Homework #6

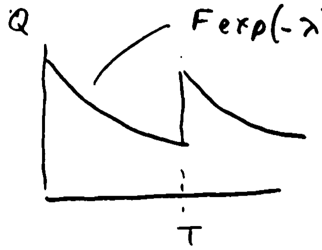
EMA 545, Spring 2013

1.) **Problem 3.41** in Ginsberg. Check your answer for λ=1.0 using FFT techniques with the fft_easy.m Matlab function from the course website.

2.) **Problem 3.50** in Ginsberg. DO PART (a) ONLY.

3.) (**20 points**) Find the steady-state response of the system in **Problems 3.45** and **3.46** from Ginsberg using FFT techniques. Perform your analysis with $\tau = \pi/(3\omega_n)$ as stated in the problem and also repeat the analysis for $\tau = 3\pi/\omega_n$. Which harmonic is dominant in the response in each case? Why? Create a plot of the steady-state displacement for each case.

Exercise 3.41



$$Q(t \pm T) = Q(t)$$

$$Q = \frac{1}{2}\sum_{n=-\infty}^{\infty} F_n \exp(2\pi i n t/T)$$

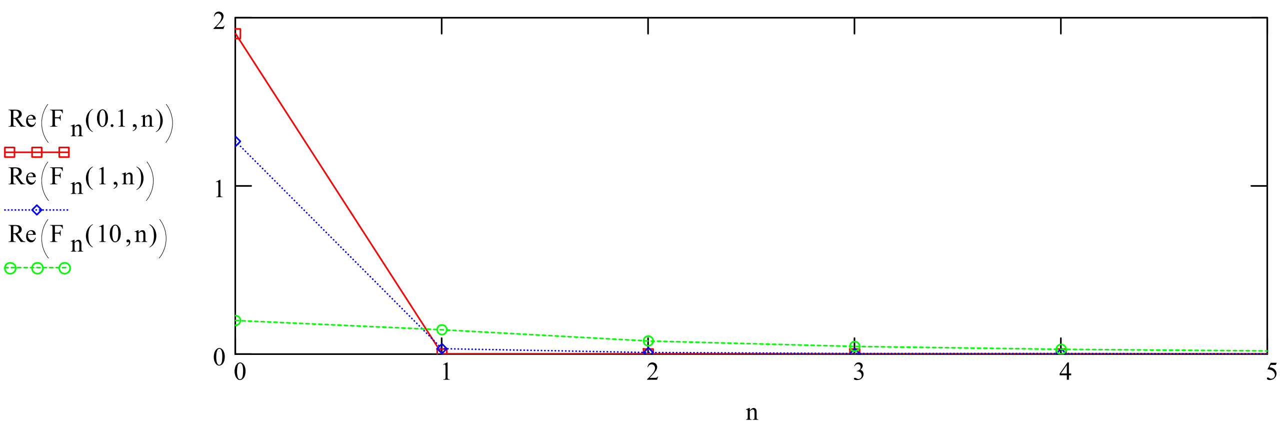
$$F_n = \frac{2}{T}\int_0^T Q \exp(-i 2\pi i n t/T)\,dt$$

