

**Homework #7**  
EMA 545, Spring 2013

**Problem 1: (20 pts)** Use FFT techniques to find the response of your vibration isolation system to the turbulent flight profile in “FlightAccel.mat” on the course website. (To do this, you will have to assume that the aircraft experiences this exact same flight profile over and over again.)



- a.) Provide at least one plot comparing the response of the aircraft to the (hopefully improved) response on the vibration isolator.
- b.) Report on the values of mass, stiffness and damping that you used and how they would be realized in practice (e.g. if you use a beam as a leaf spring, what would its dimensions be?).

**Problem 2: (20 pts)** Do **Problem 3.60** as given in the text. Comment on how you selected adequate values for  $N$  (number of samples in the time history) and the length of the time window (in seconds).

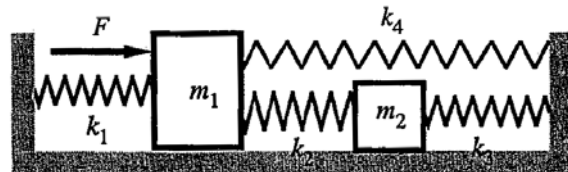
**Problem 3: Exercise 1.11** from Ginsberg. (For the proof described in (b), set  $m_2=0$  and see what your equation of motion reduces to.)

**1.11** When the system in the sketch is at its static equilibrium position, there is no axial force in each spring.

example, series or parallel—according to the equivalent spring stiffness.

(a) Derive equations of motion for the horizontal displacements  $x_1$  and  $x_2$  measured from the equilibrium position.

(b) Prove that if  $m_2 = 0$ , the equation of motion for  $x_1$  is the same as that obtained by replacing the four springs by a single equivalent spring. Identify the way in which the springs are connected—for



**EXERCISE 1.11**

**Problem 3.60.**

A SDOF system is subjected to a pulse excitation of the form:

$$Q(t) = \begin{cases} 2000t(T-t)/T^2 & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

We are told that  $m = 0.5$  kg and that the natural frequency is 100Hz; i.e., the undamped period of vibration is  $1/100 = 0.01$  sec. Since the pulse-duration is said to be equal to an undamped period,  $T = 0.01$  sec. We are to find the response of the system for the case of (a)  $\zeta = 0.2$  and (b)  $\zeta = 0.002$ .

(a) For  $\zeta = 0.2$ , the time-constant of the system is  $1/\zeta\omega_n = 1/(0.2*2*\pi*100) = 0.007958$  sec. Four time constants would be about 0.032, so choosing a maximum time of  $T_{max} = 0.05$  sec will ensure that we allow enough time for the transient response to die down to an acceptably small value. The next parameter that we need to choose is  $N$ . Let's choose  $N = 256$  and then check that our Nyquist critical frequency is high enough relative to the natural frequency of the system and the bandwidth of the input spectrum. Figure 1 shows the FFT of the input spectrum, the FFT of the output spectrum, and the system transfer function as a function of frequency ratio for  $T_{max} = 0.05$  sec and  $N = 256$ . Each curve is scaled so that its maximum value is unity. Although difficult to see whether there are enough points in the vicinity of the resonant peak, it is clearly the case that our Nyquist frequency is high enough. Figure 2 shows the displacement response of the system for  $T_{max} = 0.05$  sec and using 2 different values of  $N$ . Below  $N = 64$ , the errors in the response become much more noticeable. Note that the response is very small at  $t = 0.05$  sec, indicating that "wraparound errors" are negligible.

