HW 6

EMA 545
Mechanical Vibrations

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Contents

1 problem 1

3.41 in text: A periodic disturbance consists of a sequence of exponentially pulse repeated at intervals T, such that $Q(t) = Fe^{\frac{-\lambda t}{T}}$ for $0 < t < T$, and $Q(t \pm T) = Q(t)$. The parameter λ is nondimensional. Determine the complex Fourier series representing the force. Evaluate the first 5 coefficients when $\lambda = 0.1, 1, 10$. What does this reveal regarding the influence of λ on the frequency spectrum?

Let $\tilde{Q}(t)$ be the Fourier series approximation to $Q(t)$ given by

$$\tilde{Q}(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} F_n e^{in\frac{2\pi}{T}t} \tag{1}$$

Where

$$\begin{aligned}
F_n &= \frac{2}{T}\int_0^T Q(t)e^{-in\frac{2\pi}{T}t}dt \\
&= \frac{2}{T}\int_0^T Fe^{\frac{-\lambda t}{T}}e^{-in\frac{2\pi}{T}t}dt = \frac{2F}{T}\int_0^T e^{-t\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}dt = \frac{2F}{T}\left(\frac{e^{-t\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}}{in\frac{2\pi}{T}-\frac{\lambda}{T}}\right)_0^T \\
&= \frac{2F}{in2\pi-\lambda}\left(e^{-T\left(in\frac{2\pi}{T}-\frac{\lambda}{T}\right)}-1\right) \\
&= \frac{2F}{in2\pi-\lambda}\left(e^{-in2\pi}e^{-\lambda}-1\right)
\end{aligned}$$

But $e^{-in2\pi} = 1$, hence

$$F_n = \frac{2F}{in2\pi-\lambda}\left(e^{-\lambda}-1\right)$$

Hence Eq 1 becomes

$$\begin{aligned}
\tilde{Q}(t) &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{2F}{in2\pi-\lambda}\left(e^{-\lambda}-1\right)e^{in\frac{2\pi}{T}t} \\
&= F\sum_{n=-\infty}^{\infty}\frac{\left(e^{-\lambda}-1\right)}{in2\pi-\lambda}e^{in\frac{2\pi}{T}t} \\
&= F\sum_{n=-\infty}^{\infty}\frac{1-e^{-\lambda}}{\lambda+in2\pi}e^{in\frac{2\pi}{T}t}
\end{aligned}$$

For $n = -2, -1, 0, 1, 2$ we obtain

$$\begin{aligned}
\tilde{Q}(t) &= F\sum_{n=-2}^{2}\frac{1-e^{-\lambda}}{\lambda+in2\pi}e^{in\frac{2\pi}{T}t} \\
&= F\left(\frac{1-e^{-\lambda}}{\lambda-i4\pi}e^{-i\frac{4\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda-i2\pi}e^{-i\frac{2\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda}+\frac{1-e^{-\lambda}}{\lambda+i2\pi}e^{i\frac{2\pi}{T}t}+\frac{1-e^{-\lambda}}{\lambda+i4\pi}e^{i\frac{4\pi}{T}t}\right)
\end{aligned}$$

For $\lambda = 0.1$

$$
\begin{aligned}
\tilde{Q}(t) &= F\left(\frac{1-e^{-0.1}}{0.1-i4\pi}e^{-i\frac{4\pi}{T}t} + \frac{1-e^{-0.1}}{0.1-i2\pi}e^{-i\frac{2\pi}{T}t} + \frac{1-e^{-0.1}}{0.1} + \frac{1-e^{-0.1}}{0.1+i2\pi}e^{i\frac{2\pi}{T}t} + \frac{1-e^{-0.1}}{0.1+i4\pi}e^{i\frac{4\pi}{T}t}\right) \\
&= F\{\left(6.026\times 10^{-5} + 7.572\times 10^{-3}i\right)e^{-i\frac{4\pi}{T}t} \\
&+ \left(2.41\times 10^{-4} + 1.514\times 10^{-2}i\right)e^{-i\frac{2\pi}{T}t} \\
&+ 0.952 \\
&+ \left(2.4099\times 10^{-4} - 1.5142\times 10^{-2}i\right)e^{i\frac{2\pi}{T}t} \\
&+ \left(6.026\times 10^{-5} - 7.572\times 10^{-3}i\right)e^{i\frac{4\pi}{T}t}\}
\end{aligned}
$$

For $\lambda = 1$

$$
\begin{aligned}
\tilde{Q}(t) &= F\left(\frac{1-e^{-1}}{1-i4\pi}e^{-i\frac{4\pi}{T}t} + \frac{1-e^{-1}}{1-i2\pi}e^{-i\frac{2\pi}{T}t} + \frac{1-e^{-1}}{1} + \frac{1-e^{-1}}{1+i2\pi}e^{i\frac{2\pi}{T}t} + \frac{1-e^{-1}}{1+i4\pi}e^{i\frac{4\pi}{T}t}\right) \\
&= F\{(0.00398 + 0.05i)e^{-i\frac{4\pi}{T}t} + (0.016 + 0.098i)e^{-i\frac{2\pi}{T}t} + 0.632 + (0.016 + 0.098i)e^{i\frac{2\pi}{T}t} + (0.00398 + 0.05i)e^{i}
\end{aligned}
$$

For $\lambda = 10$

$$
\begin{aligned}
\tilde{Q}(t) &= F\left(\frac{1-e^{-10}}{10-i4\pi}e^{-i\frac{4\pi}{T}t} + \frac{1-e^{-10}}{10-i2\pi}e^{-i\frac{2\pi}{T}t} + \frac{1-e^{-10}}{10} + \frac{1-e^{-10}}{10+i2\pi}e^{i\frac{2\pi}{T}t} + \frac{1-e^{-10}}{10+i4\pi}e^{i\frac{4\pi}{T}t}\right) \\
&= F\{\left(3.877\times 10^{-2} + 4.872\times 10^{-2}i\right)e^{-i\frac{4\pi}{T}t} \\
&+ \left(7.169\times 10^{-2} + 4.505\times 10^{-2}i\right)e^{-i\frac{2\pi}{T}t} \\
&+ 0.1 \\
&+ \left(7.169\times 10^{-2} - 4.505\times 10^{-2}i\right)e^{i\frac{2\pi}{T}t} \\
&+ \left(3.877\times 10^{-2} - 4.872\times 10^{-2}i\right)e^{i\frac{4\pi}{T}t}\}
\end{aligned}
$$

We notice that as λ became larger, the DC term became smaller. Since the DC term represents average value of the whole signal, then we can say that as λ gets larger, then the average becomes smaller. This means the energy of the signal becomes smaller as λ becomes larger.

1.1 Verification using Matlab ffteasy.m

From above, we found for $\lambda = 1$

$$
\begin{aligned}
F_n &= \frac{2F}{in2\pi - \lambda}\left(e^{-\lambda} - 1\right) \\
&= \frac{2F}{in2\pi - 1}\left(e^{-1} - 1\right)
\end{aligned}
$$

and the first 5 found to be

n	F_n
-2	$0.00398 + 0.05i$
-1	$0.016 + 0.098i$
0	0.632
1	$0.016 - 0.098i$
2	$0.00398 - 0.05i$

To verify the result with ffteasy.m using $\lambda = 1$, Using $F = 1$, and using $T = 1$. This below shows the result for F_0, F_1, F_2 and we see that the DC term F_0 agrees, and that complex component of F_1, F_2 also agrees. The real parts are little larger than what I obtained using the above. This might be a scaling issue, and I was not able to determine the reason for it at this time.