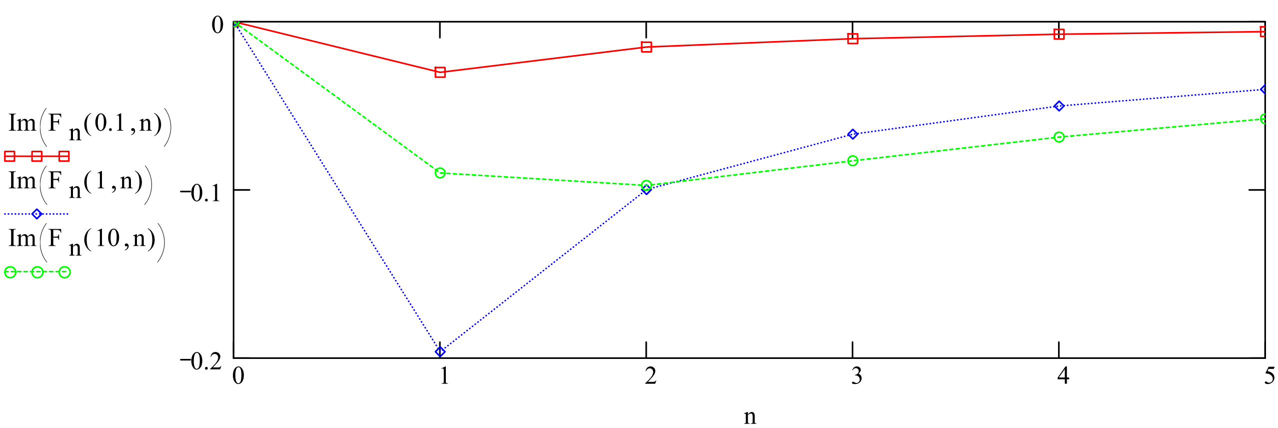
$$F_n = 2F\int_0^1 \exp[-(\lambda + 2\pi i n)t']\,dt'$$

$$= 2F \frac{1-\exp[-(\lambda+2\pi i n)]}{\lambda+2\pi i n} \quad \text{but } \exp(-i2\pi n) = 1$$

$$\text{so } F_n = F\frac{2[1-\exp(-\lambda)]}{\lambda+2\pi i n}$$

$$n := 0..5 \qquad F_n(\lambda, n) := \frac{2}{(\lambda + 2i \cdot n \cdot \pi)} \cdot (1 - \exp(-\lambda))$$





As λ increases, the higher harmonic amplitudes increase.

Matlab solution to HW#8, Problem 3.41

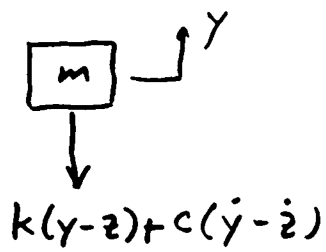
```
%% Solution to 3.41 using Matlab
% EMA 545, Spring 2011
clear all; close all

N = 2^5; % 2^13; % number of samples for FFT
tau = pi/3; % 3*pi; % pi/3;
T = 1
    w1 = 2*pi/T;
F=1; % Force will be non-dimensional
lam = 1;

dt = T/N;
ts_fft = [0:dt:(T-dt)].';

% Define Input Force in Time Domain--f(t)
ft = F*exp(-lam*ts_fft);
ft = ft(:); % make sure these are column vectors

% Use FFT easy to perform analysis
[D_fft,ws_fft] = fft_easy(ft,ts_fft);

F_fft = (2/N)*D_fft; % this only has those for positive frequencies.

ns = [0:5].';
F_fs = 2*(1-exp(-lam))./(lam+1i*2*pi*ns);

disp('n, Frequency, Fourier Coeff, FFT estimate (abs)');
[ns, ws_fft(1:6), abs(F_fs(1:6)), abs(F_fft(1:6))]
disp('Frequency, Fourier Coeff, FFT estimate (angle in deg)');
[ns, ws_fft(1:6), angle([F_fs(1:6), F_fft(1:6)])*180/pi]
```

Results with n=2^5=32

```
            n,    Frequency, Fourier Coeff, FFT estimate (abs)
ans =
            0            0        1.2642        1.2841
            1       6.2832       0.19871       0.20215
            2       12.566       0.10029       0.10252
            3        18.85      0.066976      0.069021
            4       25.133      0.050263      0.052388
            5       31.416      0.040222      0.042541
            n,    Frequency, Fourier Coeff, FFT estimate (angle in deg)
ans =
            0            0             0             0
            1       6.2832       -80.957       -75.361
            2       12.566        -85.45       -74.259
            3        18.85       -86.963       -70.177
            4       25.133       -87.721        -65.34
            5       31.416       -88.177       -60.201
```

With only 32 samples, this is already a pretty good approximation of the Fourier Coefficients. With n=2^13, the FFT is accurate to about three decimal places.