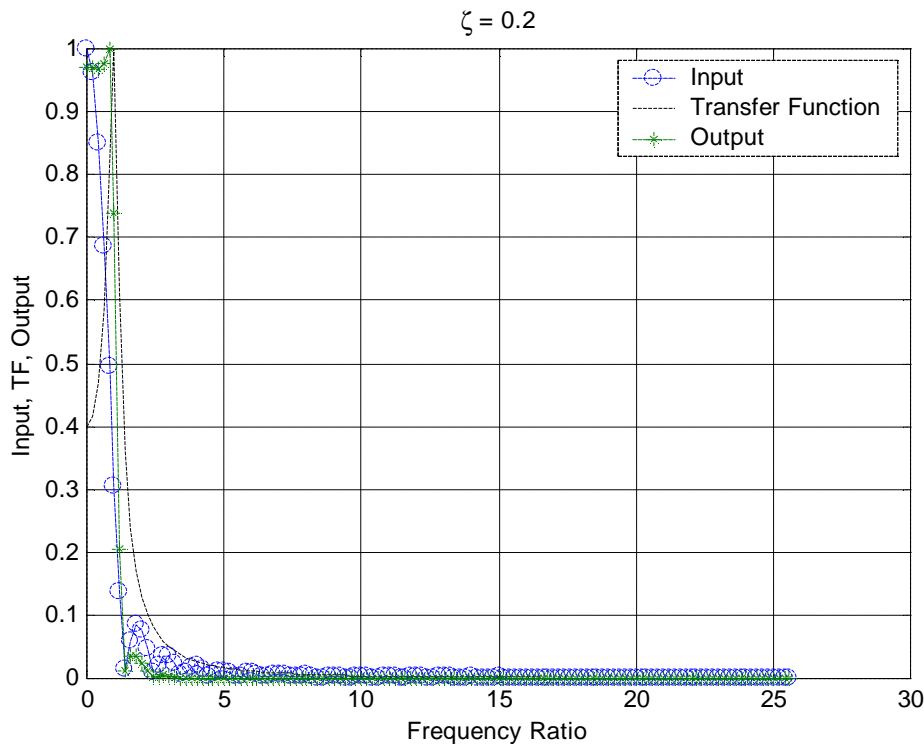


Figure 1.

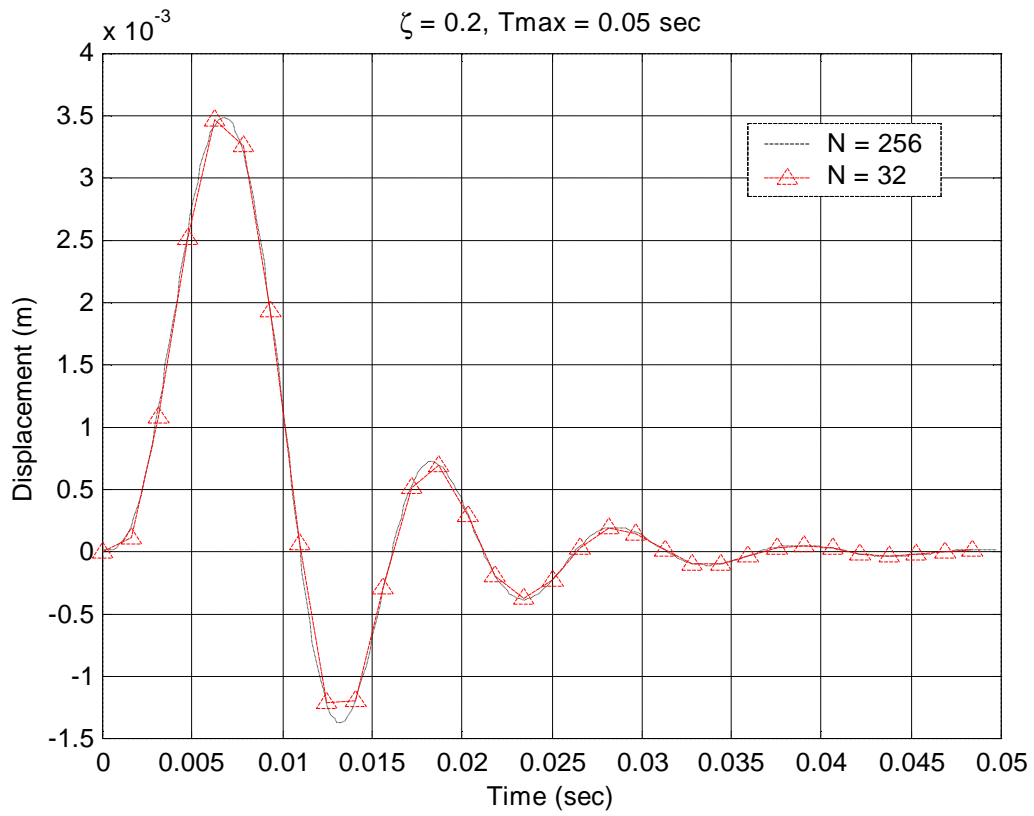


Figure 2.

Figure 3 shows the effect of varying  $T_{max}$ , keeping  $N = 256$ . It is seen that as  $T_{max}$  is reduced, the solution begins to degrade. In particular, examining the curve for  $T_{max} = 0.02 \text{ sec}$ , it is seen that the response no longer appears to begin with zero initial conditions. Thus, using a  $T_{max}$  which is twice the duration of the pulse is inadequate in this case to avoid warparound errors.

