# HW 4

# EMA 545
# Mechanical Vibrations

# Spring 2013
# University of Wisconsin, Madison

Nasser M. Abbasi

Spring 2013 Compiled on April 19, 2021 at 2:56am [public]

# Contents

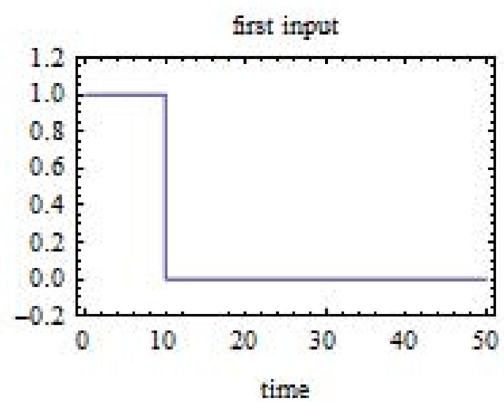
# 1 problem 1

**2.54** An undamped system is subjected to a constant force for an interval $T$, after which it begins to roll off exponentially. Specifically, $Q(t) = F_0$ if $0 < t < T$, $Q(t) = F_0 \exp[-\beta(t-T)]$ if $t > T$. Derive an expression for the response.

Assuming zero initial conditions. The input to the system is made up of two inputs. We find the response to the first input, then add this response to the response due to the second input. The first input is

$$\begin{aligned} u_1(t) &= F_0 h(t) - F_0 h(t-T) \\ &= F_0 (h(t) - h(t-T)) \end{aligned}$$

Which is a rectangular pulse of width $T$ starting at $t = 0$. For example for $T = 10$ sec. and $F_0 = 1$



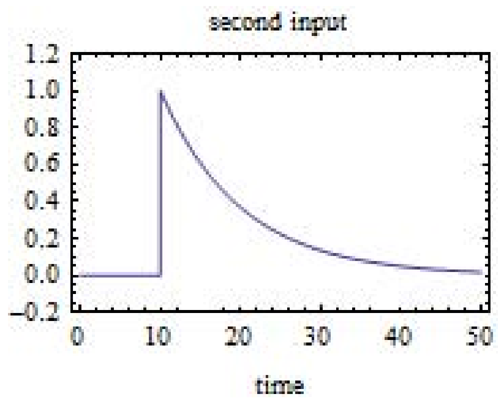

Assuming the response to unit step is $g_s(t)$ then the response to $u_1(t)$ is

$$g_1(t) = F_0 \left( \overbrace{g_s(t) h(t)} - \overbrace{g_s(t-T) h(t-T)} \right)$$

From appending B, $g_s(t) = \frac{1}{m\omega_n^2}(1-\cos(\omega_n t))$, hence the above becomes

$$g_1(t) = F_0\left(\overbrace{\frac{1}{m\omega_n^2}(1-\cos(\omega_n t))h(t)} - \overbrace{\frac{1}{m\omega_n^2}(1-\cos(\omega_n(t-T)))h(t-T)}\right) \tag{1}$$

Looking at the second input given by $u_2(t) = F_0 e^{-\beta(t-T)}h(t-T)$




From appendix B, the response to an exponential $F_0 e^{-\beta t}h(t)$ is

$$\frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta t} - \left(\cos(\omega_n t) - \frac{\beta}{\omega_n}\sin(\omega_n t)\right)\right)h(t)$$

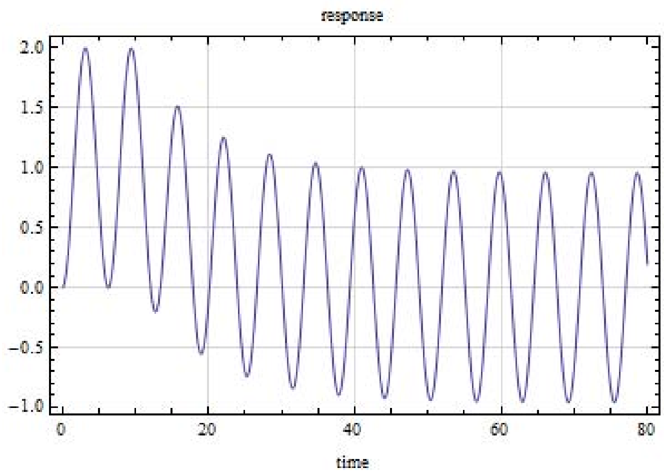
Therefore the response to $u_2(t)$ is

$$g_2(t) = \frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta(t-T)} - \left(\cos(\omega_n(t-T)) - \frac{\beta}{\omega_n}\sin(\omega_n(t-T))\right)\right)h(t-T) \tag{2}$$