Exercise 3.50



$\omega_{nat} = 10\pi$ rad/s, $\zeta = 0.4$, $m = 1500$ kg

$m\ddot{y} + c\dot{y} + ky = c\dot{z} + kz = Q$

$z(x) = (x - 5x^2)[h(x) - h(x-0.2)]$ if $0 < x < 4$ meter

$z(x+4) = z(x)$

Substitute $x = vt \Rightarrow$ Period $= \frac{4}{v} \Rightarrow \omega_1 = \frac{2\pi}{(4/v)} = \frac{\pi v}{2}$

Resonance if any harmonic matches natural frequency

$$n\left(\frac{\pi v}{2}\right) = \omega_{nat} \Rightarrow v = \frac{2\omega_{nat}}{n\pi} = 20, 10, 6.67, \ldots \text{ m/s} \Longleftarrow$$

Response @ $v = 5$ m/s $\Rightarrow T = 0.8$ sec

Use $N = 128 \Rightarrow t_n = (n-1)\frac{T}{N}$

Evaluate

$$z(t_n) = (vt_n - 5v^2t_n^2)[h(t_n) - h(t_n - \tfrac{0.2}{v})]$$

Take FFT $\Rightarrow Z(\omega_n)$ where $\omega_n = n\omega_1$, $\omega_1 = \frac{2\pi}{T}$

The FFT of the effective force will be

$$F_n = c[i\omega_n Z(\omega_n)] + k Z(\omega_n)$$

The DFT of the response is

$$Y_n = \frac{F_n}{k} \frac{1}{1-(nr_1)^2 + 2i\zeta n r_1}$$

where $k = m\omega_{nat}^2$, $c = 2m\omega_{nat}\zeta$, & $r_1 = \frac{\omega_1}{\omega_{nat}} = \frac{2\pi n}{\omega_{nat} T}$

Then find $y(t_n)$ from an IFFT.

$m := 1500$ $\omega_{nat} := 10\cdot\pi$ $\zeta := 0.4$ $k := m\cdot\omega_{nat}^{\;2}$ $c := 2\cdot m\cdot\omega_{nat}\cdot\zeta$

$v := 5$ $T := \frac{4}{v}$ $\omega_{fund} := \frac{2\cdot\pi}{T}$

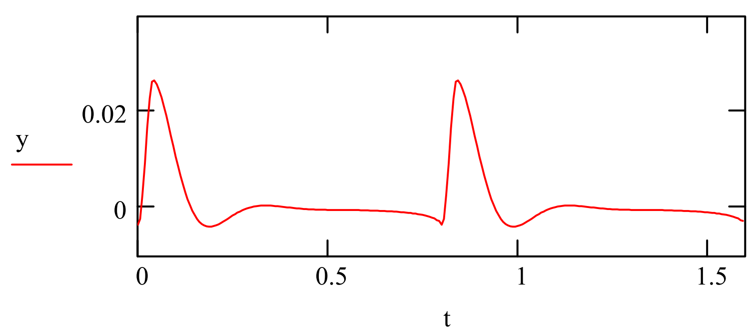
$elev(x) := \left(x - 5\cdot x^2\right)\cdot(x>0)\cdot(x<0.2)$

$N := 128$ $j := 1 .. N$ $t_j := (j-1)\cdot\frac{T}{N}$ $z_j := elev\left(v\cdot t_j\right)$ $Z := FFT(z)$

