HW 8

EMA 545
Mechanical Vibrations

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1 problem 1

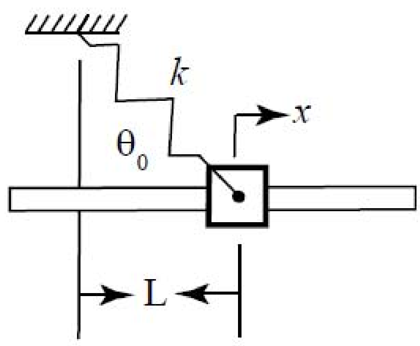
Problem 1: (40 points)

a.) Find the nonlinear equation of motion for the system pictured below. The block has mass m and the guide can be approximated as frictionless. In the position shown the spring is unstretched and the angle between the spring and guide bar is θ_0.

b.) Linearize your equation of motion for small deflections from the position shown (i.e. using a Taylor series expansion on $k(x)$ about x=0). Use a computer to plot k(x) versus the linear approximation for L=1 m, k=1000 N/m and θ_0 = 45 degrees for x ranging from -1 m to +1 m.

c.) Find the equations of motion for the system using the stiff spring approximation and assuming small displacements from an equilibrium position defined by L=1 m, k=1000 N/m and θ_0 = 45 degrees. Compare your result with your linearized result from part (b).

d.) Using m=1, find the response of the nonlinear system (in part a) using ode45 and plot the displacement of the mass over a few cycles when it is released from rest at x(0)=0.1 and also at x(0)=0.5 meters. Overlay both curves on the same set of axes. How does the period of the response compare with the linearized natural frequency in each case? In what other way(s) does the nonlinearity manifest itself in the response of the system when x(0)=0.5?



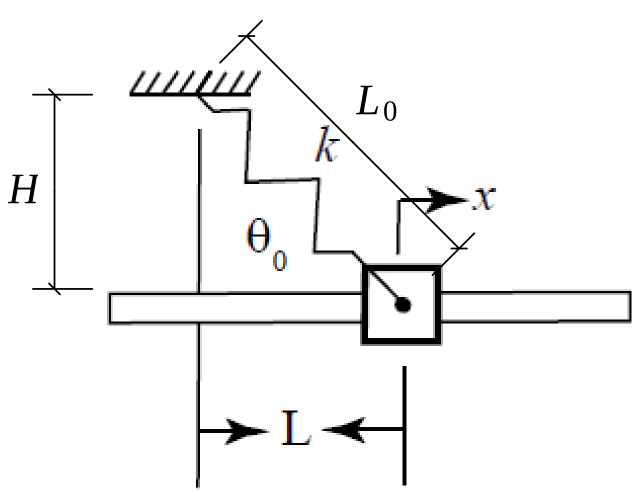
1.1 part(a)

Let initial length of the spring (un stretched length) be L_0 and when the mass m has moved to the right by an amount x then let the current length be L_{cur}.

Therefore the stretch in the spring is

$$\Delta = L_{cur} - L_0$$

Let the height of the bar by H, where $\tan\theta_0 = \frac{H}{L}$ or $H = L\tan\theta_0$



Hence from the above diagram we see that $L_0 = \sqrt{H^2 + L^2}$ and $L_{cur} = \sqrt{H^2 + (L+x)^2}$, therefore

$$\Delta = \sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}$$
$$\Delta^2 = \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2$$

Now we can derive the equation of motion using energy methods.

Let T be the current kinetic energy in the system, and let V be the current potential energy. This system is one degree of freedom, since we only need one generalized coordinate to determine the position of the mass m. This coordinate is x.

$$\begin{aligned} T &= \frac{1}{2} m \dot{x}^2 \\ V &= \frac{1}{2} k \Delta^2 \\ &= \frac{1}{2} k \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2 \end{aligned}$$

Hence the Lagrangian Φ is

$$\begin{aligned} \Phi &= T - V = T - V \\ &= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2 \end{aligned}$$

Now the equation of motion for coordinate x is (using the standard Lagrangian form)

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} = Q_x$$

But Q_x, then generalized force, is zero since there is no external force and no damping. Now we just need to evaluate each part of the above expression to obtain the EOM.

$$\begin{aligned} \frac{\partial \Phi}{\partial \dot{x}} &= m \dot{x} \\ \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) &= m \ddot{x} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right)^2\right) \\ &= -k \left(\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}\right) \frac{1}{2} \left(H^2 + (L+x)^2\right)^{\frac{-1}{2}} 2(L+x) \\ &= -k \left(\frac{\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}}{\sqrt{H^2 + (L+x)^2}}\right)(L+x) \end{aligned}$$

Hence EOM becomes

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} &= 0 \\ m \ddot{x} + k \left(\frac{\sqrt{H^2 + (L+x)^2} - \sqrt{H^2 + L^2}}{\sqrt{H^2 + (L+x)^2}}\right)(L+x) &= 0 \end{aligned}$$

1.2 part(b)

For small x we need to expand $f(x) = k\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)$ around $x=0$ in Taylor series and let higher powers of x go to zero.

$$f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + HOT.$$

$$\begin{aligned}
f(0) &= k\left(\frac{\sqrt{H^2+(L+0)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+0)^2}}\right)(L+0)\\
&= k\left(\frac{\sqrt{H^2+L^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+L^2}}\right)L\\
&= 0
\end{aligned}$$

and now for $f'(0)$

$$\begin{aligned}
f'(0) &= \frac{d}{dx}f(x)_{x=0}\\
&= k\frac{d}{dx}\left[\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)\right]_{x=0}\\
&= k\left((L+x)\frac{d}{dx}\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)+\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)\frac{d}{dx}(L+x)\right)_{x=0}\\
&= k\left((L+x)\left(\sqrt{H^2+L^2}\,\frac{L+x}{\left(H^2+L^2+2Lx+x^2\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)\right)_{x=0}
\end{aligned}$$