Adding Eqs 1 and 2 results in the final response

$$\begin{aligned} g(t) &= g_1(t) + g_1(t) \\ &= F_0\left(\frac{1}{m\omega_n^2}(1-\cos(\omega_n t))h(t) - \frac{1}{m\omega_n^2}(1-\cos(\omega_n(t-T)))h(t-T)\right) + \\ &\frac{F_0}{m\left(\omega_n^2+\beta^2\right)}\left(e^{-\beta(t-T)} - \left(\cos(\omega_n(t-T)) - \frac{\beta}{\omega_n}\sin(\omega_n(t-T))\right)\right)h(t-T) \end{aligned}$$

For illustration, the following plot shows the response using some values. Using $m = 1$ kg, $\omega_n = 1$ rad/sec, $T = 10$ sec, $\beta = 1$, $F_0 = 1$ Volt.

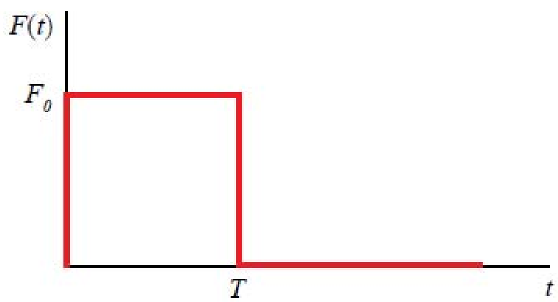

response
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
0
20
40
60
80
time

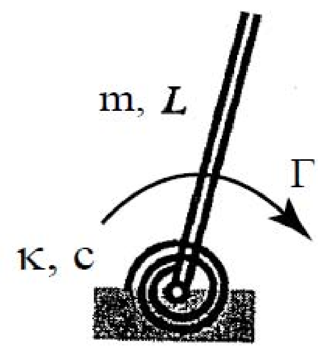
# 2 Problem 2

**Problem 2:** (**30 pts**, each part below is worth 10 pts).
The read head on a Hard Disk Drive (HDD) can be modeled as a pinned bar with a torsional spring at its base as shown below with L = 2 cm, m = 3 grams and $\kappa$ = 20 N/rad. The damping ratio for the system is $\zeta = 0.02$. The equation of motion for this system is: (later we will discuss how to find the EOM for a system like this)

$$\frac{1}{3}mL^2\ddot{\theta} + c\dot{\theta} + \kappa\theta = \Gamma(t)$$

A certain read operation involves applying a step torque $\Gamma(t) = F(t)$ with amplitude $F_0$ and duration T as shown below, where $F_0$ is the static torque required to displace the bar 30 degrees.




(a) Find the response of the system numerically over the time interval 0 < 1 < 10*T, with T = 2.5*$T_d$, where $T_d$ is the damped period of the system. Use a numerical procedure, preferably Matlab's "ode45" function together with a suitably modified version of eom_2_12.m, which is available on the class website.
(b) Assuming an underdamped response, write down a closed-form solution for the response in terms of Heaviside-step functions, and unit step responses, $q_s(t)$. Compare this with the response that you found numerically.
(c) Plot the displacement as a function of time for the case where T = 3*$T_d$ and T = 2.5*$T_d$. What do you observe? Why is the residual vibration larger in the latter case? (Hint –an undamped version of your analysis in (b) may make this easier to see.)

## 2.1 part(a)

The differential equation is

$$\frac{1}{3}ML^2\theta''(t) + c\theta'(t) + k\theta(t) = F_0(h(t) - h(t-T)) \tag{3}$$

The initial conditions are not given, and assumed to be zero, therefore $\theta(0) = 0^\circ$ and $\theta'(0) = 0$ rad/sec. The system is underdamped, hence

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Let $T_d$, be the damped period of oscillation given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

To obtain an expression for $\omega_n$, Eq 3 is changed to a standard form $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{F_0(h(t)-h(t-T))}{\frac{1}{3}ML^2}$

$$\theta''(t) + \overbrace{\frac{3c}{ML^2}}^{2\zeta\omega_n}\theta'(t) + \overbrace{\frac{3k}{ML^2}}^{\omega_n^2}\theta(t) = \frac{F_0(h(t)-h(t-T))}{\frac{1}{3}ML^2} \tag{4}$$