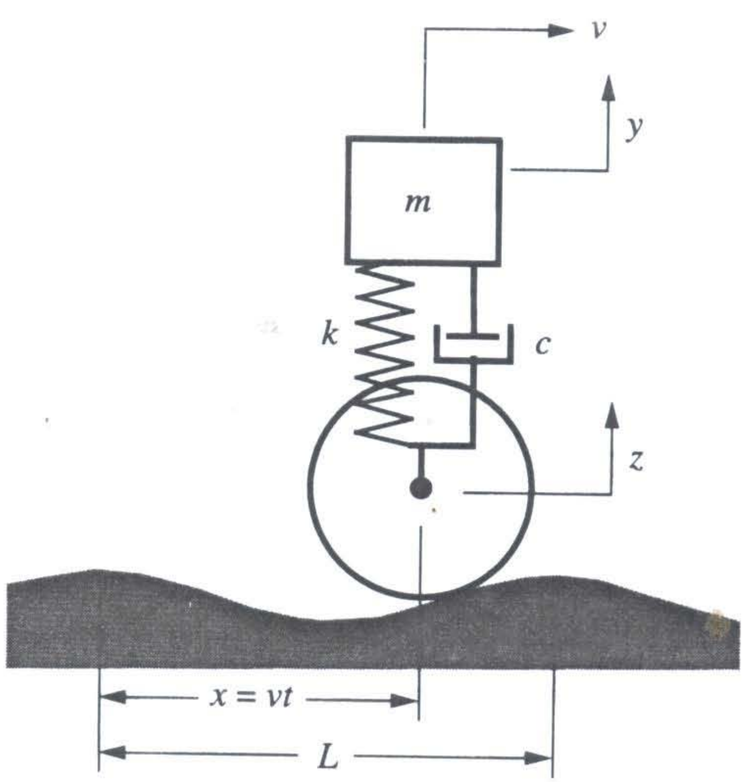
```
EDU>> T=1; del=0.01; t=0:del:T; lambda=1; xt=exp(-lambda*t/T);
EDU>> (1/length(t))*fft_easy(xt,t)

ans =

   0.6326 + 0.0000i
   0.0190 - 0.0986i
   0.0072 - 0.0502i
```

2 problem 2

3.50 The sketch depicts a one-degree-of-freedom model of an automobile traveling to the right at constant speed v when the road is not smooth. The mass is 1200 kg, the natural frequency of the system is 5 Hz, and the critical damping ratio is 0.4. The elevation of a certain road is a sequence of periodic 50 mm high bumps spaced at a distance of 4 m, specifically, $z = (x - 5x^2)$ if $0 < x < 0.2$ m, $z = 0$ if $0.2 < x < 4$ m, $z(x + 4) = z(x)$.

(a) What speeds v would cause the vertical displacement y to be resonant if the dashpot were not present?

(b) Determine the steady-state displacement y when $v = 5$ m/s.

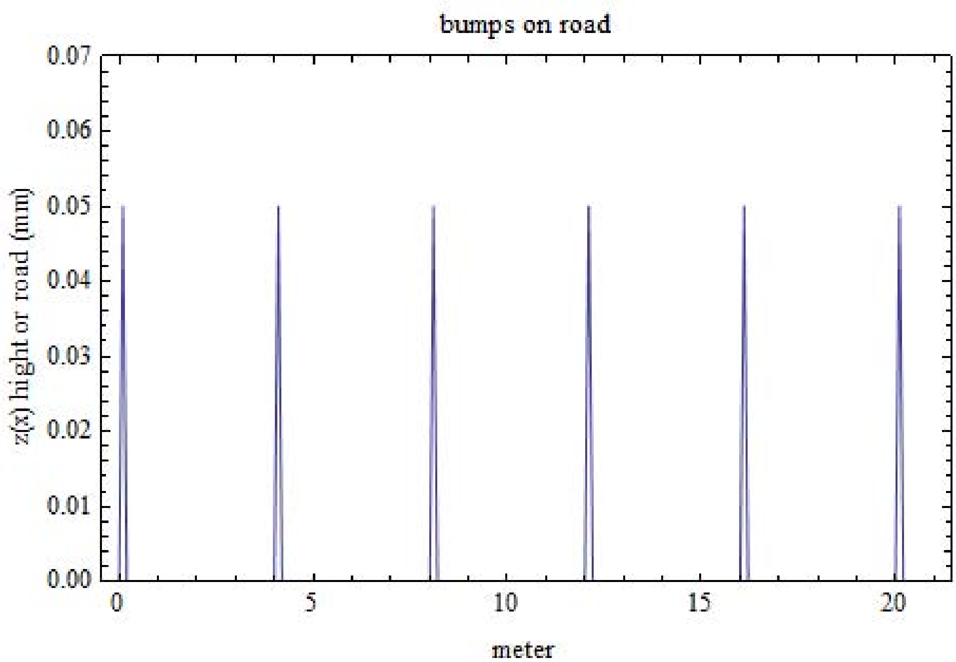


We are given that $m = 1200$ kg, $f = 5$ Hz, $\zeta = 0.4$ and

$$z(x) = \begin{cases} x - 5x^2 & 0 < x < 0.2 \\ 0 & 0.2 < x < 4 \end{cases}$$

A plot of $z(x)$ for first 20 meters is

```
z[x_] := Piecewise[{{x - 5 x^2, 0 <= x < 0.2}, {0, 0.2 <= x <= 4}}]
z[x_] /; x > 4 := z[Mod[x, 4]];
Table[{x, z[x]}, {x, 0, 21, .1}];
ListLinePlot[%, PlotRange -> {All, {0, .07}}, Frame -> True,
 FrameLabel -> {{"z(x) hight or road (mm)", None}, {"meter",
    "bumps on road"}}]
```



We need to be able to express $z(t)$ as $\mathrm{Re}\left\{Ze^{i\frac{2\pi}{T}t}\right\}$ where T is the period of the function $z(t)$. Hence we need to represent $z(x)$ as Fourier series approximation then replace $x = vt$ and use the result.

The period $T = 4$ meter. Let $\tilde{z}(x)$ be the Fourier series approximation to $z(x)$, hence

$$\tilde{z}(x) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)$$

Where

$$F_n = \frac{2}{T}\int_0^T z(x)e^{-in\frac{2\pi}{T}x}dx = \frac{1}{2}\int_0^{2/10}\left(x-5x^2\right)e^{-in\frac{\pi}{2}x}dx = \frac{1}{2}\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx - \frac{5}{2}\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx$$

Using integration by parts $\int udv = uv - \int vdu$, letting $u = x$ and $dv = e^{-in\frac{\pi}{2}x}$ then $v = \int e^{-in\frac{\pi}{2}x}dx = \frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}$hence

$$\begin{aligned}
\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx &= x\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}\Bigg|_0^{\frac{2}{10}} - \int_0^{2/10}\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}dx \\
&= \frac{2}{10}\frac{ie^{-in\frac{\pi}{2}\frac{2}{10}}}{n\frac{\pi}{2}} - \frac{2}{n\pi}\int_0^{2/10} ie^{-in\frac{\pi}{2}x}dx \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} - \frac{i2}{n\pi}\left(\frac{e^{-in\frac{\pi}{2}x}}{-in\frac{\pi}{2}}\right)_0^{\frac{2}{10}} \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} + \frac{4}{n^2\pi^2}\left(e^{-in\frac{\pi}{2}x}\right)_0^{\frac{2}{10}} \\
&= \frac{4}{10}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} + \frac{4}{n^2\pi^2}\left(e^{-in\frac{\pi}{2}\frac{2}{10}} - 1\right) \\
&= \frac{4i}{10n\pi}e^{-in\frac{\pi}{10}} + \frac{4}{n^2\pi^2}e^{-in\frac{\pi}{10}} - \frac{4}{n^2\pi^2} \\
&= e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2} + \frac{2i}{5n\pi}\right) - \frac{4}{n^2\pi^2}
\end{aligned}$$

Now we do the second integral $\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx$.

Integration by parts, $\int udv = uv - \int vdu$, letting $u = x^2$ and $dv = e^{-in\frac{\pi}{2}x}$ then $v = \frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}$ hence

$$\begin{aligned}
\int_0^{2/10} x^2e^{-in\frac{\pi}{2}x}dx &= \left[x^2\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}\right]_0^{\frac{2}{10}} - \int_0^{2/10} 2x\frac{ie^{-in\frac{\pi}{2}x}}{n\frac{\pi}{2}}dx \\
&= \frac{8}{100}\frac{ie^{-in\frac{\pi}{10}}}{n\pi} - \frac{4i}{n\pi}\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx
\end{aligned}$$

But $\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx$ was solved before and its results is Eq 2.1, hence

$$\begin{aligned}\int_0^{2/10} x^2 e^{-in\frac{\pi}{2}x}dx &= \frac{4}{100}\frac{ie^{-in\frac{\pi}{10}}}{n\frac{\pi}{2}} - \frac{4i}{n\pi}\left(e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2}+\frac{2i}{5n\pi}\right)-\frac{4}{n^2\pi^2}\right)\\ &= \frac{8i}{100n\pi}e^{-in\frac{\pi}{10}} - e^{-in\frac{\pi}{10}}\left(\frac{16i}{n^3\pi^3}-\frac{8}{5n^2\pi^2}\right)+\frac{16i}{n^3\pi^3}\\ &= e^{-in\frac{\pi}{10}}\left(\frac{8i}{100n\pi}-\frac{16i}{n^3\pi^3}+\frac{8}{5n^2\pi^2}\right)+\frac{16i}{n^3\pi^3}\end{aligned}$$