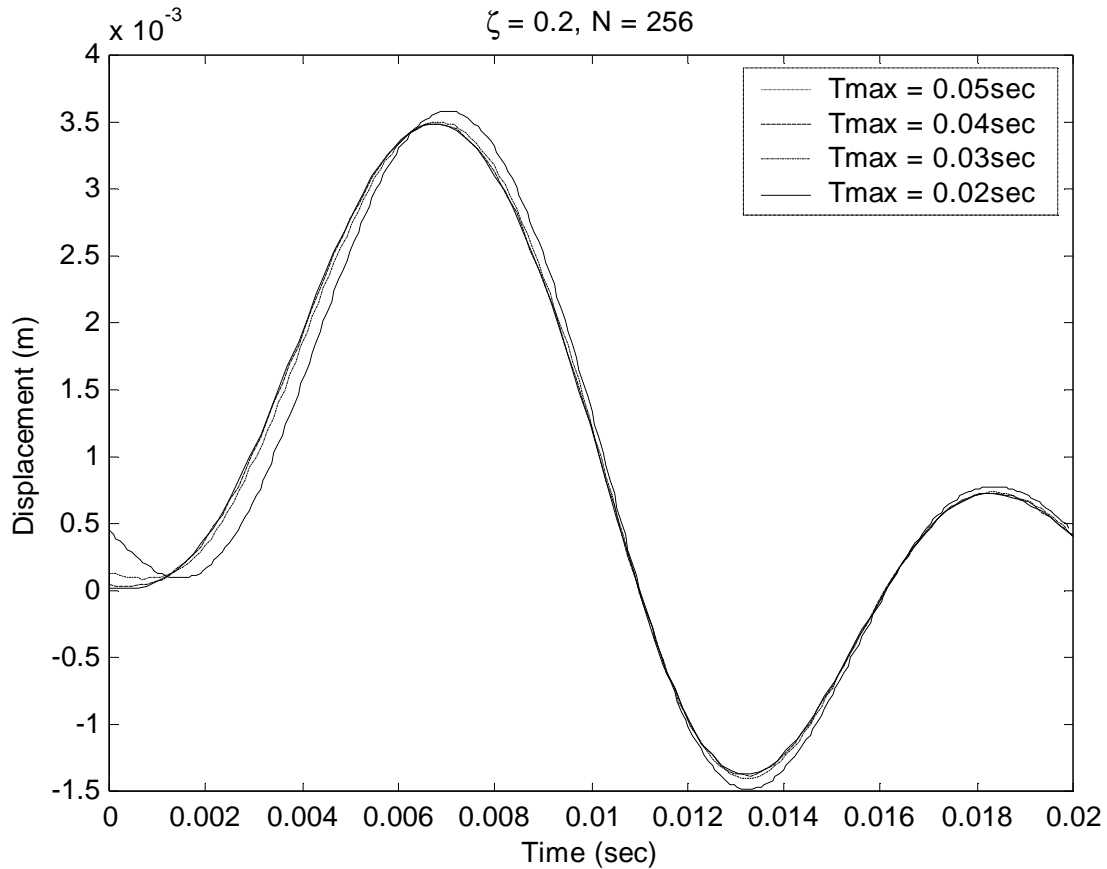


Figure 3.

(b) For  $\zeta = 0.002$ , the time-constant of the system is  $1/\zeta\omega_n = 1/(0.2*2*\pi*100) = 0.7958$  sec. Four time constants would be about 3.2, so choosing a maximum time of  $T_{max} = 5$  sec should ensure that we allow enough time for the transient response to die down to an acceptably small value. As before, we will choose a value for  $N$  (4096) and then check that our Nyquist critical frequency is high enough relative to the natural frequency of the system and the bandwidth of the input spectrum. Figure 4 shows the FFT of the input spectrum, the FFT of the output spectrum, and the system transfer function as a function of frequency ratio for  $T_{max} = 5$  sec and  $N = 4096$ . As before, each curve is scaled so that its maximum value is unity. We observe that there appear to be enough points in the vicinity of the resonant peak and that the Nyquist frequency appears to be high enough. Figure 5 shows the displacement response of the system for  $T_{max} = 5$  sec and  $N = 4096$ . The result was checked using `ode45` and found to be very close to the values produced using the FFT analysis. Figure 6 shows the first 0.01 seconds of response from Figure 5 on a larger scale. The fact that the response begins with nearly zero displacement and velocity shows that wraparound error has been avoided.