Therefore

$$\omega_n^2 = \frac{3k}{ML^2}$$

Using $k = 20$ N/rad, $L = 0.02$ meter, $M = 0.003$ kg

$$\omega_n^2 = \frac{3(20)}{(0.003)(0.02)^2} = 5.0\times 10^7$$

or

$$\omega_n = \sqrt{5.0\times 10^7} = \boxed{7071 \text{ rad/sec}}$$

and

$$T_d = \frac{2\pi}{7071.1\sqrt{1-0.02^2}} = \boxed{0.8888 \text{ ms}}$$

Therefore

$$T = 2.5T_d = 2.5\times 0.88857 = \boxed{2.221 \text{ ms}}$$

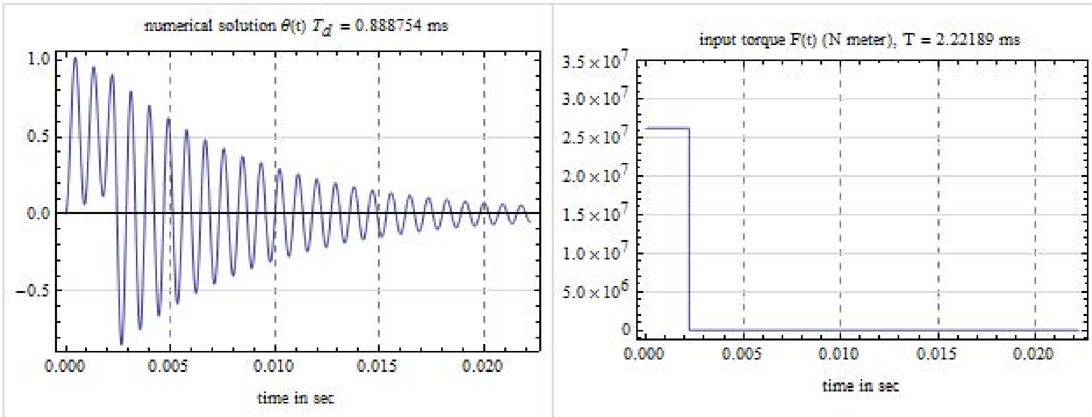
To find $F_0$ it is assumed the head was initially at rest. Therefore

$$\begin{aligned} F_0 &= k\theta_0 \\ &= 20\left(\frac{\pi}{6}\right) = \boxed{10.472 \text{ N-meter}} \end{aligned}$$

Eq 4 becomes

$$\begin{aligned} \theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) &= \frac{F_0(h(t)-h(t-2.5T_d))}{\frac{1}{3}ML^2} \\ \theta''(t) + 2(0.02)(7071)\theta'(t) + \left(5\times 10^7\right)\theta(t) &= \frac{3\times 20\left(\frac{\pi}{6}\right)(h(t)-h(t-2.5T_d))}{(0.003)(0.02)^2} \\ \theta''(t) + 283\theta'(t) + 5\times 10^7\theta(t) &= 2.618\times 10^7(h(t)-h(t-0.0022219)) \end{aligned}$$

This is solved numerically for $0 < t < 10T$ with the initial conditions $\theta(0) = 0^\circ$ and $\theta'(0) = 0$ rad/sec. Here is a plot of the solution and the input on a second plot.




A computational software was used to numerically solve the above differential equation for the solution $\theta(t)$.

```
params = {m → 0.003, L -> 0.02, ξ -> 0.02, k → 20};

ωn = Sqrt[3 k/(m L^2)]; ωd = ωn Sqrt[1 - ξ^2]; f0 = k π/6; Td = 2 π/ωd; T = 2.5 Td;

input = f0/((1/3) m L^2) (UnitStep[t] - UnitStep[t - 2.5 Td]);

eq = θ''[t] + 2 ξ ωn θ'[t] + (ωn)^2 θ[t] == input;
sol = θ[t] /. First@NDSolve[{eq /. params, θ'[0] == 0, θ[0] == 0}, θ[t], {t, 0, 10*T /. params}];

Grid[{
  {Plot[sol, {t, 0, 10*T /. params}, Frame → True, GridLines → Automatic,
    GridLinesStyle → {Dashed, LightGray},
    FrameLabel →
     {{None, None},
      {"time in sec", Row[{"numerical solution θ(t) Td = ", (Td /. params)*1000, " ms"}]}},
    ImageSize → 300],
   Plot[input /. params, {t, 0, 10*T /. params}, Frame → True, GridLines → Automatic,
    GridLinesStyle → {Dashed, LightGray}, PlotRange → {All, 3.5*10^7},
    FrameLabel →
     {{None, None}, {Row[{"time in sec"}],
       Row[{"input torque F(t) (N meter), T = ", (T /. params)*1000, " ms"}]}}, ImageSize → 300]
  }
 }, Spacings → {1, 1}, Frame → All, FrameStyle → LightGray]
```