Putting all the above together, we obtain F_n as

$$\begin{aligned}F_n &= \frac{1}{2}\int_0^{2/10} xe^{-in\frac{\pi}{2}x}dx - \frac{5}{2}\int_0^{2/10} x^2 e^{-in\frac{\pi}{2}x}dx\\ &= \frac{1}{2}\left[e^{-in\frac{\pi}{10}}\left(\frac{4}{n^2\pi^2}+\frac{2i}{5n\pi}\right)-\frac{4}{n^2\pi^2}\right]-\frac{5}{2}\left[e^{-in\frac{\pi}{10}}\left(\frac{8i}{100n\pi}-\frac{16i}{n^3\pi^3}+\frac{8}{5n^2\pi^2}\right)+\frac{16i}{n^3\pi^3}\right]\\ &= e^{-in\frac{\pi}{10}}\left(\frac{2}{n^2\pi^2}+\frac{i}{5n\pi}\right)-\frac{2}{n^2\pi^2}-e^{-in\frac{\pi}{10}}\left(\frac{20i}{100n\pi}-\frac{40i}{n^3\pi^3}+\frac{20}{5n^2\pi^2}\right)-\frac{40i}{n^3\pi^3}\\ &= e^{-in\frac{\pi}{10}}\left[\frac{2}{n^2\pi^2}+\frac{i}{5n\pi}-\frac{20i}{100n\pi}+\frac{40i}{n^3\pi^3}-\frac{4}{n^2\pi^2}\right]-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3}\\ &= e^{-in\frac{\pi}{10}}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3}\end{aligned}$$

Now

$$F_0 = \frac{2}{T}\int_0^T z(x)dx = \frac{1}{2}\int_0^{2/10}\left(x-5x^2\right)dx = \frac{1}{300}$$

Hence

$$\begin{aligned}\tilde{z}(x) &= \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right)\\ &= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{N}\left(e^{-in\frac{\pi}{10}}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-\frac{2}{n^2\pi^2}-\frac{40i}{n^3\pi^3}\right)e^{in\frac{\pi}{2}x}\right)\\ &= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{N} e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}\left(\frac{40i}{n^3\pi^3}-\frac{2}{n^2\pi^2}\right)-e^{in\frac{\pi}{2}x}\left(\frac{2}{n^2\pi^2}+\frac{40i}{n^3\pi^3}\right)\right)\\ &= \frac{1}{600} + \mathrm{Re}\left(\sum_{n=1}^{N}\frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi}{2}x-\frac{n\pi}{10}\right)}-\frac{2}{n^2\pi^2}e^{in\frac{\pi}{2}x}+\frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi}{2}x}\right)\end{aligned}$$

But $x = vt$, hence

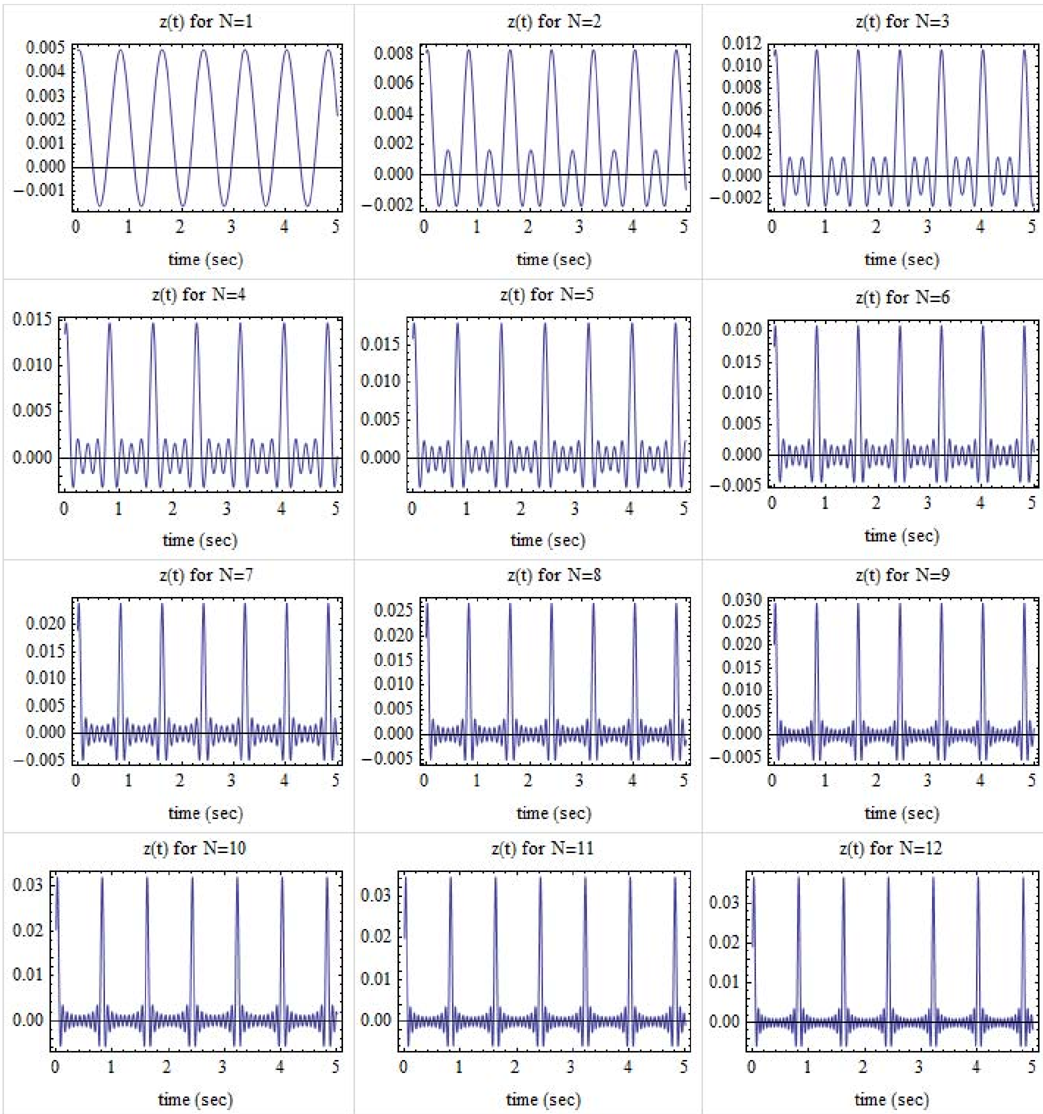
$$\tilde{z}(t) = \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{N} \frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi v}{2}t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi v}{2}t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{in\frac{\pi v}{2}t} + \frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi v}{2}t}\right)$$

Therefore the forcing frequency is $n\varpi_1 = n\frac{\pi v}{2}$ or from $2\pi f_1 = \frac{\pi v}{2}$, hence $f_1 = \frac{v}{4}$Hz.The above can be written as

$$\begin{aligned}
\tilde{z}(t) = \frac{1}{600} &+ \sum_{n=1}^{N} \operatorname{Re}\left(\frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(\frac{n\pi v}{2}t - \frac{n\pi}{10}\right)}\right) - \sum_{n=1}^{N} \operatorname{Re}\left(\frac{2}{n^2\pi^2}e^{i\left(\frac{n\pi v}{2}t - \frac{n\pi}{10}\right)}\right) \\
&- \sum_{n=1}^{N} \operatorname{Re}\left(\frac{2}{n^2\pi^2}e^{in\frac{\pi v}{2}t}\right) + \sum_{n=1}^{N} \operatorname{Re}\left(\frac{40}{n^3\pi^3}\frac{1}{i}e^{in\frac{\pi v}{2}t}\right) \\
&= \frac{1}{600} + \sum_{n=1}^{N} \frac{-40}{n^3\pi^3}\sin\left(n\varpi_1 t - \frac{n\pi}{10}\right) - \sum_{n=1}^{N} \frac{2}{n^2\pi^2}\cos\left(n\varpi_1 t - \frac{n\pi}{10}\right) \\
&- \sum_{n=1}^{N} \frac{2}{n^2\pi^2}\cos(n\varpi_1 t) + \sum_{n=1}^{N} \frac{40}{n^3\pi^3}\sin(n\varpi_1 t) \\
&= \frac{1}{600} - \frac{40}{\pi^3}\sum_{n=1}^{N} \frac{1}{n^3}\sin\left(n\varpi_1 t - \frac{n\pi}{10}\right) - \frac{2}{\pi^2}\sum_{n=1}^{N} \frac{1}{n^2}\cos\left(n\varpi_1 t - \frac{n\pi}{10}\right) \\
&- \frac{2}{\pi^2}\sum_{n=1}^{N} \frac{1}{n^2}\cos(n\varpi_1 t) + \frac{40}{\pi^3}\sum_{n=1}^{N} \frac{1}{n^3}\sin(n\varpi_1 t)
\end{aligned}$$

Where $\varpi_1 = \frac{\pi v}{2}$

To verify the above, here is a plot for different number of fourier series terms showing that approximation improves as N increases. This was done for $v = 5m/s$ and for 5 seconds.