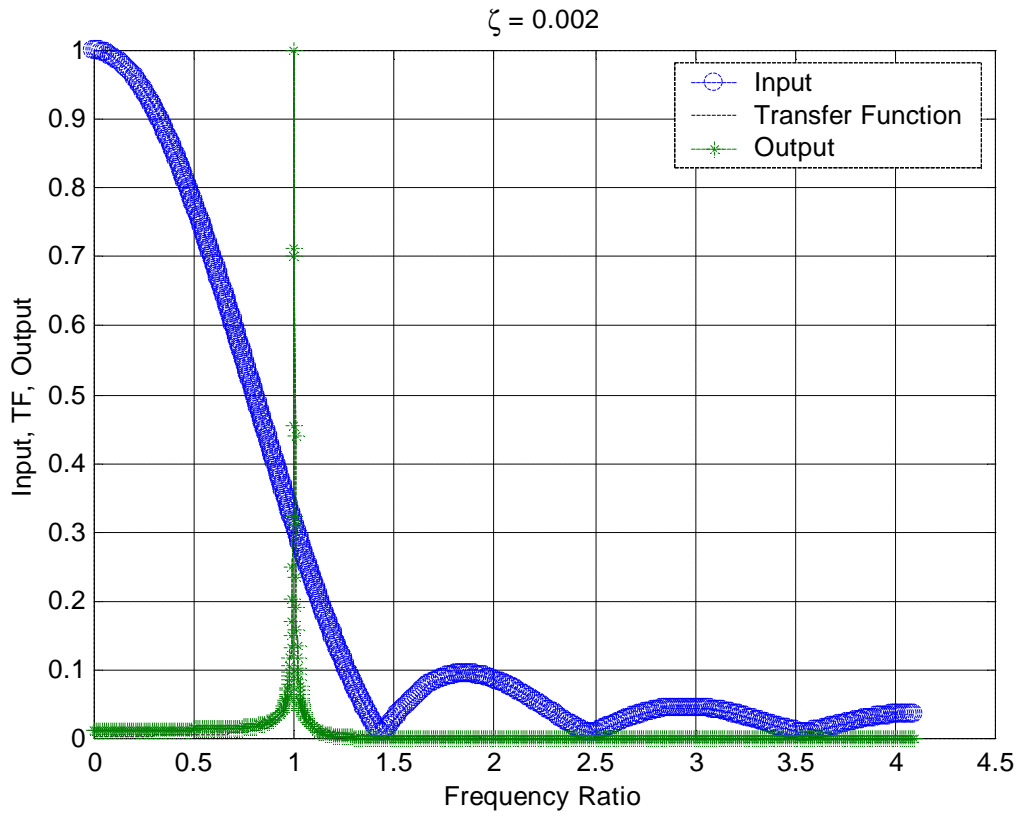


Figure 4.

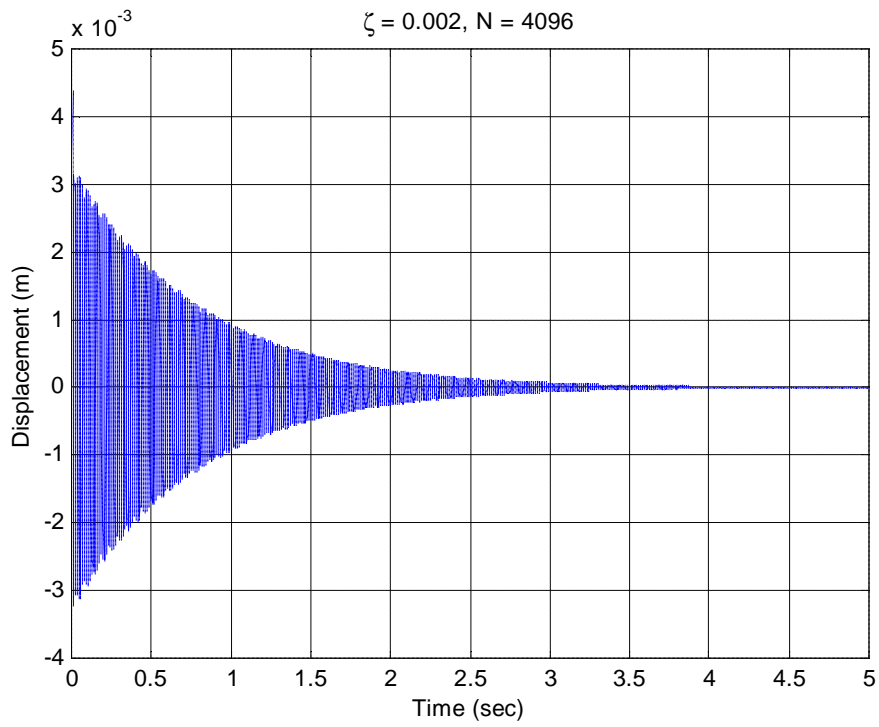


Figure 5. Tmax = 5 sec, N = 4096 points.