$n := 1 .. \frac{N}{2} + 1$ $\omega_n := (n-1)\cdot\omega_{fund}$ $F_n := \left(i\cdot\omega_n\cdot c + k\right)\cdot Z_n$ $r_{fund} := \frac{2\cdot\pi}{\omega_{nat}\cdot T}$

$$Y_n := \frac{F_n}{k}\cdot\frac{1}{1 - \left(n\cdot r_{fund}\right)^2 + 2i\cdot\zeta\cdot n\cdot r_{fund}}$$

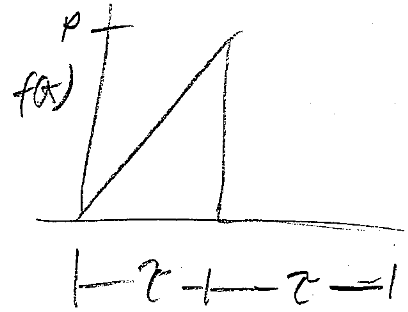
$y := IFFT(Y)$ $y := stack(y,y)$ $t_{j+N} := t_j + T$



3.46 Solution

$f(t)$ = Triangle pulse

$\zeta = 0.04, \ \omega_n$



$\tau = \frac{\pi}{3\omega_n}, \quad \tau = \frac{3\pi}{\omega_n}$

$$m\ddot{x} + c\dot{x} + kx = f(t) = Re(F e^{i\omega t})$$

$$x(t) = Re(X e^{i\omega t}) \longrightarrow$$

$$X = \frac{F/m}{\omega_n^2 - \omega^2 + i\omega 2\zeta\omega_n}$$

FFT ~ $T = 2\tau$

$$\omega_1 = \frac{2\pi}{T} = \frac{2\pi \, 3\omega_n}{2\pi} = 3\omega_n \quad \text{for } \tau = \frac{\pi}{3\omega_n}$$

$$\omega_1 = \frac{2\pi \, \omega_n}{2 \cdot 3\pi} = \frac{1}{3}\omega_n \quad \text{for } \tau = \frac{\pi}{3\omega_n}$$

Nondimensionalize:

$$t = t'/\omega_n$$

$$x(t) = Re(X e^{i(\omega/\omega_n)t'}) = Re(X e^{irt})$$

$$\dot{x}(t) = \frac{\partial x}{\partial t'}\frac{\partial t'}{\partial t} = Re(irX e^{irt}) \cdot \omega_n \longrightarrow \text{same}$$

$$X = \frac{F/(m\omega_n^2)}{1 - r^2 + i2\zeta r}$$

$$X_a = \frac{-\omega^2 F/(m\omega_n^2)}{1 - r^2 + i2\zeta r} = \frac{-r^2 F/(m\omega_n^4)}{1 - r^2 + i2\zeta r}$$

plot $x(t)m\omega_n^2$ and $x_a(t')m\omega_n^4$

$$t = \tau = \frac{t'}{\omega_n} \longrightarrow t' = \omega_n\tau \longrightarrow \tau' = \frac{\pi}{3} \text{ and } 3\pi$$