2.1 Part(a)

The equation of motion is

$$\begin{aligned} my'' + c\left(y' - z'\right) + k\left(y - z\right) &= 0 \\ my'' + cy' + ky &= cz' + kz \end{aligned} \tag{2.1}$$

From earlier, we found that fourier series approximation to $z(t)$ is

$$\begin{aligned} z(t) &= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{-40}{n^3\pi^3}\frac{1}{i}e^{i\left(n\varpi t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{i\left(n\varpi t - \frac{n\pi}{10}\right)} - \frac{2}{n^2\pi^2}e^{in\varpi t} + \frac{40}{n^3\pi^3}\frac{1}{i}e^{in\varpi t}\right) \\ &= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{-40}{n^3\pi^3}e^{-i\frac{n\pi}{10}}\frac{1}{i}e^{in\varpi t} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}}e^{in\varpi t} - \frac{2}{n^2\pi^2}e^{in\varpi t} + \frac{40}{n^3\pi^3}\frac{1}{i}e^{in\varpi t}\right) \\ &= \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{\infty} e^{in\varpi t}\left[\frac{-40}{n^3\pi^3}e^{-i\left(\frac{n\pi}{10}+\frac{\pi}{2}\right)} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}} - \frac{2}{n^2\pi^2} + \frac{40}{n^3\pi^3}e^{-i\frac{\pi}{2}}\right]\right) \end{aligned}$$

Let

$$Z_n = \frac{-40}{n^3\pi^3}e^{-i\left(\frac{n\pi}{10}+\frac{\pi}{2}\right)} - \frac{2}{n^2\pi^2}e^{-i\frac{n\pi}{10}} - \frac{2}{n^2\pi^2} + \frac{40}{n^3\pi^3}e^{-i\frac{\pi}{2}}$$

Then above can be simplified to

$$z(t) = \frac{1}{600} + \operatorname{Re}\left(\sum_{n=1}^{\infty} e^{in\varpi t}Z_n\right)$$

Where $\varpi = \frac{\pi v}{2}$, hence

$$z'(t) = \operatorname{Re}\left(\sum_{n=1}^{\infty} in\varpi e^{in\varpi t}Z_n\right)$$

Hence, let

$$y_{ss}(t) = \operatorname{Re}\sum_{n=1}^{\infty} Y_n e^{in\varpi t}$$

Hence Eq 2.1 becomes

$$\begin{aligned} \sum_{n=1}^{\infty} - mn^2\varpi^2 Y_n e^{in\varpi t} + \sum_{n=1}^{\infty} icn\varpi Y_n e^{in\varpi t} + \sum_{n=1}^{\infty} kY_n e^{in\varpi t} &= \sum_{n=1}^{\infty} icn\varpi e^{in\varpi t}Z_n + \frac{k}{600} + \sum_{n=1}^{\infty} ke^{in\varpi t}Z_n \\ \sum_{n=1}^{\infty}\left(-mn^2\varpi^2 + icn\varpi + k\right)Y_n e^{in\varpi t} &= \sum_{n=1}^{\infty}(icn\varpi + k)Z_n e^{in\varpi t} + \frac{k}{600} \\ \sum_{n=1}^{\infty}\left(-mn^2\varpi^2 + icn\varpi + k\right)Y_n e^{in\varpi t} &= \frac{k}{600} + \sum_{n=1}^{\infty}(icn\varpi + k)Z_n e^{in\varpi t} \end{aligned}$$

Hence

$$Y_n = \frac{(icn\varpi + k)}{-m(n\varpi)^2 + icn\varpi + k}Z_n \tag{2}$$

Let

$$\begin{aligned}D(r_n,\zeta) &= \frac{icn\varpi + k}{-m(n\varpi)^2 + icn\varpi + k}\\ &= \frac{i2\zeta m\omega_{nat}n\varpi + \omega_{nat}^2 m}{-m(n\varpi)^2 + i2\zeta m\omega_{nat}n\varpi + \omega_{nat}^2 m}\\ &= \frac{i2\zeta n\frac{\varpi}{\omega_{nat}} + 1}{-\left(n\frac{\varpi}{\omega_{nat}}\right)^2 + i2\zeta n\frac{\varpi}{\omega_{nat}} + 1}\\ &= \frac{1 + i2\zeta r_n}{\left(1 - r_n^2\right) + i2\zeta r_n}\end{aligned}$$

Where in the above $r_n = \frac{n\varpi}{\omega_{nat}}$ where ϖ is $\frac{2\pi}{T}$ which means it is the fundamental frequency of the forcing function and ω_{nat} is the natural frequency.

Then Eq 2 becomes

$$Y_n = D(r_n,\zeta)Z_n$$

And the steady state solution $y_{ss}(t)$ becomes

$$y_{ss}(t) = \frac{k}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} D(r_n,\zeta)Z_n e^{in\varpi t}\right)$$

Now we can answer the question. When $c = 0$ then $D(r_n,\zeta)$ reduces to $\frac{k}{-m(n\varpi)^2+k} = \frac{1}{1-\left(n\frac{\varpi}{\omega_{nat}}\right)^2} = \frac{1}{1-r_n^2}$, hence

$$y_{ss}(t) = \frac{k}{600} + \mathrm{Re}\left(\sum_{n=1}^{\infty} \frac{1}{1 - r_n^2} Z_n e^{in\varpi t}\right)$$

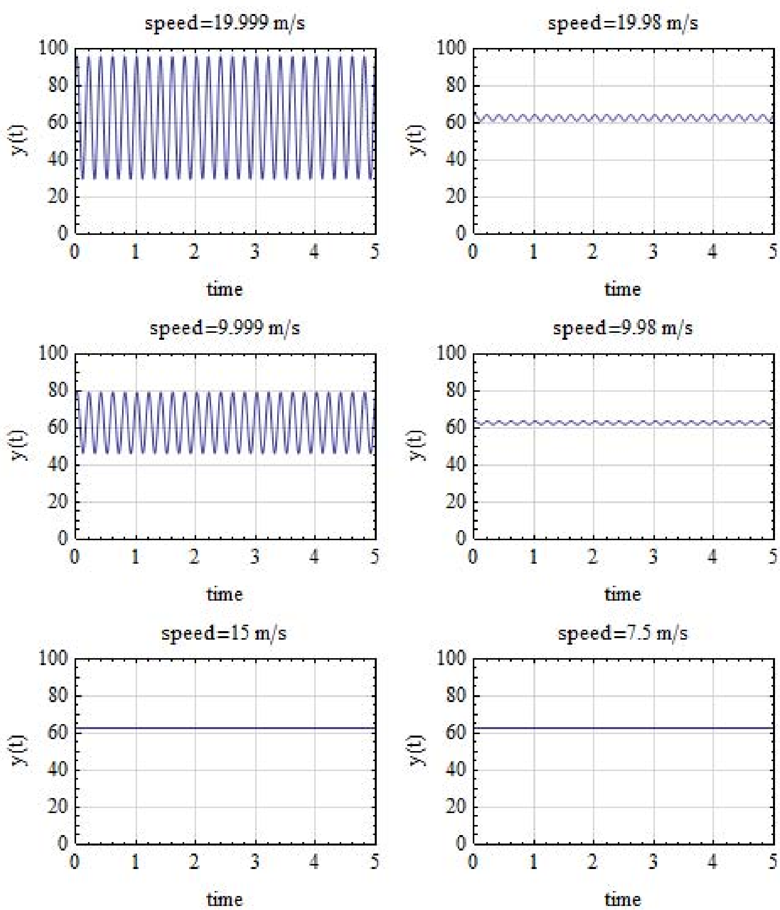
So the displacement $y_{ss}(t)$ will be resonant when $r_n = 1$ or $\frac{n\pi v}{2\omega_{nat}} = 1$ or $v = \frac{2\omega_{nat}}{n\pi}$

Hence

$$v = \frac{2(2\pi 5)}{n\pi} = \frac{20}{n}$$

Hence $v = 20, 10, 5, 2.5, 1.25, \cdots$ meter/sec will each cause resonance. To verify, here is a plot of $y_{ss}(t)$ with no damper for speed near resonance $v = 19.99$ and comparing this for speeds away from resonance speed. This plot shows that when speed v is close to any of the above speeds, then the displacement $y_{ss}(t)$ becomes very large. Once the speed is away from those values, then $y_{ss}(t)$ quickly comes down to steady state F/k value.