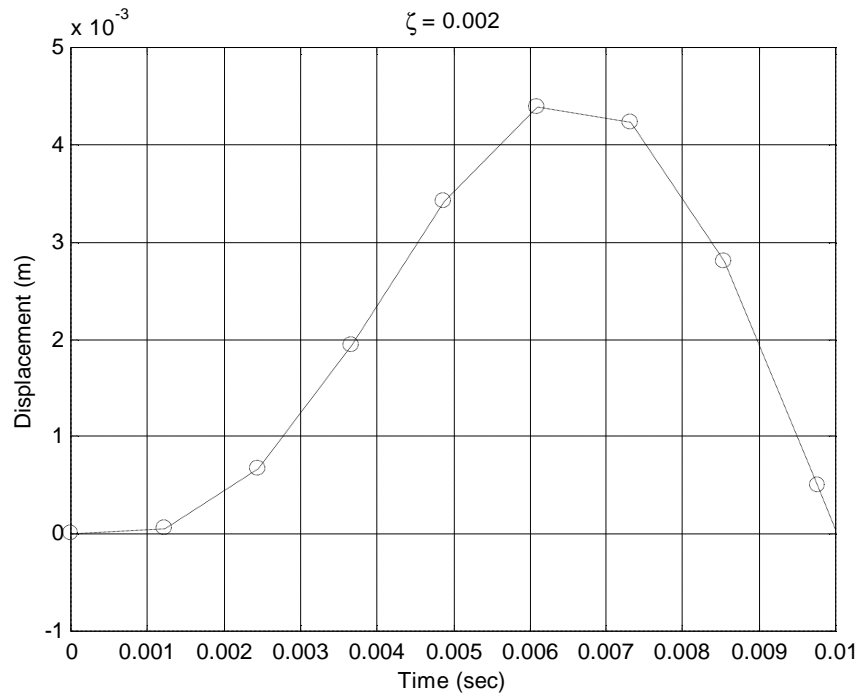


Figure 6.  $T_{max} = 5$  sec,  $N = 4096$ . Same result as shown in Figure 5, but over a smaller time range.

The Matlab code used to generate these results is shown below:

```

% ME6442, Spring 2002
% Dr. Ferri
% Problem 3.60

% Q is a short-pulse parabolic input. The system is
% a damped SDOF system with  $m = 0.5$ kg, and  $f_n = 100$  Hz. The duration of
% the pulse is  $T_n$ , an undamped period of motion.

m = 0.5 ; %kg
zeta = 0.2; % or zeta = 0.002 for part (b)
wn = 100*2*pi; % rad/s
timeconstant = 1/(zeta*wn)
T = 2*pi/wn; % duration of pulse
Tmax = input('Enter Tmax... ');
N = input('Enter N... ');
w1 = 2*pi/Tmax;
delt = Tmax/N;
t = 0:delt:(N-1)*delt;

% Define input pulse

Q = zeros(1,N);
for k = 1:N;
    if t(k) > T; break; end

```

```

    Q(k) = 2000*t(k).*(T-t(k))/T^2;
end

plot(t,Q);
xlabel('Time (sec)')
ylabel('Q')
title(['N = ',num2str(N),' , \zeta = ',num2str(zeta)])
grid
pause

Q_trans = fft(Q);

r = (0:(N/2))*w1/wn;
D = 1./(1 - r.^2 + i*2*zeta*r);
Xhalf = Q_trans(1:(N/2+1)).*D/(m*wn^2);
rev_index = (N/2):-1:2;
X = [Xhalf conj(Xhalf(rev_index))];

% Plot input fft, output fft, and TF on same plot.

Qplot = Q_trans(1:(N/2+1))/max(abs(Q_trans(1:(N/2+1))));
Dplot = D/max(abs(D));
Xplot = Xhalf/max(abs(Xhalf));
plot(r,abs(Qplot),'o--',r,abs(Dplot),'k-',r,abs(Xplot),'*--')
xlabel('Frequency Ratio')
ylabel('Input, TF, Output')
title(['\zeta = ',num2str(zeta)]);
grid
legend('Input','Transfer Function','Output')
pause

xifft = ifft(X);

% Check that the ifft is "mostly real-valued." The quantity
% imag_check is the ratio of the norms of the imaginary and
% real parts. This term should be very small (<1e-5).

imag_check = norm(imag(xifft))/norm(real(xifft))

% Assuming that imag_check is small, discard the imaginary part
% of xifft:

xifft=real(xifft);

plot(t,xifft);
xlabel('Time');
ylabel('x');
title(['N = ',num2str(N),' , \zeta = ',num2str(zeta)])
grid
zoom on

