y}_2 - \dot{y}_1}{\cancel{L}}\right)^2$$

$$T = \tfrac{1}{2} m \left[\tfrac{1}{4}(\dot{y}_1^2 + 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2) + 0.4^2(\dot{y}_1^2 - 2\dot{y}_1\dot{y}_2 + \dot{y}_2^2)\right]$$

$$= \tfrac{1}{2} m \left[(0.25 + 0.4^2)\dot{y}_1^2 + 2(0.25 - 0.4^2)\dot{y}_1\dot{y}_2 + (0.25 + 0.4^2)\dot{y}_2^2\right]$$

$$M = \begin{bmatrix} 0.41 & 0.09 \\ 0.09 & 0.41 \end{bmatrix} m \qquad K = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} K$$

$\left(V_{spr} = \tfrac{1}{2}K\Delta_2^2 + \tfrac{1}{2}K \cdot 1.5\Delta_1^2\right)$

$V_{grav} \Rightarrow$ no contribution to K matrix

$$[M]\ddot{y} + [K]y = 0 \qquad \frac{dy}{dt} = \frac{\partial y}{\partial t'}\frac{\partial t'}{\partial t} = y' \cdot \sqrt{\frac{K}{m}}$$

$$\rightarrow [M]\cancel{m} \cdot \frac{\cancel{K}}{\cancel{m}} y'' + \cancel{K}[K]y = 0$$ – K, m cancel with $t = t'$

Solve EVP in Matlab

$$\{y\} = [\Phi]\{\eta\} \longrightarrow \{\eta(0)\} = [\Phi]^T[M]\begin{Bmatrix}1\\0\end{Bmatrix} \cdot \frac{mg}{K}$$

$$\eta_1(t) = c_1\cos(\omega_1 t) + 0 \qquad \eta_2(t) = c_2\cos(\omega_2 t) + 0$$

$$\{y(t)\} = [\Phi]\begin{Bmatrix} c_1\cos(\omega_1 t) \\ c_2\cos(\omega_2 t)\end{Bmatrix}$$ (See Matlab results)

$$\{y(t)\} = \begin{Bmatrix} 0.1595 \\ 0.4485 \end{Bmatrix}\cos(1.504\,t') + \begin{Bmatrix} 0.8405 \\ -0.4485 \end{Bmatrix}\cos(2.036\,t')$$