Now we evaluate it at $x=0$

$$\begin{aligned}
f'(0) &= k\left((L+0)\left(\sqrt{H^2+L^2}\,\frac{L+0}{\left(H^2+L^2+2L0+0\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+(L+0)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+0)^2}}\right)\right)\\
&= k\left(L\left(\sqrt{H^2+L^2}\,\frac{L}{\left(H^2+L^2\right)^{\frac{3}{2}}}\right)+\left(\frac{\sqrt{H^2+L^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+L^2}}\right)\right)\\
&= k\left(L^2\left(\frac{\left(H^2+L^2\right)^{\frac{1}{2}}}{\left(H^2+L^2\right)^{\frac{3}{2}}}\right)\right)\\
&= k\left(\frac{L^2}{\left(H^2+L^2\right)}\right)
\end{aligned}$$

Therefore, EOM of motion becomes (notice we ignored higher order terms, which contains x^2 in them)

$$m\ddot{x} + \left(f(0) + xf'(0)\right) = 0$$

Hence the linearized EOM is

$$\boxed{m\ddot{x} + k\frac{L^2}{\left(H^2+L^2\right)}x = 0}$$

Or in terms of θ_0 the EOM can be written as

$$m\ddot{x} + k\frac{L^2}{\left((L\tan\theta_0)^2 + L^2\right)}x = 0$$

$$m\ddot{x} + k\frac{1}{1+\tan^2\theta_0}x = 0$$

This is the linearized EOM around $x = 0$. Using numerical values given in the problem $L = 1, m = 1, k = 1000N/m, \theta_0 = \frac{\pi}{4}$, it becomes

$$\ddot{x} + 1000\frac{1}{1+\left(\tan\frac{\pi}{4}\right)^2}x = 0$$

$$\ddot{x} + 500x = 0$$

Therefore the $\boxed{\text{linearized stiffness is } 500x}$ while the nonlinearized stiffness is

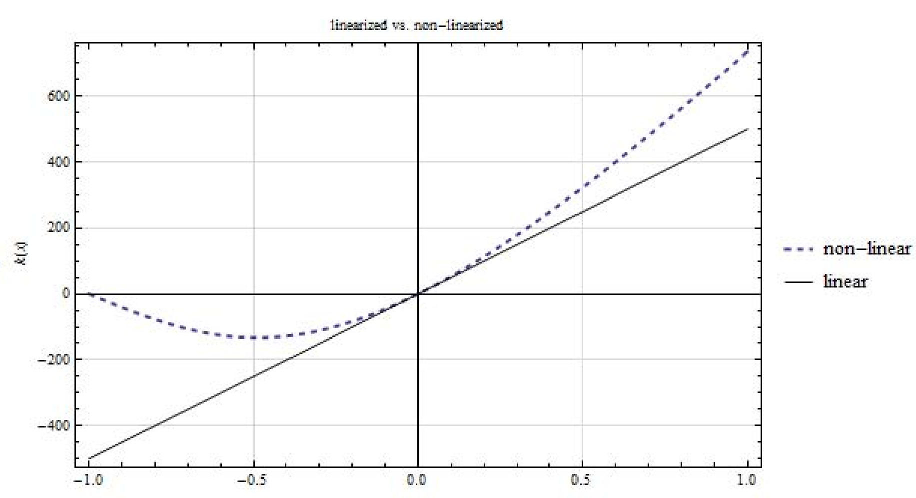
$$\left[k\left(\frac{\sqrt{H^2+(L+x)^2}-\sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x)\right]_{L=1,\theta=45^0}$$

$$= 1000\left(\frac{\sqrt{\left(\tan\frac{\pi}{4}\right)^2+(1+x)^2}-\sqrt{\left(\tan\frac{\pi}{4}\right)^2+1}}{\sqrt{\left(\tan\frac{\pi}{4}\right)^2+(1+x)^2}}\right)(1+x)$$

$$= 1000\left(\frac{\sqrt{(x+1.0)^2+1.0}-1.4142}{\sqrt{(x+1.0)^2+1.0}}\right)(1+x)$$

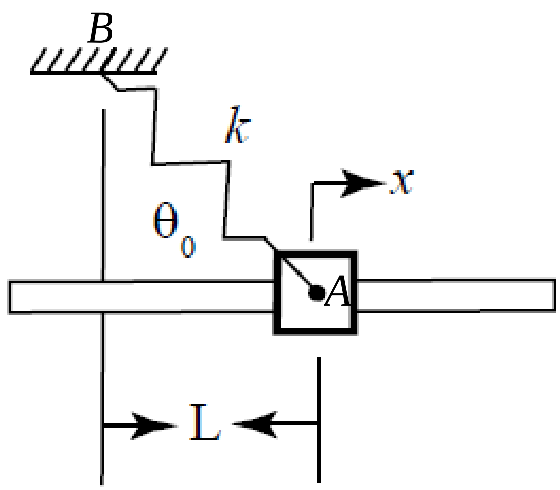
Here is a plot of linearized vs. non-linearized stiffness for $x = -1 \cdots 1$

```
In[13]:= nonlinear[x_] := k (L + x) ((Sqrt[h^2 + (L + x)^2] - Sqrt[h^2 + L^2]) / Sqrt[h^2 + (L + x)^2]);
linear[x_] := x 500;
values = {L → 1, θ → 45 Degree, h → Tan[45 Degree], k → 1000};
Plot[{nonlinear[x] /. values, linear[x] /. values}, {x, -1, 1},
 PlotStyle → {{Dashed, Thick}, Black}, Frame → True, PlotLegends → {"non-linear", "linear"},
 FrameLabel → {{k[x], None}, {x, "linearized vs. non-linearized"}}, ImageSize → 500,
 GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[16]=


1.3 part(c)

The spring extension Δ is first found by assuming there is a point A at $x = 0$ and point B where the spring is attached to the ceiling. Hence



$$\begin{aligned}
\dot{\Delta} &= (\dot{u}_A - \dot{u}_B) e_{A/B} \\
&= \left(\dot{x}\hat{\imath} - 0\hat{\jmath}\right) \cdot \left(\cos\theta_0 \hat{\imath} - \sin\theta_0 \hat{\jmath}\right) \\
&= \dot{x}\cos\theta_0
\end{aligned}$$

Therefore

$$\Delta = x\cos\theta_0$$

Now we repeat the same calculations but using $\Delta = x\cos\theta_0$ for the spring extension.

$$\begin{aligned}
T &= \frac{1}{2} m\dot{x}^2 \\
V &= \frac{1}{2} k\Delta^2 \\
&= \frac{1}{2} k(x\cos\theta_0)^2
\end{aligned}$$

Hence the Lagrangian Φ is

$$\begin{aligned}
\Phi &= T - V \\
&= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} k(x\cos\theta_0)^2
\end{aligned}$$

Now the equation of motion for coordinate x is (using the standard Lagrangian form)

$$\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) - \frac{\partial\Phi}{\partial x} = 0$$

It is equal to zero above, since there is no generalized force associated with coordinate x. Now we just need to evaluate each part of the above expression to obtain the EOM.

$$\begin{aligned}
\frac{\partial\Phi}{\partial\dot{x}} &= m\dot{x} \\
\frac{d}{dt}\left(\frac{\partial\Phi}{\partial\dot{x}}\right) &= m\ddot{x}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial\Phi}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} k(x\cos\theta_0)^2\right) \\
&= -k(x\cos\theta_0)\cos\theta_0 \\
&= -kx\cos^2\theta_0
\end{aligned}$$

Hence EOM becomes

$$\frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{x}}\right) - \frac{\partial \Phi}{\partial x} = 0$$
$$m\ddot{x} + kx\cos^2\theta_0 = 0$$

But $\cos\theta_0 = \frac{L}{\sqrt{H^2+L^2}}$ hence

$$\boxed{m\ddot{x} + kx\frac{L^2}{H^2+L^2} = 0}$$

This is the same as the EOM for the linearized case found in part(c)

1.4 part(d)

Now we need to solve numerically the nonlinear EOM found in part(a) which is

$$m\ddot{x} + k\left(\frac{\sqrt{H^2+(L+x)^2} - \sqrt{H^2+L^2}}{\sqrt{H^2+(L+x)^2}}\right)(L+x) = 0$$

using $m = 1, k = 1000, L = 1, \theta_0 = 45^0$. For IC we use $x(0) = 0.1, x'(0) = 0$ for first case, and for second case using $x(0) = 0.5, x'(0) = 0$. This is a plot showing both responses on same diagram