Solution: Problem 4 (3.45-46) Spring 2011

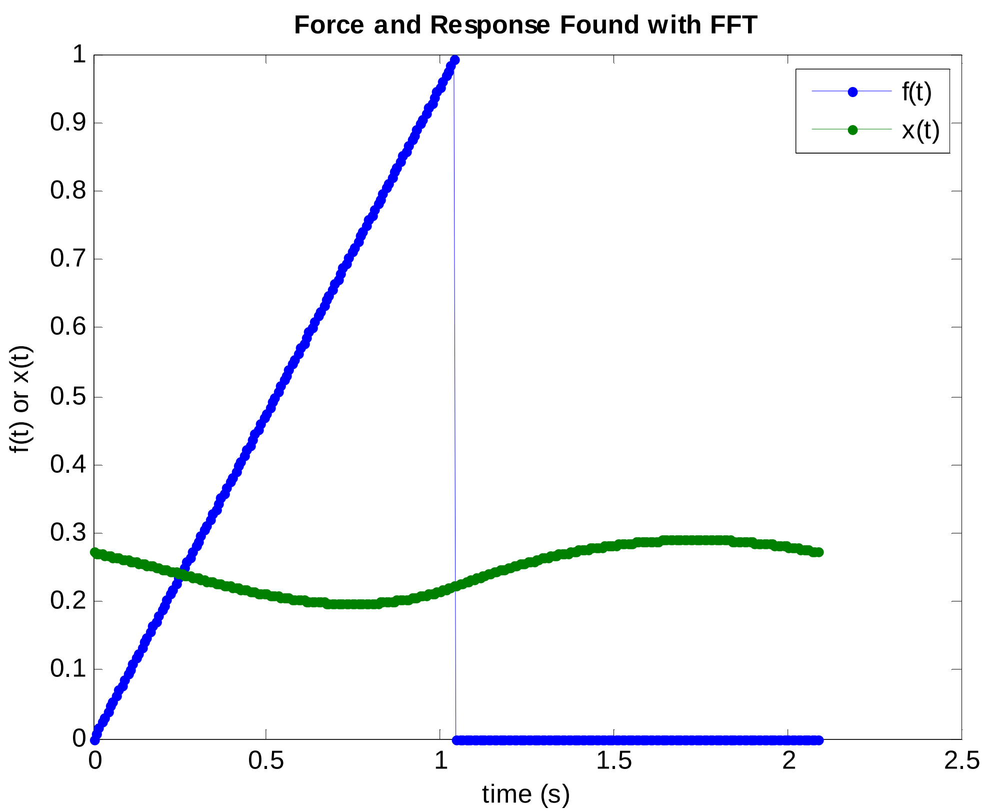
Matlab code given below, which is modified only slightly from FFT_Square_Ex_545_v2.m

```
%% Solution to P4, HW#8 (based on 3.45 and 3.46 in Ginsberg)
clear all; close all

N = 256; % number of samples for FFT
tau = pi/3; % 3*pi; % pi/3;
T = 2*tau;
    w1 = 2*pi/T;
m=1;
wn=1; % since time is non-dimensional
zt = 0.04;
P = 1; % Force will be non-dimensional

dt = T/N;
ts_fft = [0:dt:(T-dt)];

% Define Input Force in Time Domain--f(t) = mean at discontinuity
for k = 1:1:length(ts_fft)
    if ts_fft(k) < tau;
        ft(k,1) = P*ts_fft(k)/tau;
    else
        ft(k,1) = 0;
    end
end

% Use FFT easy to perform analysis
[F_fft,ws_fft] = fft_easy(ft,ts_fft);

% Each coefficient in F_fft is the complex amplitude of a harmonic
whose
% frequency is given in ws_fft.  We could scale these to obtain
estimates
% of the Fourier coefficients, but we do not need to since we are just
% going to take the IFFT later.

% Make everything a column vector:
F_fft = F_fft(:); ws_fft = ws_fft(:);

% Now form a vector of transfer function values at each frequency:
H = (1/m)./((i*ws_fft).^2 + i*ws_fft*(2*zt*wn) + wn^2); % dot or term
by term multiply
    % Same as doing a for loop over each frequency.

% Now the response is just the force times the transfer function.
X = H.*F_fft(:);

% Plot everything in the frequency domain to understand what's
happening.
figure(3)
semilogy(ws_fft,abs(F_fft),'o',ws_fft,abs(X),'*',ws_fft,abs(H),'-');
grid on;
```