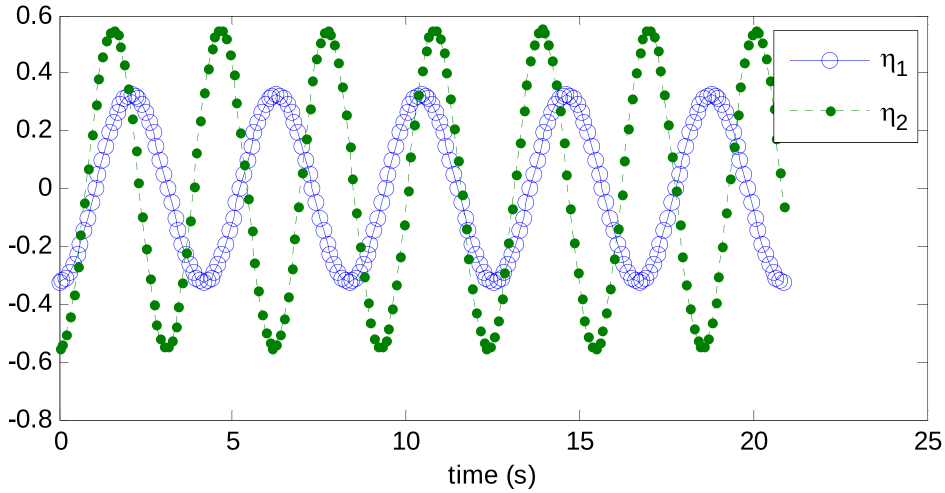
Solution: Ch. 4, Problem 4.29

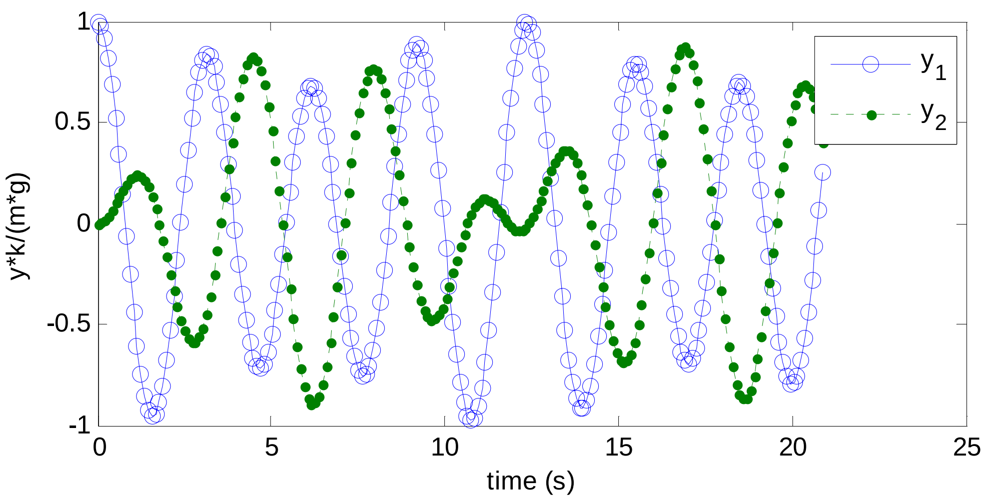
See handwritten notes for derivation.
Matlab Code:

```
% Define Mass and Stiffness matrices:
M = [0.5+0.4^2, 0.5-0.4^2;
    0.5-0.4^2, 0.5+0.4^2];
K = [1.5, 0; 0, 1];
% q_0 = [1; 1];
% q_0 = [1; 0];
q_0 = [1; 0];
q_dot_0 = [0; 0];
% Solve the eigenvalue problem:
[Phi,Lam] = eig(K,M);
% Matlab solves an eigenvalue problem [A]{x} = lam*[B]{x}, where lam is
a
% scalar if we input eig(A,B).  the matrix Lam output by Matlab is a
% diagonal matrix of eigenvalues lam, so by comparing with our
eigenvalue
% problem, we see that each nat. freq wn = sqrt(lam), or the vector of
% natural frequencies is wns = diag(Lam).^(1/2)
Phi
wns = diag(Lam).^(1/2)
% Check that the eigenvectors are mass normalized.  If this is not an
% identity, then we need to normalize
Phi.'*M*Phi
% Note that Matlab sometimes gives the first eigenvector as the
negative of
% what we found.  Either is a valid mode for the system.

% Now the initial conditions give:
eta_0 = Phi.'*M*q_0
eta_dot_0 = Phi.'*M*q_dot_0

% and the constants in the solutions eta(t) = a1*cos(wns(1)*t)+etc...
a1 = eta_0(1); a2 = eta_dot_0(1)/wns(1);
b1 = eta_0(2); b2 = eta_dot_0(2)/wns(2);

% Define a time vector with 5 cycles of the lowest frequency:
ts = [0:1:200]/200*5*(2*pi/wns(1));
eta_t = [a1*cos(wns(1)*ts)+a2*sin(wns(1)*ts);
    b1*cos(wns(2)*ts)+b2*sin(wns(2)*ts)];
% each column of the matrix above gives {eta(t)} at some instant t.
Since
% the eta values are in columns, we obtain {q} by multiplying by [Phi]
q_t = Phi*eta_t;

figure(1);
subplot(2,1,1)
plot(ts,eta_t(1,:),'-o',ts,eta_t(2,:),':.'); legend('\eta_1','\eta_2');
xlabel('time (s)');
subplot(2,1,2)
plot(ts,q_t(1,:),'-o',ts,q_t(2,:),':.'); legend('y_1','y_2');
xlabel('time (s)');  ylabel('y*k/(m*g)');
```

Results:

```
>> M
M =
          0.41          0.09
          0.09          0.41
>> K
K =
           1.5             0
             0             1
Phi =
       -0.4905       -1.5238
       -1.3789       0.81302
wns =
        1.5041
        2.0357
ans =
             1  5.5511e-017
 -5.5511e-017             1
eta_0 =
      -0.32521
      -0.55158
eta_dot_0 =
     0
     0
```





Response plotted over 5 cycles. Notice that the initial conditions are satisfied. Both modes are excited and oscillate at different frequencies. The superposition of both modes causes the response in y1, y2 coordinates to look quite complicated.