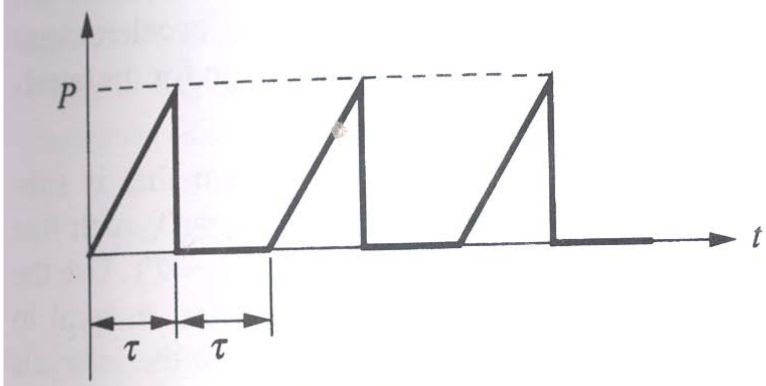
```
k = 10 π * 1200;
Grid[
 Partition[
  Plot[y[t, k, 10, #], {t, 0, 5}, PlotRange → {{0, 5}, {0, 100}},
     AxesOrigin → {0, 0}, Frame → True, GridLines → Automatic,
     GridLinesStyle → LightGray,
     FrameLabel → {{"y(t)", None},
       {"time", Row[{"speed=", #, " m/s"}]}}] & /@
   {19.999, 19.98, 9.999, 9.98, 15, 7.5}, 2]]
```



3 problem 3

3.) (**20 points**) Find the steady-state response of the system in **Problems 3.45** and **3.46** from Ginsberg using FFT techniques. Perform your analysis with $\tau = \pi/(3\omega_n)$ as stated in the problem and also repeat the analysis for $\tau = 3\pi/\omega_n$. Which harmonic is dominant in the response in each case? Why? Create a plot of the steady-state displacement for each case.

3.45 A one-degree-of-freedom, underdamped system having mass m, natural frequency ω_{nat}, and critical damping ratio $\zeta = 0.04$ is subjected to cyclical triangular pulse excitation, as shown below. What is the largest harmonic in the response when $\tau = \pi/3\omega_{\text{nat}}$?



EXERCISES 3.45, 3.46

3.46 Use FFT techniques to determine and graph the steady-state displacement and acceleration of the system in Exercise 3.45 for the parameters stated there.

The function is periodic with period $T = 2\tau$

$$f(t) = \begin{cases} \frac{P}{\tau}t & 0 < t < \tau \\ 0 & \tau < t < 2\tau \end{cases}$$

and $f(t \pm T) = f(t)$. Let $\tilde{f}(t)$ be the Fourier series approximation to $f(t)$, hence

$$\tilde{f}(t) = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right) \tag{3}$$

Where

$$\begin{aligned} F_n &= \frac{2}{T}\int_0^T f(t)e^{-in\frac{2\pi}{T}t}dt \\ &= \frac{2}{2\tau}\int_0^\tau \frac{P}{\tau}te^{-in\frac{\pi}{\tau}t}dt \\ &= \frac{P}{\tau^2}\int_0^\tau te^{-in\frac{\pi}{\tau}t}dt \end{aligned}$$

Using integration by parts $\int udv = uv - \int vdu$, letting $u = t$ and $dv = e^{-in\frac{\pi}{\tau}t}$ then $v = \int e^{-in\frac{\pi}{\tau}t}dt = \frac{ie^{-in\frac{\pi}{\tau}t}}{n\frac{\pi}{\tau}}$ hence

$$\begin{aligned} F_n &= \frac{P}{\tau^2}\left[\left(t\frac{ie^{-in\frac{\pi}{\tau}t}}{n\frac{\pi}{\tau}}\right)_0^\tau - \frac{i}{n\frac{\pi}{\tau}}\int_0^\tau e^{-in\frac{\pi}{\tau}t}dt\right] \\ &= \frac{P}{\tau^2}\left[\left(\tau\frac{ie^{-in\frac{\pi}{\tau}\tau}}{n\frac{\pi}{\tau}}\right) - \frac{i}{n\frac{\pi}{\tau}}\left(\frac{e^{-in\frac{\pi}{\tau}t}}{-in\frac{\pi}{\tau}}\right)_0^\tau\right] \\ &= \frac{P}{\tau^2}\left[\left(\tau^2\frac{ie^{-in\pi}}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left(e^{-in\frac{\pi}{\tau}t}\right)_0^\tau\right] \\ &= \frac{P}{\tau^2}\left[\left(\tau^2\frac{ie^{-in\pi}}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left(e^{-in\pi} - 1\right)\right] \end{aligned}$$

$e^{-in\pi} = \cos(n\pi) = (-1)^n$, hence

$$F_n = \frac{P}{\tau^2}\left[\left(\tau^2\frac{i(-1)^n}{n\pi}\right) + \frac{\tau^2}{n^2\pi^2}\left((-1)^n - 1\right)\right]$$

Hence for even n

$$\begin{aligned} F_n &= \frac{P}{\tau^2}\left[\left(\tau^2\frac{i}{n\pi}\right)\right] \\ &= P\frac{i}{n\pi} \end{aligned}$$

and for odd n

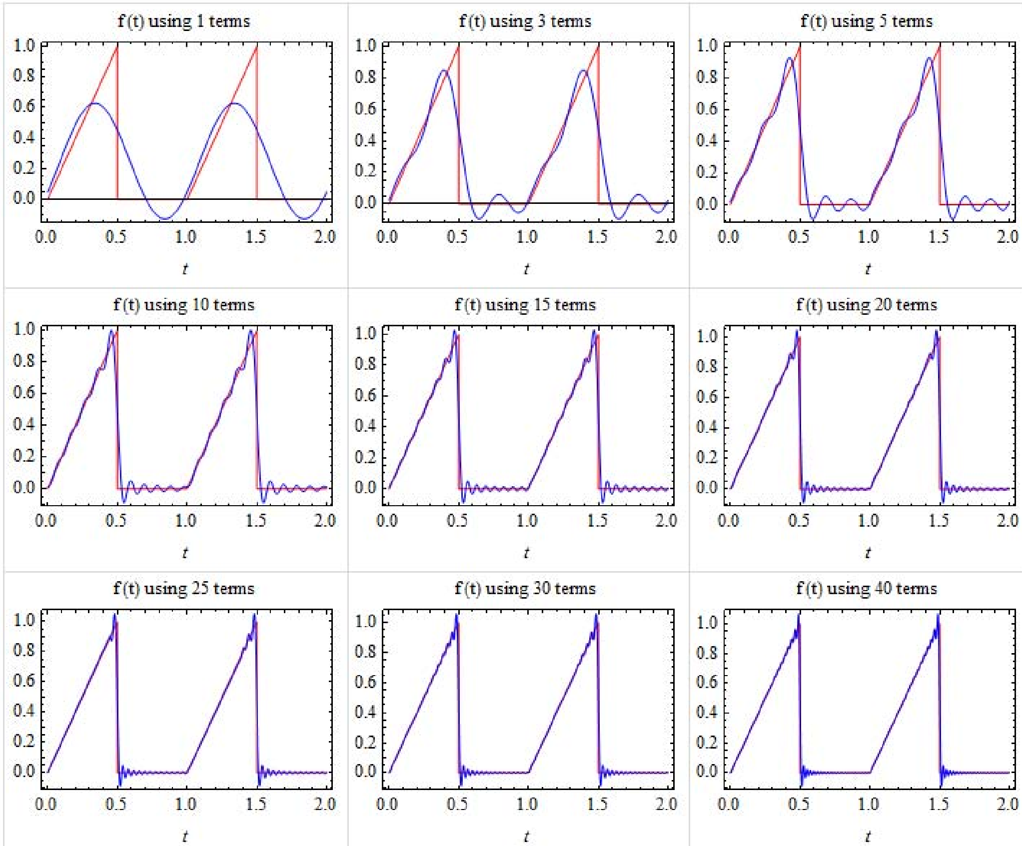
$$
\begin{aligned}
F_n &= \frac{P}{\tau^2}\left[\left(-\tau^2\frac{i}{n\pi}\right) - 2\frac{\tau^2}{n^2\pi^2}\right] \\
&= -\frac{P}{n\pi}\left(\frac{2}{n\pi} + i\right)
\end{aligned}
$$

$$
\begin{aligned}
F_0 &= \frac{P}{\tau^2}\int_0^\tau t dt \\
&= \frac{P}{\tau^2}\left(\frac{t^2}{2}\right)_0^\tau = \frac{P}{\tau^2}\left(\frac{\tau^2}{2}\right) \\
&= \frac{P}{2}
\end{aligned}
$$