```



### Exercise 3.60

Superposition solution:  $t' = t - \tau$

$$Q(t) = 2000 \left( \frac{t}{\tau} - \frac{t^2}{\tau^2} \right) h(t) - 2000 \left[ \frac{t'+\tau}{\tau} - \left( \frac{t'+\tau}{\tau} \right)^2 \right] h(t')$$

$$= \frac{2000}{\tau} t h(t) - \frac{2000}{\tau^2} t^2 h(t) + \frac{2000}{\tau} t' h(t') + \frac{2000}{\tau^2} (t')^2 h(t')$$

Let  $r(t)$  and  $s(t)$  denote the unit ramp and quadratic responses. Then

$$q = \frac{2000}{\tau} [r(t) + r(t-\tau)] - \frac{2000}{\tau^2} [s(t) - s(t-\tau)]$$

$M := 0.5$        $\omega_{\text{nat}} := 200 \cdot \pi$        $\tau := \frac{2 \cdot \pi}{\omega_{\text{nat}}}$       Note: Change pulse duration to  $\tau$

#### Superposition solution

$$\text{ramp}(t, \zeta) := (t > 0) \cdot \frac{1}{M \cdot \omega_{\text{nat}}^3} \left[ \omega_{\text{nat}} \cdot t - 2 \cdot \zeta + \exp(-\zeta \cdot \omega_{\text{nat}} \cdot t) \cdot \left[ 2 \cdot \zeta \cdot \cos(\sqrt{1 - \zeta^2} \cdot \omega_{\text{nat}} \cdot t) \dots \right. \right. \\ \left. \left. + \frac{1 - 2 \cdot \zeta^2}{\sqrt{1 - \zeta^2}} \cdot \sin(\sqrt{1 - \zeta^2} \cdot \omega_{\text{nat}} \cdot t) \right] \right]$$

$$\text{quad}(t, \zeta) := (t > 0) \cdot \frac{1}{M \cdot \omega_{\text{nat}}^4} \left[ (\omega_{\text{nat}} \cdot t)^2 - 4 \cdot \zeta \cdot \omega_{\text{nat}} \cdot t - 2 \cdot (1 - 4 \cdot \zeta^2) \dots \right. \\ \left. + \exp(-\zeta \cdot \omega_{\text{nat}} \cdot t) \cdot \left[ 2 \cdot (1 - 4 \cdot \zeta^2) \cdot \cos(\sqrt{1 - \zeta^2} \cdot \omega_{\text{nat}} \cdot t) \dots \right. \right. \\ \left. \left. + \frac{6 \cdot \zeta - 8 \cdot \zeta^3}{\sqrt{1 - \zeta^2}} \cdot \sin(\sqrt{1 - \zeta^2} \cdot \omega_{\text{nat}} \cdot t) \right] \right]$$

$$q_{\text{super}}(t, \zeta) := \frac{2000}{\tau} \cdot (\text{ramp}(t, \zeta) + \text{ramp}(t - \tau, \zeta)) - \frac{2000}{\tau^2} \cdot (\text{quad}(t, \zeta) - \text{quad}(t - \tau, \zeta))$$



**First case**  $\zeta := 0.20$

$$\frac{4}{\zeta \cdot \omega_{\text{nat}}} = 0.032 \quad \text{This is greater than } \tau = 0.01, \text{ so let } T' = 4/\zeta^* \omega_{\text{nat}} \quad T' := \frac{4}{\zeta \cdot \omega_{\text{nat}}}$$

Because damped period is larger than the undamped period, select  $\Delta' > \pi/2\omega_{\text{nat}}$   $\Delta' := \frac{\pi}{8 \cdot \omega_{\text{nat}}}$

$$N' := \frac{2 \cdot T'}{\Delta'} \quad N' = 101.859 \quad N := 128 \quad T := 2 \cdot T' \quad \Delta := \frac{T}{N}$$

$$j := 1..N \quad t_j := (j - 1) \cdot \Delta \quad Q_j := 2000 \cdot \frac{t_j \cdot (\tau - t_j)}{\tau^2} \cdot (t_j < \tau)$$