```
eq = m x''[t] + k (L + x[t]) ( (Sqrt[h^2 + (L + x[t])^2] - Sqrt[h^2 + L^2]) / Sqrt[h^2 + (L + x[t])^2] ) = 0;
ic = {{x[0] = 0.1, x'[0] = 0}, {x[0] = 0.5, x'[0] = 0}};
values = {L → 1, h → Tan[45 Degree], m → 1, k → 1000};
sol = First@NDSolve[{Evaluate[eq /. values], #}, x[t], {t, 0, 1}] & /@ ic;
Plot[Evaluate[x[t] /. sol], {t, 0, 1}, PlotStyle → {{Dashed, Thick}, Black}, Frame → True,
 PlotLegends → {"x[0]=0.1", "x[0]=0.5"},
 FrameLabel → {{x[t], None}, {"t sec", "numerical found nonlinear solution for 2 initial conditions"}},
 ImageSize → 600, GridLines → Automatic, GridLinesStyle → LightGray]
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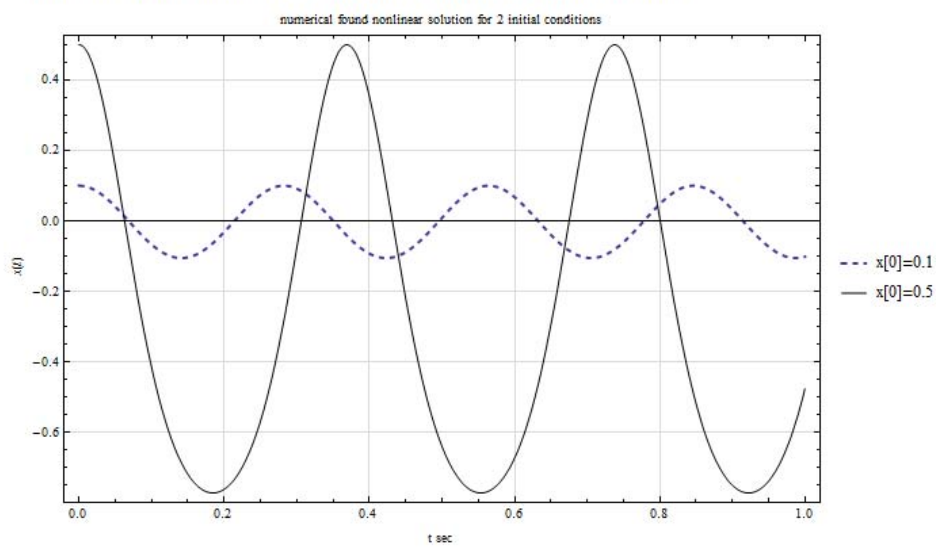


The period for the response for case of IC given by x(0) = 0.5 is seen to be about 0.375 seconds and for the case $x(0) = 0.1$ it is 0.275 sec.

The linearized EOM is $\ddot{x} + 500x = 0$ and hence $\omega_n^2 = 500$ or $\omega_n = \sqrt{500} = 22.361$ rad/sec, hence $T = \frac{2\pi}{\omega_n} = \frac{2\pi}{22.361} = \boxed{0.281 \text{ sec}}$.

We notice this agrees well with the period of the response of the nonlinear equation for only the case $x = 0.1$.This is because $x = 0.1$ is very close to $x = 0$ the point at which the linearization happened. Therefore, the linearized EOM gave an answer of 0.281 sec that is very close the more exact value of 0.275 seconds. But when the initial conditions changed to $x(0) = 0.5$, then T found from linearized EOM does not agree with the exact value of 0.375 seconds.