Now Eq 3 becomes

$$
\begin{aligned}
\tilde{f}(t) &= \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{n=1}^{N} F_n e^{in\frac{2\pi}{T}x}\right) \\
&= \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{even\ n} F_n e^{in\frac{2\pi}{T}t} + \sum_{odd\ n} F_n e^{in\frac{2\pi}{T}t}\right) \\
&= \frac{p}{4} + \operatorname{Re}\left(\sum_{even\ n} P\frac{i}{n\pi}e^{in\frac{2\pi}{T}t} + \sum_{odd\ n} -\frac{P}{n\pi}\left(\frac{2}{n\pi} + i\right)e^{in\frac{2\pi}{T}t}\right) \\
&= \frac{p}{4} + \operatorname{Re}\left(\frac{P}{\pi}\sum_{even\ n}\frac{i}{n}e^{in\frac{2\pi}{T}t} - \frac{P}{\pi}\sum_{odd\ n}\frac{1}{n}\left(\frac{2}{n\pi} + i\right)e^{in\frac{2\pi}{T}t}\right) \\
&= \frac{P}{4} + \operatorname{Re}\left(\frac{P}{\pi}\sum_{even\ n}\frac{i}{n}e^{in\frac{2\pi}{T}t} - \frac{P}{\pi}\sum_{odd\ n}\left(\frac{2}{n^2\pi} + \frac{i}{n}\right)e^{in\frac{2\pi}{T}t}\right)
\end{aligned}
$$

To verify, here is a plot of the above, using $P = 1$ and $\tau = 0.5$ sec for $t = 0 \cdots 2$ seconds. This shows as more terms are added, the approximation becomes very close to the function. At $N = 40$ the approximation appears very good.



Now we need to write $f(t)$ as sum of exponential to answer the question.

$$\tilde{f}(t)=\frac{1}{2}F_0+\text{Re}\left(\sum_{n=1}^{N}F_n e^{in\frac{2\pi}{T}x}\right)$$

where ϖ is the fundamental frequency of the force given by $\frac{2\pi}{T}=\frac{2\pi}{2\tau}=\frac{\pi}{\tau}$

Hence, let $y_{ss}=\sum_{n=-\infty}^{\infty}Y_n e^{in\varpi t}$, then

$$\text{Re}\left(m\sum_{n=-\infty}^{\infty}-(n\varpi)^2 Y_n e^{in\varpi t}+c\sum_{n=-\infty}^{\infty}in\varpi Y_n e^{in\varpi t}+k\sum_{n=-\infty}^{\infty}Y_n e^{in\varpi t}\right)=\frac{1}{2}F_0+\text{Re}\left(\sum_{n=1}^{N}F_n e^{in\frac{2\pi}{T}x}\right)$$

$$\sum_{n=-\infty}^{\infty}\left(-m(n\varpi)^2+icn\varpi+k\right)Y_n e^{in\varpi t}=\frac{1}{2}F_0+\text{Re}\left(\sum_{n=1}^{N}F_n e^{in\frac{2\pi}{T}x}\right)$$

Hence

$$Y_n = \frac{F_n}{k}\frac{1}{\left(1-\left(n\frac{\varpi}{\omega_{nat}}\right)^2\right)+i2\zeta n\frac{\varpi}{\omega_{nat}}}$$
$$= \frac{F_n}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}$$

Hence

$$y_{ss} = \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

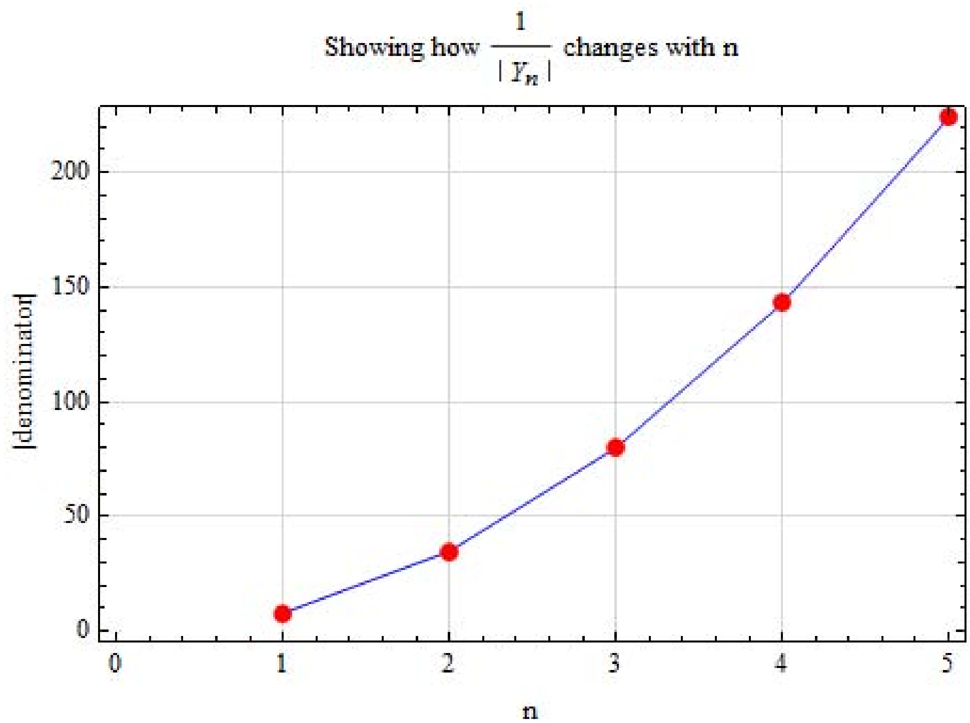
<u>Finding Y_n for $\tau = \frac{\pi}{3\omega_{nat}}$</u>

where $r = \frac{\varpi}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{\pi}{3\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{\pi}{3\omega_{nat}}\right)\omega_{nat}} = 3$, therefore

$$Y_n = \frac{F_n}{k}\frac{1}{\left(1-(3n)^2\right)+i6(0.04)n}$$
$$= \frac{F_n}{k}\frac{1}{\left(1-9n^2\right)+i0.24n}$$

The largest Y_n will occur when the denominator of the above is smallest. Plotting the modulus of the denominator $\sqrt{\left(1-9n^2\right)^2+(0.24n)^2}$ for different n values shows that $n = 1$ is the values which makes it minimum.

This happens since for any $n > 1$ the denominator will become larger due to n^2 and hence Y_n will become smaller. So $n = 1$ will be used.



For $n = 1$, we obtain

$$Y_1 = \frac{F_1}{k}\frac{1}{(1-9)+i6(0.04)}$$

But $F_1 = -\frac{P}{\pi}\left(\frac{2}{\pi}+i\right)$, hence

$$\begin{aligned}
Y_1 &= \frac{-\frac{P}{\pi}\left(\frac{2}{\pi}+i\right)}{k}\frac{1}{(1-9)+i6(0.04)} = \frac{-P}{\pi k}\frac{\left(\frac{2}{\pi}+i\right)}{-8+i0.24}\\
&= \frac{P}{\pi k}\frac{\frac{2}{\pi}+i}{8-i0.24} = \frac{P}{\pi k}\frac{\left(\frac{2}{\pi}+i\right)(8+i0.24)}{(8-i0.24)(8+i0.24)}\\
&= \frac{P}{\pi k}(0.075759+0.12727i)
\end{aligned}$$

Therefore

$$Y_1 = \frac{P}{k}(0.024115+0.0405i)$$

Here is a list of Y_n for $n = 1\cdots 10$ with the phase and magnitude of each (this was done for $\frac{p}{k} = 1$)

r=3, ζ=0.04			
n	Y_n	$\lvert Y_n \rvert$	Arg[Y_n]
1	0.0241149 + 0.0405122 i	0.0471462	59.2367
2	0.000062351 - 0.00454643 i	0.00454686	-89.2143
3	0.000269489 + 0.00132872 i	0.00135577	78.5348
4	3.73568×10^{-6} - 0.000556461 i	0.000556473	-89.6154
5	0.0000346626 + 0.000284391 i	0.000286496	83.0509
6	7.3223×10^{-7} - 0.000164243 i	0.000164245	-89.7446
7	9.00427×10^{-6} + 0.000103382 i	0.000103773	85.0223
8	2.31058×10^{-7} - 0.000069197 i	0.0000691974	-89.8087
9	3.29231×10^{-6} + 0.0000485919 i	0.0000487033	86.1239
10	9.45233×10^{-8} - 0.0000354069 i	0.000035407	-89.847

From the above we see that most of the energy in the response will be contained in Y_1 and adding more terms will not have large effect on the response shape. This is confirmed by the plot that follows.

