# Appendix, HW7, EMA 545, spring 2013

## Analytical (superposition) solution for HW7 EMA 545

```
T = 0.01;
wn = 2.0 * Pi * 100;
m = 0.5;
```

In[108]:=
```
wd[ζ_] := wn √(1 - ζ^2) ;
β = 2000 / T^2;
tp = t - T;
f1[t_, ζ_] :=
  β T 1/(m wn^3) (wn t - 2 ζ + Exp[-ζ wn t] (2 ζ Cos[wd[ζ] t] - (1 - 2 ζ^2) wn/wd[ζ] Sin[wd[ζ] t]));
f2[t_, ζ_] := β 1/(m wn^4) ((wn t)^2 - 4 ζ wn t - 2 (1 - 4 ζ^2) +
    Exp[-ζ wn t] (2 (1 - 4 ζ^2) Cos[wd[ζ] t] + (6 ζ - 8 ζ^3) wn/wd[ζ] Sin[wd[ζ] t]));
f3[t_, ζ_] := 2 β T 1/(m wn^3) (wn tp - 2 ζ + Exp[-ζ wn tp]
      (2 ζ Cos[wd[ζ] tp] - (1 - 2 ζ^2) wn/wd[ζ] Sin[wd[ζ] tp]));
f4[t_, ζ_] := β 1/(m wn^4) ((wn tp)^2 - 4 ζ wn tp - 2 (1 - 4 ζ^2) +
    Exp[-ζ wn tp] (2 (1 - 4 ζ^2) Cos[wd[ζ] tp] + (6 ζ - 8 ζ^3) wn/wd[ζ] Sin[wd[ζ] tp]));
f5[t_, ζ_] := 2 β T^2 1/(m wn^2) (1 - Exp[-ζ wn tp] (Cos[wd[ζ] tp] + ζ wn/wd[ζ] Sin[wd[ζ] tp]));
f6[t_, ζ_] :=
  β T 1/(m wn^3) (wn tp - 2 ζ + Exp[-ζ wn tp] (2 ζ Cos[wd[ζ] tp] - (1 - 2 ζ^2) wn/wd[ζ] Sin[wd[ζ] tp]));
freeResponse[t_, q0_, v0_, ζ_] := Exp[-ζ wn t] (q0 Cos[wd[ζ] t] + (v0 + ζ wn q0)/wd[ζ] Sin[wd[ζ] t]);
```

In[118]:=
```
impulseResponse[t_, ζ_] := f1[t, ζ] UnitStep[t] - f2[t, ζ] UnitStep[t] - f3[t, ζ] UnitStep[tp] +
   f4[t, ζ] UnitStep[tp] + f5[t, ζ] UnitStep[tp] - f6[t, ζ] UnitStep[tp];
```

## case $\zeta$=0.2

In[121]:=
```
ζ = 0.2;
p1 = Plot[impulseResponse[t, ζ] (UnitStep[t] - UnitStep[tp]), {t, 0, T}];
```

## Evaluate IC at end of impulse to use for free vibration response

In[123]:=
```
q0 = (f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ]) /. t → T
```

Out[123]= 0.00120186

In[124]:=
```
v0 = D[f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ], t] /. t → T
```

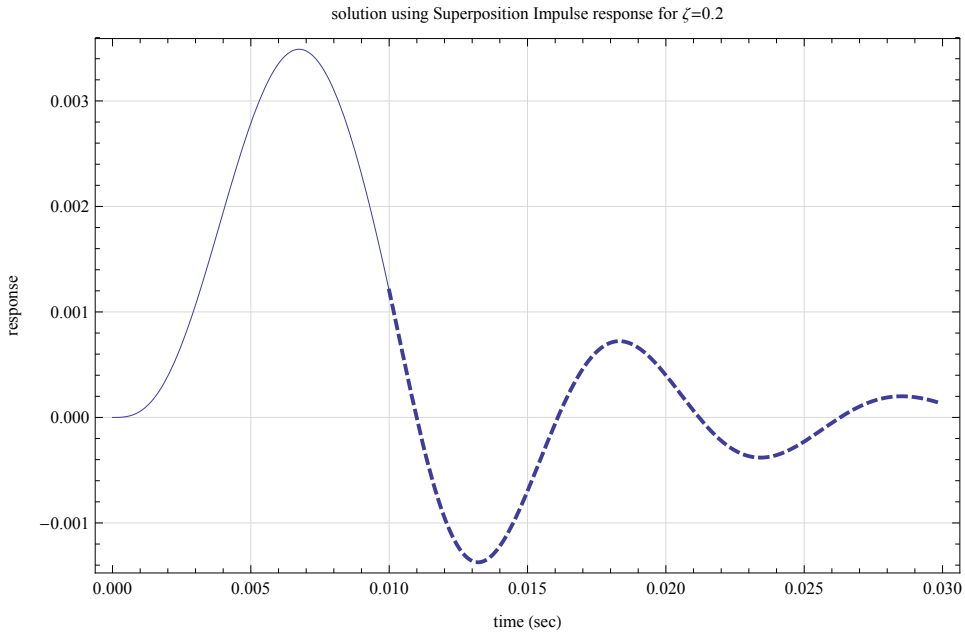
Out[124]=
```
-1.21014
```

In[125]:=
```
p2 = Plot[freeResponse[tp, q0, v0, ζ] UnitStep[tp],
    {t, T, 3 * T}, Exclusions → None, PlotStyle → {Dashed, Thick}];
```