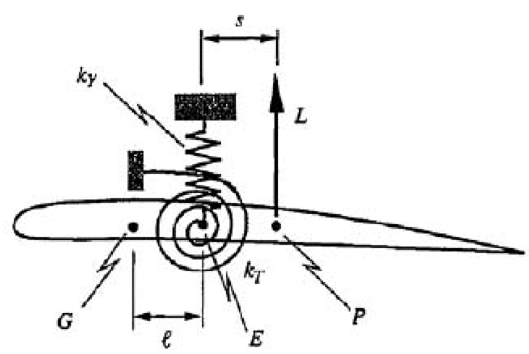
This is because $x = 0.5$ is far away from the point $x = 0$ where the linearized was done. Hence the linearized EOM can be used for only initial conditions that are close to the point where the linearization was done.

Additionally, the nonlinearity manifests itself in the response of the system by noticing that the frequency of the free vibration response has actually changed depending on initial conditions. In a linear system, only the phase and amplitude of the free vibration response will change as initial conditions is changed, while the natural frequency of vibrations does not change.

2 problem 2

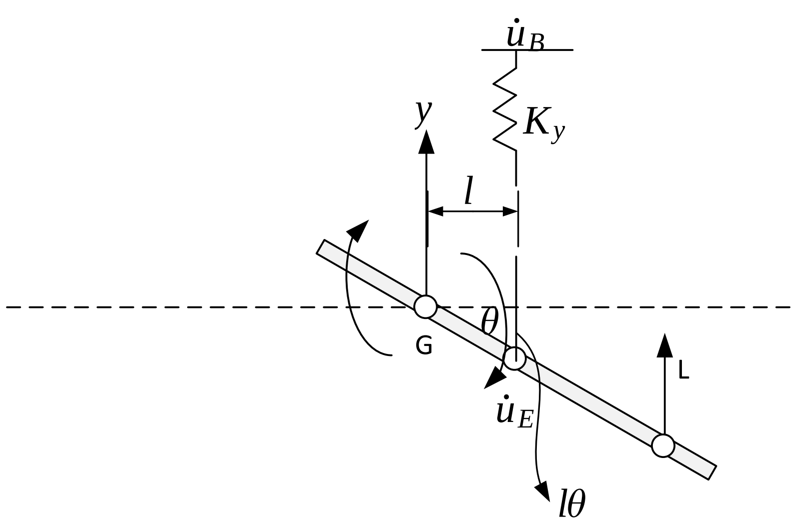
Problem 2: Exercise 1.27 from Ginsberg.

A standard model for a wing has a translational spring k_y and a torsional spring k_T representing the elastic rigidity. Point E represents the elastic center because static application of a vertical force at that point results in upward displacement without an associated rotation. The design of the wing is such that horizontal movement of point E is negligible. The lift force L acts at point P, which is called the center of pressure. The lift force may be treated as known. When the wing is in its static equilibrium position, points G, E and P form a horizontal line. Point G is the center of mass, and the radius of gyration of the wing about that point is r_G. Denote the mass of the wing m. Derive the equations of motion for the wing, assuming small displacements (and small rotational displacements). Put the equations in matrix form and check the units and sign of each term in your EOM. (Hint: use the displacement of the center of gravity and the rotation of the wing as generalized coordinates.)



EXERCISE 1.27

Use y and θ as generalized coordinates as shown in this diagram in the positive direction



Using Lagrangian method, we start by finding the kinetic energy of the system, then the potential energy.

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(mr_G^2\right)\dot{\theta}^2$$

For the potential energy, there will be potential energy due to k_y spring extension and due to k_T spring angle of rotation in system. From the diagram above, we see that, for small angle θ

$$V = \frac{1}{2}k_y\Delta^2 + \frac{1}{2}k_T\theta^2$$

To find Δ we use the stiff spring approximation. Let the point the spring is attached at the top be B, then

$$\begin{aligned}\dot{\Delta} &= (\dot{u}_E - \dot{u}_B) e_{E/B} \\ &= \left(\left(l\dot{\theta} - \dot{y}\right)\hat{\jmath} - 0\right) \cdot \left(-\hat{\jmath}\right) \\ &= \left(l\dot{\theta} - \dot{y}\right)\hat{\jmath} \cdot \left(-\hat{\jmath}\right) \\ &= \left(\dot{y} - l\dot{\theta}\right)\end{aligned}$$

Hence

$$\Delta = y - l\theta$$

Therefore, the potential energy now can be found to be

$$V = \frac{1}{2}k_y\left(y - l\theta\right)^2 + \frac{1}{2}k_T\theta^2$$

Therefore, the Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V \\ &= \frac{1}{2}m\dot{y}^2 + \frac{1}{2}\left(mr_G^2\right)\dot{\theta}^2 - \frac{1}{2}k_y\left(y - l\theta\right)^2 - \frac{1}{2}k_T\theta^2\end{aligned}$$

We now find the equations for each coordinate. For y

$$\begin{aligned}\frac{\partial \Phi}{\partial \dot{y}} &= m\dot{y} \\ \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{y}} &= m\ddot{y} \\ \frac{\partial \Phi}{\partial y} &= -k_y\left(y - l\theta\right)\end{aligned}$$

Hence EOM is

$$\begin{aligned}\frac{d}{dt}\frac{\partial \Phi}{\partial \dot{y}} - \frac{\partial \Phi}{\partial y} &= Q_y \\ m\ddot{y} + k_y\left(y - l\theta\right) &= Q_y\end{aligned}$$

We just need to find Q_y the generalized force in the y direction. Using virtual work, we make small virtual displacement δy in positive y direction while fixing all other generalized coordinates from moving (in this case θ) and then find out the work done by external forces. In this case, there is only one external force which is L. Hence

$$\delta W = L\delta y$$

Therefore $Q_y = L$ since that is the force that is multiplied by δy. Hence EOM for y is now found

$$\boxed{m\ddot{y} + k_y\left(y - l\theta\right) = L}$$

verification: As L increases, then we see that y'' gets larger. This makes sense since y is upwards acceleration, so wing accelerates in the same direction.