Plot for the steady state

Since

$$y_{ss} = \frac{1}{2}F_0 + \mathrm{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

Where now $r = \frac{\varpi}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{\pi}{3\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{\pi}{3\omega_{nat}}\right)\omega_{nat}}$

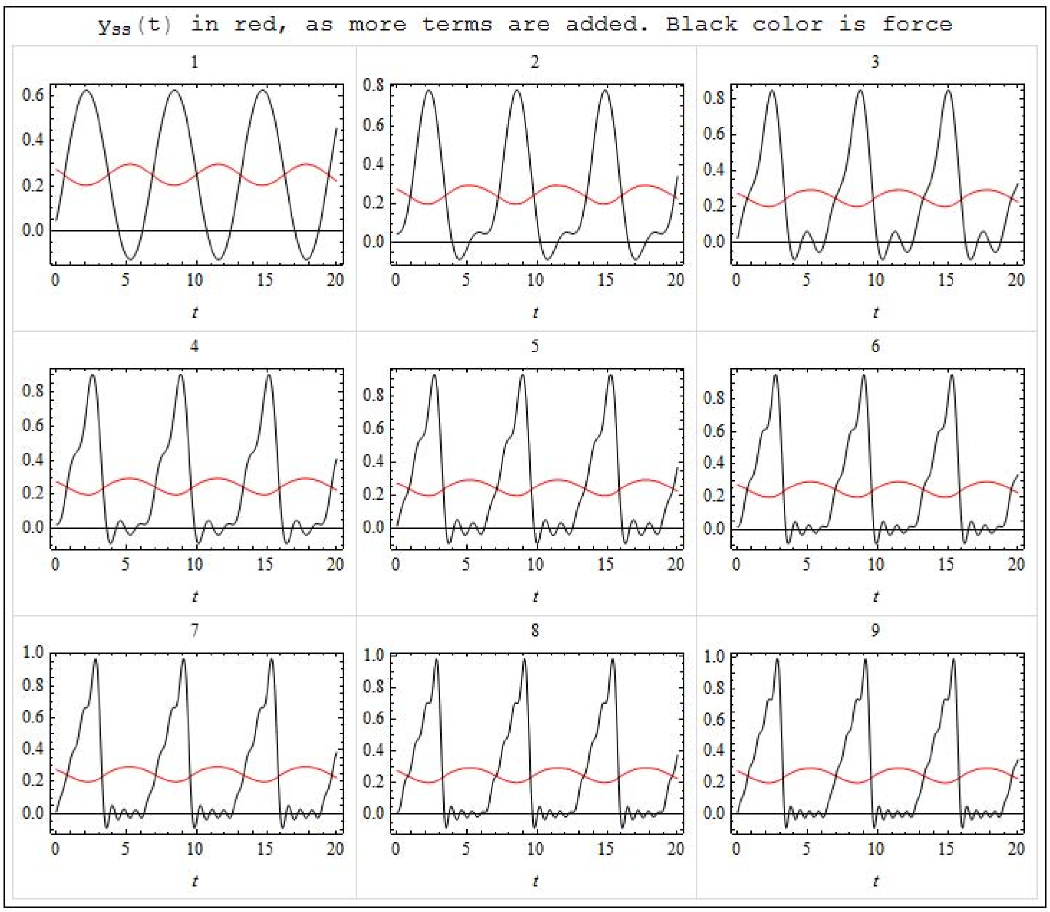
therefore $\boxed{r = 3}$

$$\begin{aligned}
y_{ss} &= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} Y_n e^{in\varpi t} + \sum_{n=2,4,6\cdots}^{\infty} Y_n e^{in\varpi t}\right) \\
&= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} \frac{F_{n_{odd}}}{k} \frac{1}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{F_{n_{even}}}{k} \frac{1}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t}\right) \\
&= \frac{p}{4} + \mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} \frac{-\frac{P}{n\pi}\left(\frac{2}{n\pi} + i\right)}{k} \frac{1}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{P\frac{i}{n\pi}}{k} \frac{1}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t}\right) \\
&= \frac{p}{4} + \frac{p}{k}\mathrm{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{\frac{1}{n\pi}\left(\frac{2}{n\pi} + i\right)}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t} + \sum_{n=2,4,6\cdots}^{\infty} \frac{\frac{i}{n\pi}}{\left(1-(nr)^2\right) + i2\zeta nr} e^{in\varpi t}\right)
\end{aligned}$$

Now let $r = 3$, $\zeta = 0.04$. Normalizing the equation for $\varpi = 1$ which implies $\tau = \pi$ and $k = 1$ and $p = 1$, then the above becomes

$$y_{ss} = \frac{1}{4} + \operatorname{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{\frac{1}{n\pi}\left(\frac{2}{n\pi} + i\right)}{\left(1-(3n)^2\right) + i2(0.04)3n} e^{int} + \sum_{n=2,4,6\cdots}^{\infty} \frac{\frac{i}{n\pi}}{\left(1-(3n)^2\right) + i2(0.04)3n} e^{int} \right)$$

Here is a plot of the above for $t = 0 \cdots 20$ seconds for different values of n



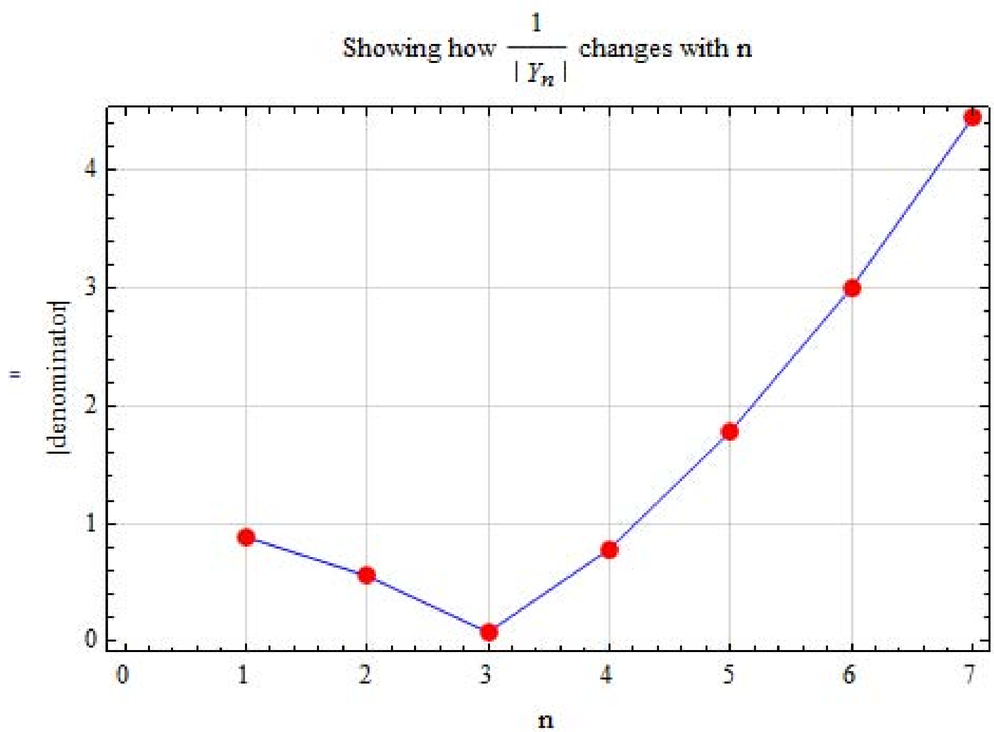
We see from the above plot, that $y_{ss}(t)$ does not change too much as more terms are added, since when $r = 3$, then Y_n for $n = 1$ contains most of the energy, hence adding more terms did not have an effect.