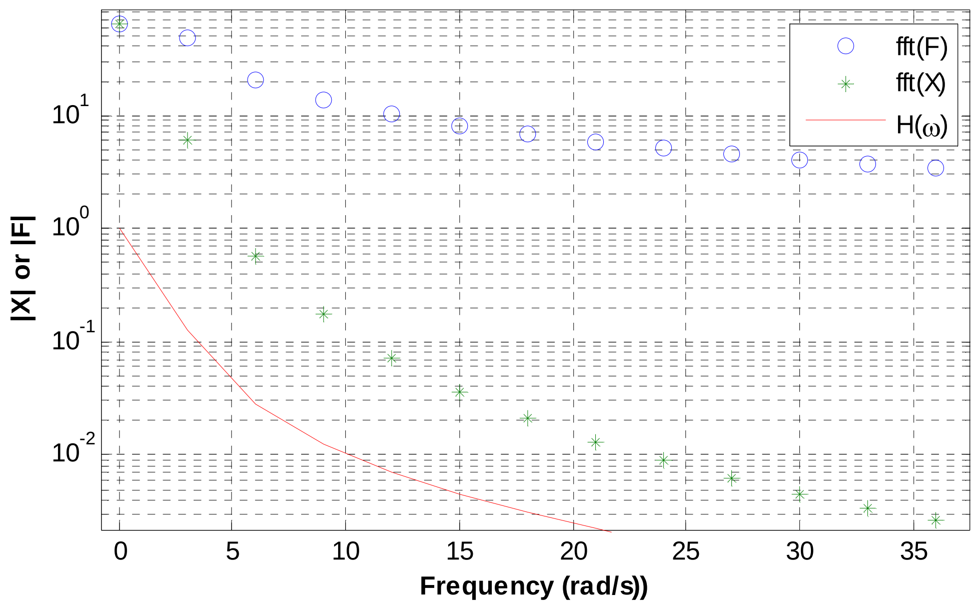
```
xlabel('\bfFrequency (rad/s))'); ylabel('\bf|X| or |F|');
legend('fft(F)','fft(X)','H(\omega)');

% take IFFT of the coefficients X to find the time signal x(t)
xt = ifft_easy(X,ws_fft);
% note ifft_easy(F_fft,ws_fft) = ft, exactly with no approximation

figure(4)
% [ax,h1,h2] = plotyy(ts_fft,gt,ts_fft,xt);
plot(ts_fft,ft,'.-',ts_fft,xt,'.-');
xlabel('time (s)'); ylabel('f(t) or x(t)');
legend('f(t)','x(t)');
title('\bfForce and Response Found with FFT');
```

Results:

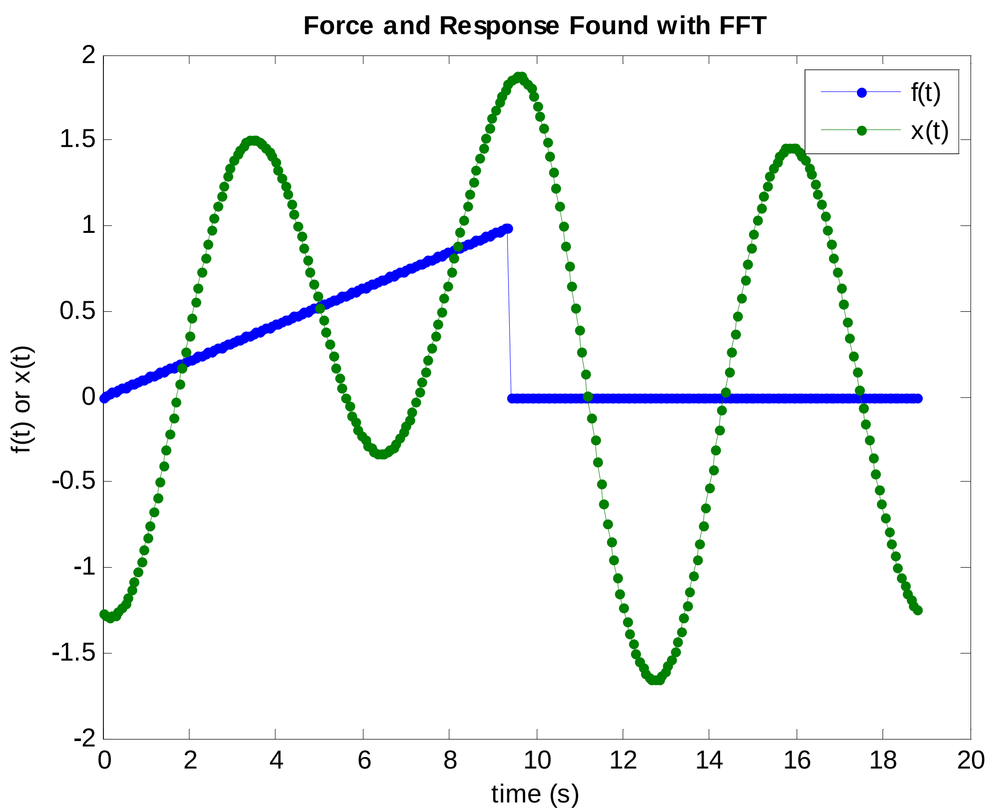
Case 1: $\tau = \pi/(3\omega_n)$ **(r = 3)**