Now we find EOM for θ

$$\begin{aligned}\frac{\partial \Phi}{\partial \dot{\theta}} &= mr_G^2\dot{\theta} \\ \frac{d}{dt}\frac{\partial \Phi}{\partial \dot{\theta}} &= mr_G^2\ddot{\theta} \\ \frac{\partial \Phi}{\partial \theta} &= k_y l\left(y - l\theta\right) - k_T\theta\end{aligned}$$

Therefore the EOM is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} - \frac{\partial\Phi}{\partial\theta} &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l\left(y - l\theta\right) + k_T\theta &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l y + k_y l^2\theta + k_T\theta &= Q_\theta \\
mr_G^2\ddot{\theta} - k_y l y + \left(k_T + k_y l^2\right)\theta &= Q_\theta
\end{aligned}
$$

We just need to find Q_θ the generalized force in the θ direction. Using virtual work, we make small virtual displacement $\delta\theta$ in positive θ direction (i.e. anticlock wise) while fixing all other generalized coordinates from moving (in this case y) and then find out the work done by external forces. In this case, there is only one external force which is L. When we make $\delta\theta$ rotation in the positive θ direction, the displacement where the force L acts is $(l+s)\delta\theta$ for small angle. But this displacement is in the downward direction, hence it is negative, since we are using y as positive upwards. Hence

$$
\delta W = -L(l+s)\delta\theta
$$

Therefore $Q_\theta = -L(l+s)$ since that is the force that is multiplied by $\delta\theta$. Hence EOM for θ is now found

$$
mr_G^2\ddot{\theta} - k_y l y + \left(k_T + k_y l^2\right)\theta = -L(l+s)
$$

Verification: As L gets larger, then $\ddot{\theta}$ gets negative (since L has negative sign). This makes sense, since as L gets larger, the rotation as shown in the positive direction will change sign and the wing will now swing the opposite direction (i.e. anticlockwise).

Now we can make the matrix of EOM

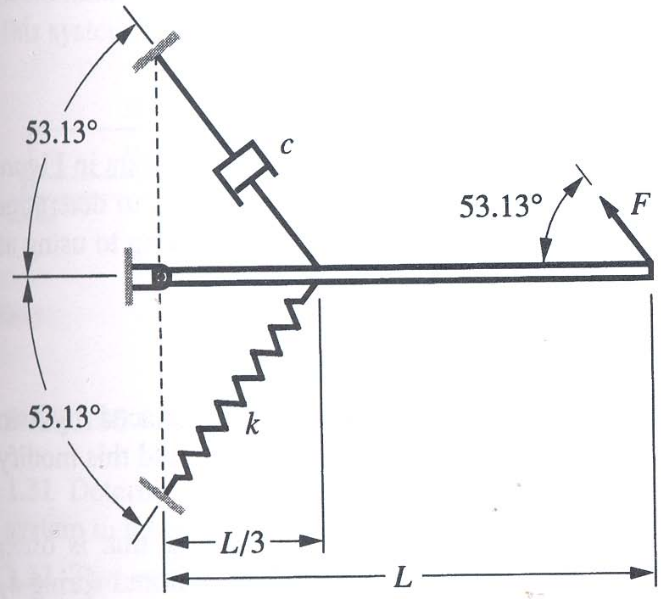
$$
MX'' + kX = Q
$$

$$
\begin{pmatrix} m & 0 \\ 0 & mr_G^2 \end{pmatrix}\begin{pmatrix} \ddot{y} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} k_y & -lk_y \\ -lk_y & k_T + k_y l^2 \end{pmatrix}\begin{pmatrix} y \\ \theta \end{pmatrix} = \begin{pmatrix} L \\ -L(l+s) \end{pmatrix}
$$

Notice that for [k] the matrix is symmetric as expected, and also positive on the diagonal as expected. The mass matrix [m] is symmetric and positive definite as well.

3 problem 3

1.16 The bar executes small rotations in the vertical plane relative to the static equilibrium position depicted in the sketch. Let the rotation of the bar be the generalized coordinate. Determine the damping coefficient C_{11}.



EXERCISE 1.16

Problem 3: Use the power balance method and the stiff spring approximation to find the equation of motion of the system pictured in Problem 1.16.

Let θ be the small angle of rotation that the rod rotates by in the anti clockwise direction. Let the point the spring is fixed be B and the moving point where the spring is attached to the rod be A.To find spring extension Δ we use the stiff spring approximation. Let the angle $\alpha = 53.13^0$, hence

$$\begin{aligned}
\dot{\Delta} &= (\dot{u}_A - \dot{u}_B) \cdot \mathbf{e}_{A/B} \\
&= \left(\left(\frac{L}{3} \dot{\theta} \right) \hat{\jmath} - 0 \right) \cdot \left(\cos \alpha \hat{\imath} + \sin \alpha \hat{\jmath} \right) \\
&= \frac{L}{3} \dot{\theta} \sin \alpha
\end{aligned}$$

Hence

$$\Delta = \frac{L}{3} \theta \sin \alpha$$

Using Lagrangian method, we start by finding the kinetic energy of the system, then the potential energy. θ is the only generalized coordinate. Assume bar has mass m and hence $I = \frac{mL^2}{3}$

$$T = \frac{1}{2} I \dot{\theta}^2$$

For the potential energy, there will be potential energy due to k spring extension. From the diagram above, we see that

$$V = \frac{1}{2} k \left(\frac{L}{3} \theta \sin \alpha \right)^2$$

Therefore, the Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V\\ &= \frac{1}{2}I\dot{\theta}^2 - \frac{1}{2}k\frac{L^2}{9}\theta^2\sin^2\alpha\end{aligned}$$

Now we find EOM for θ

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{\theta}} &= I\dot{\theta}\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} &= I\ddot{\theta}\\ \frac{\partial\Phi}{\partial\theta} &= -\frac{kL^2\sin^2\alpha}{9}\theta\end{aligned}$$

Therefore the EOM is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}} - \frac{\partial\Phi}{\partial\theta} &= Q_\theta\\ I\ddot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= Q_\theta\end{aligned}$$

We now need to find the generalized force due to virtual $\delta\theta$ rotation using the virtual work method. There are 2 external forces, the damping force which will have negative sign since it takes energy away from the system, and the external force F which will add energy hence will have positive sign.

We start by making $\delta\theta$ and then find the work done by these 2 forces.