Repeating the calculations for $\tau = \frac{3\pi}{\omega_{nat}}$

$r=\frac{\varpi}{\omega_{nat}}$. When $\zeta=0.04$ and $\tau=\frac{3\pi}{\omega_{nat}}$, hence now $r=\frac{2\pi}{(2\tau)\omega_{nat}}=\frac{2\pi}{\left(2\frac{3\pi}{\omega_{nat}}\right)\omega_{nat}}=\frac{1}{3}$, therefore

$$
\begin{aligned}
Y_n &= \frac{F_n}{k}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}\\
&= \frac{F_n}{k}\frac{1}{\left(1-\left(\frac{1}{3}n\right)^2\right)+i\frac{2}{3}(0.04)n}\\
&= \frac{F_n}{k}\frac{1}{\left(1-\frac{n^2}{9}\right)+i0.0267n}
\end{aligned}
$$

The largest Y_n will occur when the denominator of the above is smallest. Similar to above, we can either find n which minimizes the denominator (by taking derivative and setting it to zero and solve for n) or we can make a plot and see how the function behaves. Making a plot shows this



From the above we see that the smallest value of the denominator happens when $n=3$.

so using $n = 3$ we find

$$Y_3 = \frac{F_3}{k}\frac{1}{\left(1-(3r)^2\right)+i2\zeta 3r}$$
$$= \frac{F_3}{k}\frac{1}{\left(1-\left(3\frac{1}{3}\right)^2\right)+i2(0.04)3\frac{1}{3}}$$
$$= \frac{F_3}{k}\frac{1}{i0.08}$$

But $F_n = -\frac{P}{n\pi}\left(\frac{2}{n\pi}+i\right)$, hence

$$F_3 = -\frac{P}{3\pi}\left(\frac{2}{3\pi}+i\right)$$

Therefore

$$Y_3 = \frac{-\frac{P}{3\pi}\left(\frac{2}{3\pi}+i\right)}{k}\frac{1}{i0.08}$$

Hence

$$Y_3 = \frac{p}{k}(-1.3263+0.28145i)$$

Here is a list of Y_n for $n = 1\cdots 10$ with the phase and magnitude of each (this was done for $\frac{p}{k} = 1$)

r=1/3, ζ=0.04			
n	Y_n	$\lvert Y_n \rvert$	$Arg[Y_n]$
1	-0.238501 - 0.350944 i	0.424316	-124.2
2	0.0272508 + 0.283863 i	0.285168	84.5164
3	-1.32629 + 0.281448 i	1.35582	168.019
4	0.0137726 - 0.100425 i	0.101365	-82.191
5	0.00186323 + 0.0359496 i	0.0359979	87.0331
6	0.000940465 - 0.0176337 i	0.0176588	-86.9471
7	0.0004999 + 0.0102524 i	0.0102646	87.2085
8	0.000227012 - 0.00650296 i	0.00650692	-88.0007
9	0.000179929 + 0.00442637 i	0.00443002	87.6722
10	0.0000829696 - 0.00314593 i	0.00314703	-88.4893

We see from the above that $\left|Y_3\right|$ is the largest harmonic.

Plot for the steady state

Since

$$y_{ss} = \frac{1}{2}F_0 + \operatorname{Re}\left(\sum_{n=1}^{\infty} Y_n e^{in\varpi t}\right)$$

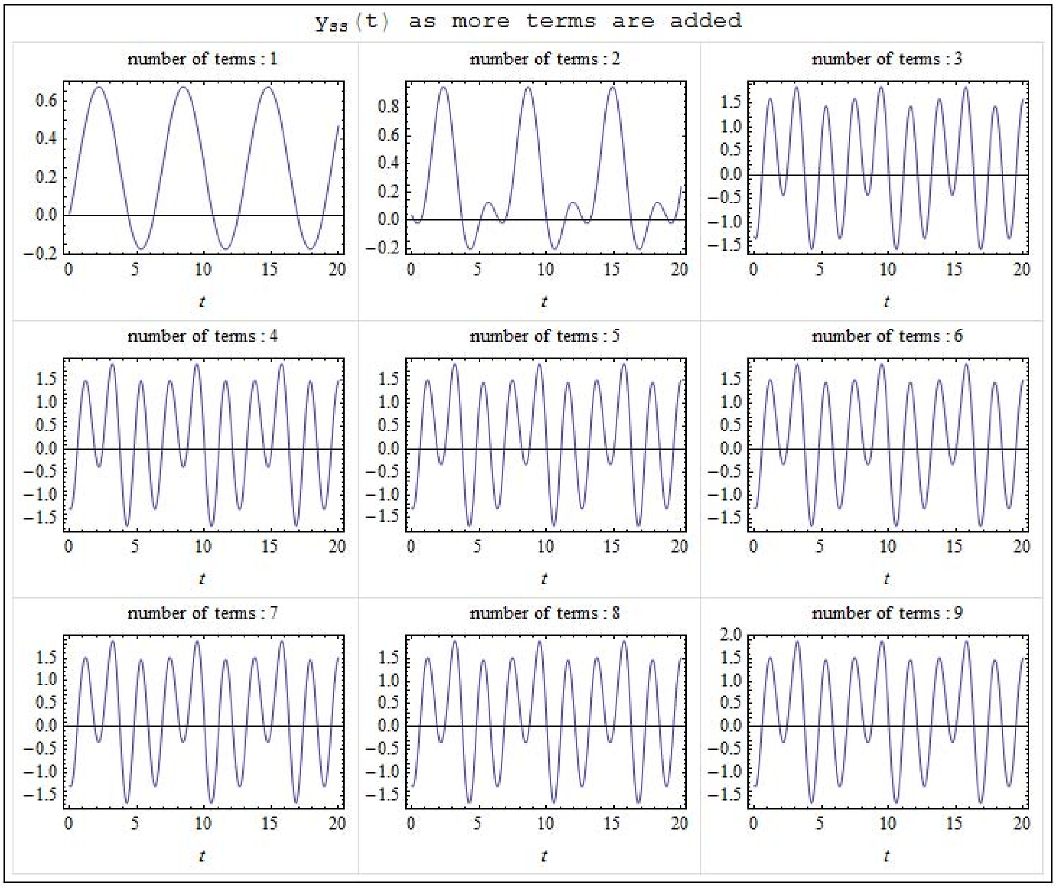
Where now $r = \frac{\varpi}{\omega_{nat}}$. When $\zeta = 0.04$ and $\tau = \frac{3\pi}{\omega_{nat}}$, hence now $r = \frac{2\pi}{(2\tau)\omega_{nat}} = \frac{2\pi}{\left(2\frac{3\pi}{\omega_{nat}}\right)\omega_{nat}} = \frac{1}{3}$, therefore from above

$$y_{ss} = \frac{p}{4} + \frac{p}{k}\operatorname{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{1}{n\pi}\left(\frac{2}{n\pi} + i\right)\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\varpi t} + \sum_{n=2,4,6\cdots}^{\infty}\frac{i}{n\pi}\frac{1}{\left(1-(nr)^2\right)+i2\zeta nr}e^{in\varpi t}\right)$$

Now let $r = \frac{1}{3}$, $\zeta = 0.04$, and assuming $\tau = 0.5$ then $\varpi = \frac{2\pi}{2\tau} = \frac{\pi}{0.5}$, and assuming $k = 1$, then the above becomes

$$y_{ss} = \frac{1}{4} + \frac{1}{k}\operatorname{Re}\left(\sum_{n=1,3,5\cdots}^{\infty} -\frac{1}{n\pi}\left(\frac{2}{n\pi} + i\right)\frac{1}{\left(1-\left(n\frac{1}{3}\right)^2\right)+i2(0.04)\frac{1}{3}n}e^{in\frac{\pi}{0.5}t}\right)$$
$$+ \frac{1}{k}\operatorname{Re}\left(\sum_{n=2,4,6\cdots}^{\infty}\frac{i}{n\pi}\frac{1}{\left(1-\left(n\frac{1}{3}\right)^2\right)+i2(0.04)\frac{1}{3}n}e^{in\frac{\pi}{0.5}t}\right)$$

Here is a plot of the above for $t = 0 \cdots 20$ seconds for different values of n



We see now that after $n = 3$ that the response did not change much by adding more terms, this is because more of the energy are contained in the first 3 harmonics with Y_n being the the largest.