## 2.2 Part(b)

From appendix B the response to underdamped second order system to a unit step $u(t)$ is

$$q_s(t) = \frac{1}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right)h(t)$$

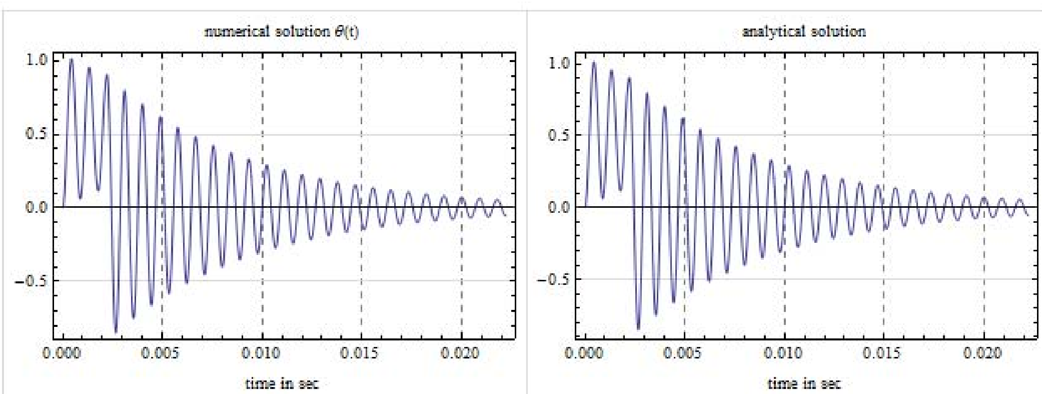
Hence the response $U(t)$ due to $\frac{F_0}{\left(\frac{1}{3}mL^2\right)}(h(t) - h(t-T))$ is given by

$$U(t) = \frac{F_0}{\left(\frac{1}{3}L^2\right)}\left(\overbrace{q_s(t)h(t)} - \overbrace{q_s(t-T)h(t-T)}\right)$$

Notice the factor $\frac{F_0}{\frac{1}{3}L^2}$. This was used since appendix B solution on based on equation of motion $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{1}{m}$ while in this case, the equation of motion is $\theta''(t) + 2\zeta\omega_n\theta'(t) + \omega_n^2\theta(t) = \frac{F_0}{\frac{1}{3}mL^2}$, hence a factor of $\frac{F_0}{\frac{1}{3}L^2}$ is needed to scale the solution. Therefore the analytical solution is

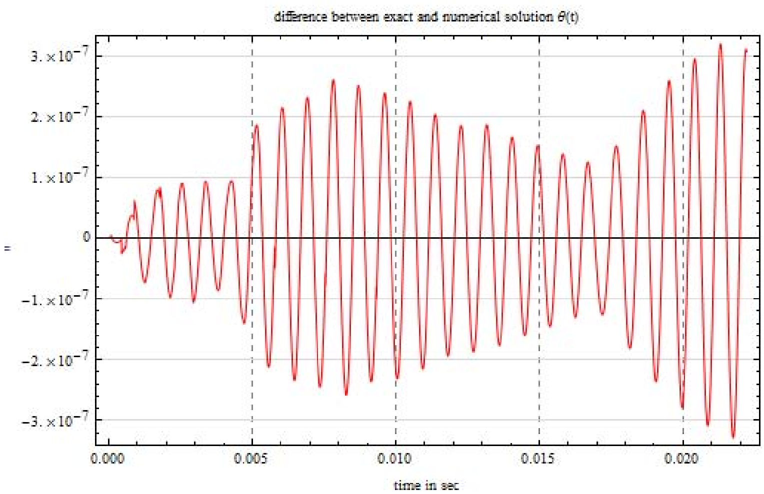
$$\begin{aligned} U(t) = & \frac{3F_0/L^2}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right)\right)h(t) \\ & - \frac{3F_0/L^2}{M\omega_n^2}\left(1 - e^{-\zeta\omega_n(t-2T)}\left(\cos(\omega_d(t-T)) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d(t-T))\right)\right)h(t-T) \end{aligned}$$

To compare this solution with the numerical solution found in part(a), the two solutions are plotted side-by-side for the case $T = 2.5T_d$