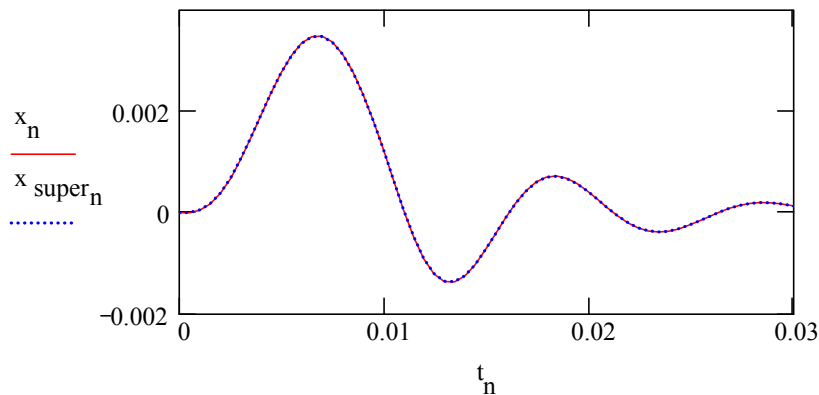
F := FFT(Q)

$$r_{\text{fund}} := \frac{2 \cdot \pi}{\omega_{\text{nat}} \cdot T} \quad K := M \cdot \omega_{\text{nat}}^2$$

$$n := 1.. \frac{N}{2} + 1$$

$$X_n := \frac{F_n}{K} \cdot \frac{1}{1 - [(n - 1) \cdot r_{\text{fund}}]^2 + 2i \cdot \zeta \cdot [(n - 1) \cdot r_{\text{fund}}]} \quad x := \text{IFFT}(X)$$

$$x_{\text{super}_n} := q_{\text{super}}(t_n, \zeta)$$



$\tau = 0.01$

**Second case**  $\zeta := 0.0020$

$$\frac{4}{\zeta \cdot \omega_{\text{nat}}} = 3.183 \quad \text{This is greater than } \tau = 0.01, \text{ so let } T' = 4/\zeta^* \omega_{\text{nat}} \quad T' := \frac{4}{\zeta \cdot \omega_{\text{nat}}}$$

Because damped is larger than the undamped period, select  $\Delta' > \pi/2\omega_{\text{nat}}$   $\Delta' := \frac{\pi}{8 \cdot \omega_{\text{nat}}}$

$$N' := \frac{2 \cdot T'}{\Delta'} \quad N' = 1.019 \cdot 10^4 \quad N := 1024 \cdot 16 \quad T := 2 \cdot T' \quad \Delta := \frac{T}{N}$$

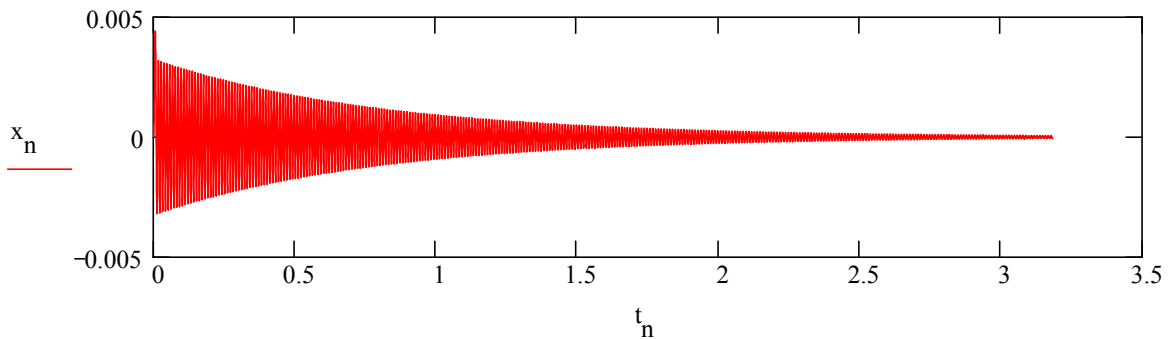
$$j := 1..N \quad t_j := (j - 1) \cdot \Delta \quad Q_j := 2000 \cdot \frac{t_j \cdot (\tau - t_j)}{\tau^2} \cdot (t_j < \tau)$$

F := FFT(Q)