Work done by F is $FL\delta\theta$ since the displacement is $L\delta\theta$ for small angle. Now the work done by damping is $\left(c\frac{L}{3}\dot{\theta}\right)\frac{L}{3}\delta\theta$ hence total work is

$$\begin{aligned}\delta W &= FL\delta\theta - \left(c\frac{L}{3}\dot{\theta}\right)\frac{L}{3}\delta\theta\\ &= \left(FL - c\frac{L^2}{9}\dot{\theta}\right)\delta\theta\end{aligned}$$

Notice that work due to damping was added with negative sign since damping removes energy from the system.

Hence $Q_\theta = \left(FL - c\frac{L^2}{9}\dot{\theta}\right)$ therefore the EOM is

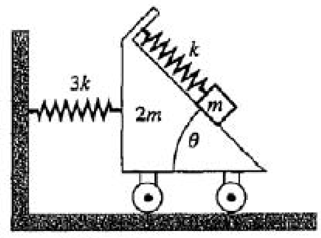
$$\begin{aligned}I\ddot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= FL - c\frac{L^2}{9}\dot{\theta}\\ I\ddot{\theta} + c\frac{L^2}{9}\dot{\theta} + \frac{kL^2\sin^2\alpha}{9}\theta &= FL\end{aligned}$$

Hence the damping coefficient is $c\frac{L^2}{9}$.

4 problem 4

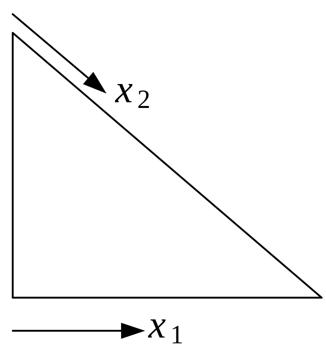
Problem 4: Exercise 1.33 from Ginsberg: (be very careful to write a correct expression for the acceleration of the small block.) Check the unit and sign of each term in your EOM.

1.33 Determine the equations of motion governing a pair of generalized coordinates that locate the position of the cart and the sliding block. Friction is negligible.



EXERCISE 1.33

Let x_1 and x_2 be the generalized coordinates as shown in this diagram

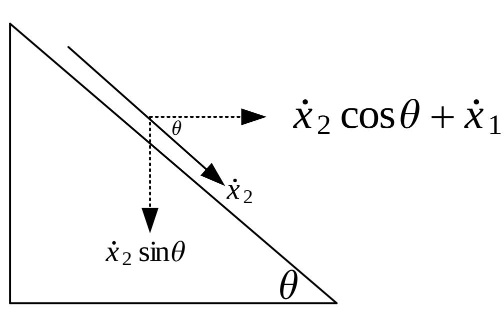


Let mass of cart be m_1 and mass of small sliding block be m_2 (at the end, they will be replaced by values given). Let k for spring attached to wall be k_1 and k for spring for small block be k_2.We start by finding the kinetic energy of the system

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2v^2$$

where v is the velocity of the block. To find this v it is easier to resolve components on the x and y direction. Therefore we find that

$$\vec{v} = \dot{x}_2 \sin\theta j + (\dot{x}_2 \cos\theta + \dot{x}_1)i$$



Hence

$$\begin{aligned}
|\vec{v}|^2 &= (\dot{x}_2 \sin\theta)^2 + (\dot{x}_2 \cos\theta + \dot{x}_1)^2 \\
&= \left(\dot{x}_2^2 \sin^2\theta\right) + \left(\dot{x}_2^2 \cos^2\theta + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta\right) \\
&= \dot{x}_2^2\left(\sin^2\theta + \cos^2\theta\right) + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta \\
&= \dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta
\end{aligned}$$

Therefore

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1 \cos\theta\right)$$

Now we find the potential energy.

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 - m_2g(x_2\sin\theta)$$

There are no external forces, hence generalized forces Q_{x_1}, Q_{x_2} are zero. The Lagrangian Φ is

$$\begin{aligned}\Phi &= T - V\\ &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{x}_1^2 + 2\dot{x}_2\dot{x}_1\cos\theta\right) - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 + m_2g(x_2\sin\theta)\end{aligned}$$

Now we find EOM for x_1 is

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_1\dot{x}_1 + m_2\dot{x}_1 + m_2\dot{x}_2\cos\theta\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_1\ddot{x}_1 + m_2\ddot{x}_1 + m_2\ddot{x}_2\cos\theta\\ &= (m_1+m_2)\ddot{x}_1 + m_2\ddot{x}_2\cos\theta\\ \frac{\partial\Phi}{\partial x_1} &= -k_1x_1\end{aligned}$$

Therefore the EOM for x_1 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} - \frac{\partial\Phi}{\partial x_1} &= 0\\ (m_1+m_2)\ddot{x}_1 + m_2\ddot{x}_2\cos\theta + k_1x_1 &= 0\end{aligned}$$

Now we replace the actual values for $m_1 = 2m, m_2 = m, k_1 = 3k$ hence

$$\boxed{3m\ddot{x}_1 + m\ddot{x}_2\cos\theta + 3kx_1 = 0}$$

Now we find EOM for x_2 is

$$\begin{aligned}\frac{\partial\Phi}{\partial\dot{x}_2} &= m_2(\dot{x}_2 + \dot{x}_1\cos\theta)\\ \frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_1} &= m_2(\ddot{x}_2 + \ddot{x}_1\cos\theta)\\ &= m_2\cos\theta\ddot{x}_1 + m_2\ddot{x}_2\\ \frac{\partial\Phi}{\partial x_2} &= -k_2x_2 + m_2g\sin\theta\end{aligned}$$

Therefore the EOM for x_2 is

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{x}_2} - \frac{\partial\Phi}{\partial x_2} &= 0\\ m_2\cos\theta\ddot{x}_1 + m_2\ddot{x}_2 + k_2x_2 - m_2g\sin\theta &= 0\end{aligned}$$