- **Plot the complete solution by combining the above 2 plots**

In[126]:=
```
Show[p1, p2, PlotRange → All, Frame → True, FrameLabel → {{"response", None},
    {"time (sec)", Row[{"solution using Superposition Impulse response for ζ=", ζ}]}},
 GridLines → Automatic, GridLinesStyle → LightGray, Axes → False, ImageSize → 500]
```

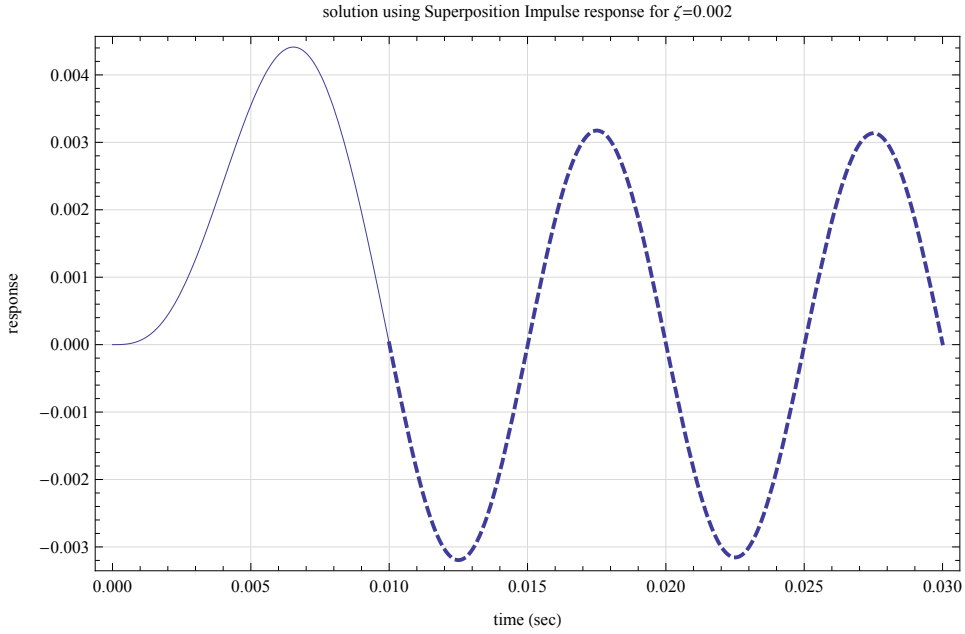
Out[126]=



## ■ case $\zeta$=0.002

In[137]:=
```
ζ = 0.002;
p1 = Plot[impulseResponse[t, ζ] (UnitStep[t] - UnitStep[tp]), {t, 0, T}];
q0 = (f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ]) /. t → T;
v0 = D[f1[t, ζ] - f2[t, ζ] - f3[t, ζ] + f4[t, ζ] + f5[t, ζ] - f6[t, ζ], t] /. t → T;
p2 = Plot[freeResponse[tp, q0, v0, ζ] UnitStep[tp],
    {t, T, 3 * T}, Exclusions → None, PlotStyle → {Dashed, Thick}];
Show[p1, p2, PlotRange → All, Frame → True, FrameLabel → {{"response", None},
    {"time (sec)", Row[{"solution using Superposition Impulse response for ζ=", ζ}]}},
 GridLines → Automatic, GridLinesStyle → LightGray, Axes → False, ImageSize → 500]
```

Out[142]=




## ■ Verify the solutions using Numerical DE solver

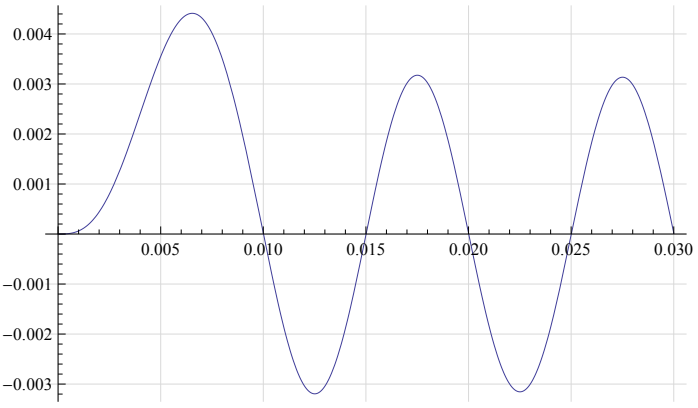
In[143]:=
```
ζ = 0.002;
eq = y''[t] + 2 ζ wn y'[t] + wn^2 y[t] == f[t] / m
```

Out[144]=
$$394\,784.\, y[t] + 2.51327\, y'[t] + y''[t] == 4.\times 10^{7}\, (0.01 - t)\, t\, (-\text{UnitStep}[-0.01 + t] + \text{UnitStep}[t])$$

In[145]:=
```
sol = First@NDSolve[{eq, y'[0] == 0, y[0] == 0}, y[t], {t, 0, 3 T}];
```

In[147]:=
```
Plot[y[t] /. sol, {t, 0, 3 T}, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[147]=



The above shows that the analytical solution using superposition is correct.