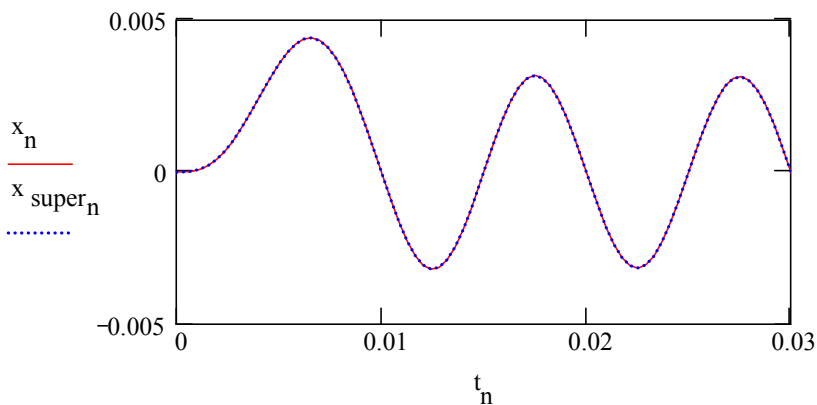
$$r_{\text{fund}} := \frac{2 \cdot \pi}{\omega_{\text{nat}} \cdot T} \quad K := M \cdot \omega_{\text{nat}}^2$$

$$n := 1.. \frac{N}{2} + 1$$

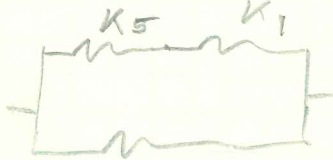
$$X_n := \frac{F_n}{K} \cdot \frac{1}{1 - [(n - 1) \cdot r_{\text{fund}}]^2 + 2i \cdot \zeta \cdot [(n - 1) \cdot r_{\text{fund}}]} \quad x := \text{IFFT}(X)$$



$$x_{\text{super}_n} := q_{\text{super}}(t_n, \zeta)$$

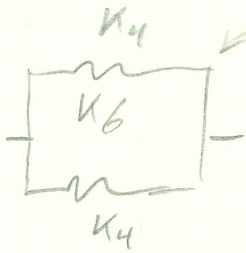


P1 (1.1) eg. to



$K_5 = K_2 + K_3$  (parallel)

$K_1, K_5$  in series

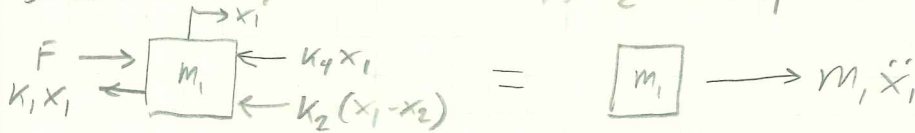


$K_6 = \frac{K_5 K_1}{K_5 + K_1}$

$\Rightarrow K_{eg} = \frac{(K_2 + K_3) K_1}{K_1 + K_2 + K_3} + K_4$

P2 (1.11)

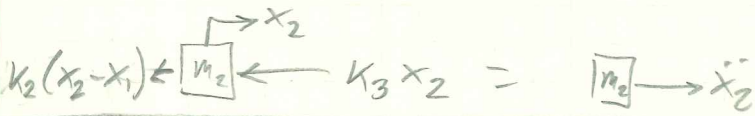
a)  $\Sigma F$  on  $m_1 \rightarrow$  let  $x_1, x_2 =$  disp. of  $m_1, m_2$  to right



check,  $x_1 > x_2$ , spring  $k_2$  pushes left  $\checkmark$

$F - K_4 x_1 - K_1 x_1 - K_2 (x_1 - x_2) = m_1 \ddot{x}_1$

$m_1 \ddot{x}_1 + (K_4 + K_1 + K_2) x_1 - K_2 x_2 = F$  (1)



$m_2 \ddot{x}_2 + (K_2 + K_3) x_2 - K_2 x_1 = 0$  (2)

$\checkmark$  2 masses, so we have 2 EOM

b) set  $m_2 = 0$  in (2), solve for  $K_2 x_2 \rightarrow$

$K_2 x_2 = \frac{K_2^2 x_1}{(K_2 + K_3)} \rightarrow$  plug into (1)

$m_1 \ddot{x}_1 + (K_4 + K_1 + K_2) x_1 - \frac{K_2^2 x_1}{(K_2 + K_3)} = F$

$m_1 \ddot{x}_1 + (K_4 + K_1) x_1 + \frac{K_2(K_2 + K_3) - K_2^2}{(K_2 + K_3)} x_1 = F$

$m_1 \ddot{x}_1 + (K_4 + K_1 + \frac{K_2 K_3}{(K_2 + K_3)}) x_1 = F$

Identical to system with  $K_1, K_4$  and  $K_5$  in //

with  $K_5 = \frac{K_2 K_3}{(K_2 + K_3)} = K_2$  and  $K_3$  in series.  $\checkmark$