Now we replace the actual values for $m_1 = 2m, m_2 = m, k_2 = k$ hence

$$m\cos\theta\ddot{x}_1 + m\ddot{x}_2 + kx_2 = m_2g\sin\theta$$

Now we can make the matrix of EOM

$$MX'' + kX = Q$$

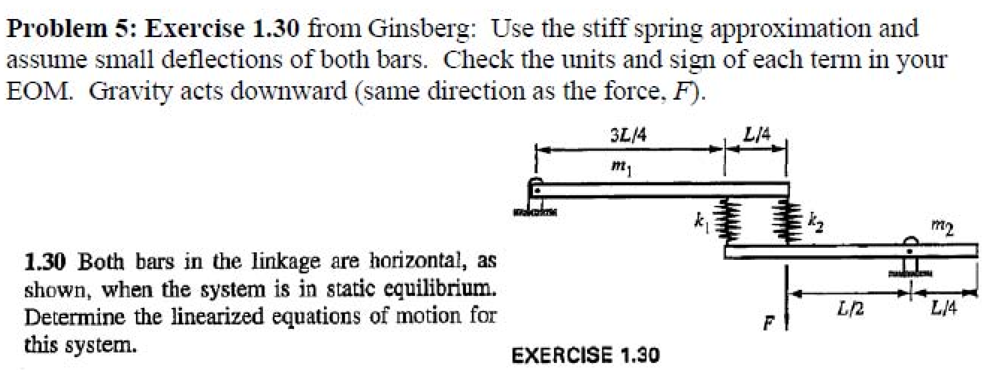
$$\begin{pmatrix} 3m & m\cos\theta\\ m\cos\theta & m\end{pmatrix}\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\end{pmatrix} + \begin{pmatrix}3k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ m_2g\sin\theta\end{pmatrix}$$

Hence

$$\boxed{m\begin{pmatrix} 3 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + k\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ m_2 g \sin\theta \end{pmatrix}}$$

Notice that there zeros now off diagonal in the $[K]$ matrix, which means the springs are not coupled. (which is expected, as motion of one is not affected by the other). But mass matrix $[m]$ has non-zeros off the diagonal. So the masses are coupled. i.e. EOM is coupled. This means we can't solve on EOM on its own and both have to be solved simultaneously.

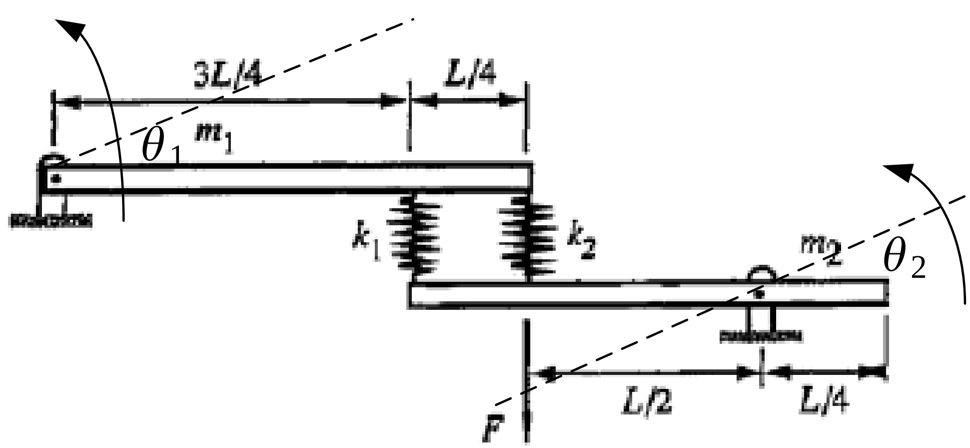
5 problem 5


Problem 5: Exercise 1.30 from Ginsberg: Use the stiff spring approximation and assume small deflections of both bars. Check the units and sign of each term in your EOM. Gravity acts downward (same direction as the force, F).

1.30 Both bars in the linkage are horizontal, as shown, when the system is in static equilibrium. Determine the linearized equations of motion for this system.

EXERCISE 1.30

There are 2 degrees of freedom, θ_1 and θ_2 as shown in this diagram, using anticlock wise rotation as positive



The Lagrangian $\Phi = T - V$ where

$$T = \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

Where $I_1 = \frac{m_1 L^2}{3}$ and $I_2 = \frac{m_2 L^2}{12} + m_2\left(\frac{L}{4}\right)^2$ (using parallel axis theorem). Hence $I_2 = \frac{m_2 L^2}{12} + m_2\frac{L^2}{16} = \frac{7}{48}L^2 m_2$

Now we find the potential energy, assuming springs remain straight (stiff spring assumption) and assuming small angles

$$V = \frac{1}{2}k_1\left(\overbrace{\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2}^{\Delta_1}\right)^2 + \frac{1}{2}k_2\left(\overbrace{L\theta_1 + \frac{L}{2}\theta_2}^{\Delta_2}\right)^2 + m_1 g\frac{L}{2}\theta_1 - m_2 g\frac{L}{4}\theta_2$$

Hence

$$
\begin{aligned}
\Phi &= T - V \\
&= \left(\frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2\right) - \left(\frac{1}{2}k_1\left(\frac{3L}{4}\theta_1 + \frac{3L}{4}\theta_2\right)^2 + \frac{1}{2}k_2\left(L\theta_1 + \frac{L}{2}\theta_2\right)^2 + m_1 g\frac{L}{2}\theta_1 - m_2 g\frac{L}{4}\theta_2\right)
\end{aligned}
$$

Now we find EOM for θ_1

$$
\begin{aligned}
\frac{\partial\Phi}{\partial\dot{\theta}_1} &= I\dot{\theta}_1 \\
\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_1} &= I\ddot{\theta}_1 \\
\frac{\partial\Phi}{\partial\theta_1} &= -k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) - k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)(L) - m_1 g\frac{L}{2} \\
&= -\frac{3L}{4}k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right) - k_2 L\left(\frac{L}{2}\theta_2 + L\theta_1\right) - m_1 g\frac{L}{2}
\end{aligned}
$$

Therefore the EOM for θ_1 is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_1} - \frac{\partial\Phi}{\partial\theta_1} &= Q_{\theta_1} \\
I_1\ddot{\theta}_1 + \frac{3L}{4}k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right) + k_2 L\left(\frac{L}{2}\theta_2 + L\theta_1\right) + m_1 g\frac{L}{2} &= 0
\end{aligned}
$$