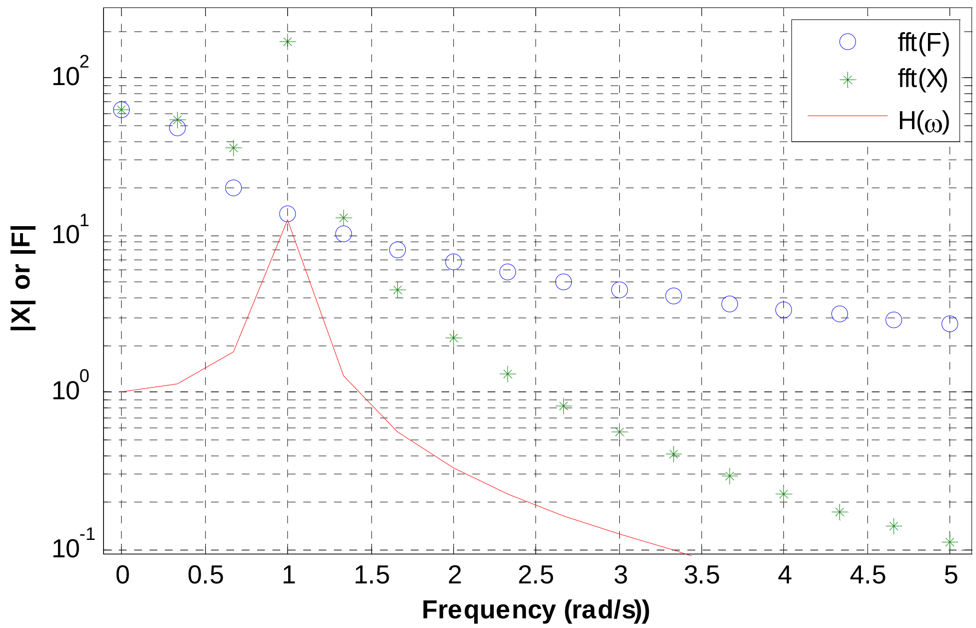


To understand this, let's look at the FFT coefficients of X, F and the transfer function H, shown in the following figure. Since the first coefficient is beyond the natural frequency, the transfer function gets smaller as omega increases. The force is also dominated by the lower frequency terms, so the low frequency terms dominate. The DC term is the largest, followed by the first harmonic (DC is Electrical Engineering terminology for Direct Current or the zero frequency). So, it shouldn't surprise us to see that the response is an offset sinusoid with low amplitude.



Case 2: $\tau = 3\pi/\omega_n$ (r = 1/3)





In this case the fundamental frequency of the sawtooth is 1/3 the natural frequency. Hence, the third harmonic is equal to the natural frequency so that harmonic is amplified in the response. So, the response looks like a 3-cycle sinusoid (per period of the force) even though the force spectrum is dominated by the lower frequency harmonics.