The generalized force is zero, since there is no direct external force acting on top rod. Hence EOM for θ_1 is from above

$$
\frac{m_1 L^2}{3}\ddot{\theta}_1 + \theta_1\left(k_1\left(\frac{3L}{4}\right)^2 + k_2 L^2\right) + \theta_2\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2}\right) = -m_1 g\frac{L}{2}
$$

Now we find EOM for θ_2

$$
\begin{aligned}
\frac{\partial\Phi}{\partial\dot{\theta}_2} &= I\dot{\theta}_2 \\
\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_2} &= I\ddot{\theta}_2 \\
\frac{\partial\Phi}{\partial\theta_2} &= -k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) - k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)\left(\frac{L}{2}\right) + m_2 g\frac{L}{4}
\end{aligned}
$$

Therefore the EOM for θ_2 is

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial\Phi}{\partial\dot{\theta}_2} - \frac{\partial\Phi}{\partial\theta_2} &= Q_{\theta_2} \\
I_2\ddot{\theta}_2 + k_1\left(\frac{3L}{4}\theta_2 + \frac{3L}{4}\theta_1\right)\left(\frac{3L}{4}\right) + k_2\left(\frac{L}{2}\theta_2 + L\theta_1\right)\left(\frac{L}{2}\right) - m_2 g\frac{L}{4} &= Q_{\theta_1}
\end{aligned}
$$

Now $Q_{\theta_2}$is found by virtual work. Making a virtual displacement $\delta\theta_2$ while fixing θ_1 and finding the work done by all external forces.

$$
\delta W = F\frac{L}{2}\delta\theta_2
$$

Hence $Q_{\theta_2} = F\frac{L}{2}$ with positive sign since it add energy to the system. Hence EOM for θ_2 is

$$
\frac{7}{48}L^2 m_2\ddot{\theta}_2 + \theta_1\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2}\right) + \theta_2\left(k_1\left(\frac{3L}{4}\right)^2 + k_2\left(\frac{L}{2}\right)^2\right) = m_2 g\frac{L}{4} + F\frac{L}{2}
$$

Now we can make the matrix of EOM

$$MX'' + kX = Q$$

$$\begin{pmatrix} \frac{m_1L^2}{3} & 0 \\ 0 & \frac{7}{48}L^2 m_2 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} k_1\frac{9L^2}{16} + k_2L^2 & k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2} \\ k_1\left(\frac{3L}{4}\right)^2 + k_2\frac{L^2}{2} & k_1\left(\frac{3L}{4}\right)^2 + k_2\left(\frac{L}{2}\right)^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -m_1 g\frac{L}{2} \\ m_2 g\frac{L}{4} + F\frac{L}{2} \end{pmatrix}$$

The matrix $[k]$ is coupled but the mass matrix $[m]$ is not.

6 problem 6

The inertia and stiffness matrices for a system are $[M] = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ kg, $[K] = \begin{bmatrix} 200 & 200 \\ 200 & 800 \end{bmatrix}$ N/m. determine the corresponding natural frequencies and modes of free vibration.

$$\left[[k] - \omega^2[M]\right]\{\Phi\} = \{0\}$$

Solving for eigenvalues

$$\det\left(\begin{bmatrix} 200 & 200 \\ 200 & 800 \end{bmatrix} - \omega^2\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 200 - 4\omega^2 & 200 \\ 200 & 800 - 2\omega^2 \end{bmatrix} = 0$$

$$\left(200 - 4\omega^2\right)\left(800 - 2\omega^2\right) - 200^2 = 0$$

$$8\omega^4 - 3600\omega^2 + 120000 = 0$$

Hence, taking the positive square root only we find

$$\omega_1 = 20.341 \text{ rad/sec}$$
$$\omega_2 = 6.0211 \text{ rad/sec}$$

When $\boxed{\omega = \omega_1}$

$$\begin{bmatrix} 200 - 4\omega_1^2 & 200 \\ 200 & 800 - 2\omega_1^2 \end{bmatrix} \begin{Bmatrix} \Phi_{11} \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{11} be the arbitrary value 1 hence

$$\begin{bmatrix} 200 - 4\omega_1^2 & 200 \\ \times & \times \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$200 - 4\omega_1^2 + 200\Phi_{21} = 0$$

$$\Phi_{21} = \frac{-200 + 4\omega_1^2}{200} = \frac{-200 + 4(20.341)^2}{200} = 7.2751$$

Hence the first mode associated with $\omega = 20.341$ rad/sec is

$$\begin{Bmatrix} 1 \\ 7.2751 \end{Bmatrix}$$

When $\boxed{\omega = \omega_2}$

$$\begin{bmatrix} 200 - 4\omega_2^2 & 200 \\ 200 & 800 - 2\omega_2^2 \end{bmatrix} \begin{Bmatrix} \Phi_{12} \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let Φ_{12} be the arbitrary value 1 hence

$$\begin{bmatrix} 200 - 4\omega_2^2 & 200 \\ \times & \times \end{bmatrix} \begin{Bmatrix} 1 \\ \Phi_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \times \end{Bmatrix}$$

$$200 - 4\omega_2^2 + 200\Phi_{22} = 0$$

$$\Phi_{22} = \frac{-200 + 4\omega_2^2}{200} = \frac{-200 + 4(6.0211)^2}{200} = -0.27493$$

Hence the first mode associated with $\omega = 6.0211$ rad/sec is

$$\begin{Bmatrix} 1 \\ -0.27493 \end{Bmatrix}$$

Summary

ω_n (rad/sec)	mode shape
6.0211	$\begin{Bmatrix} 1 \\ -0.27493 \end{Bmatrix}$
20.341	$\begin{Bmatrix} 1 \\ 7.2751 \end{Bmatrix}$

Verification using Matlab:

```
EDU>> M=[4 0;0 2]; K=[200 200;200 800];
EDU>> [phi,omega]=eig(K,M);
EDU>> sqrt(omega)

    6.0211         0
         0   20.3407
EDU>> phi(:,1)/abs(phi(1,1))

   -1.0000
    0.2749

EDU>> phi(:,2)/abs(phi(1,2))

    1.0000
    7.2749
```

Which matches the result derived. One mode shape has both displacement in phase, and the other mode shape shows the displacements to be out of phase.