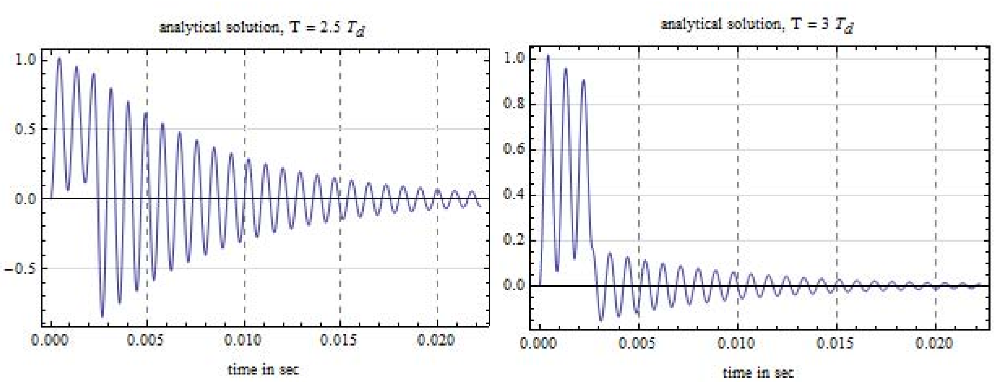

We see that solutions are in good approximate. Here is a plot of the difference. The error is in the order of $10^{-7}$

```
Plot[sol - analyticalSolution /. params, {t, 0, 10 * T /. params}, Frame → True, GridLines → Automatic,
 GridLinesStyle → {Dashed, LightGray},
 FrameLabel → {{None, None}, {"time in sec", "difference between exact and numerical solution θ(t)"}},
 ImageSize → 500, PlotStyle → Red]
```




## 2.3 Part(c)

The analytical solutions for $T = 2.5T_d$ and $T = 3.0T_d$ are




We see when the step load duration is $T = 2.5T_d$, the disk head will vibrate with larger amplitudes than when the step duration was $T = 3T_d$.

To understand the reason for this, analysis was done on the undamped version of the solution for part $b$

From appendix B the response to undamped second order system to a unit step $u(t)$ is

$$q_s(t) = \frac{1}{M\omega_n^2}(1 - \cos(\omega_n t))h(t)$$

Therefore the solution for $0 < t < T$ is $\frac{3F_0/L^2}{M\omega_n^2}(1 - \cos(\omega_n t))$. This means at $t = T$ which is when the step load is removed, $\theta(T) = \frac{3F_0/L^2}{M\omega_n^2}(1-\cos(\omega_n T))$ and $\theta'(T) = -\frac{3F_0/L^2}{M\omega_n^2}(\omega_n \sin(\omega_n T))$.

For $t > T$, the load is not present any more and we have free vibration response but with the above initial conditions obtained at the end of the $T$. The solution to free vibration of an undamped system for $\tilde{t} = t - T \geq 0$ is given by

$$\begin{aligned}
\theta(\tilde{t}) &= \frac{\theta'(T)}{\omega_n} \sin \omega_n \tilde{t} + \theta(T) \cos \omega_n \tilde{t} \\
&= \frac{-\frac{3F_0/L^2}{M\omega_n^2}(\omega_n \sin(\omega_n T))}{\omega_n} \sin \omega_n \tilde{t} + \frac{3F_0/L^2}{M\omega_n^2}(1-\cos(\omega_n T)) \cos \omega_n \tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2} \sin(\omega_n T) \sin \omega_n \tilde{t} + \left(\frac{3F_0/L^2}{M\omega_n^2} - \frac{3F_0/L^2}{M\omega_n^2}\cos(\omega_n T)\right) \cos \omega_n \tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2} \sin(\omega_n T) \sin \omega_n \tilde{t} - \frac{3F_0/L^2}{M\omega_n^2} \cos(\omega_n T) \cos \omega_n \tilde{t} + \frac{3F_0/L^2}{M\omega_n^2} \cos \omega_n \tilde{t} \\
&= -\frac{3F_0/L^2}{M\omega_n^2}\left(\sin(\omega_n T) \sin \omega_n \tilde{t} + \cos(\omega_n T) \cos \omega_n \tilde{t}\right) + \frac{3F_0/L^2}{M\omega_n^2} \cos \omega_n \tilde{t} \qquad (5)
\end{aligned}$$

We have obtained a solution for the time after the step load was removed. We now investigate the result observed. We see that when $T$ is close to an integer multiple of the period of the system, where we call the period of the system $\tilde{T}$ to differentiate it from $T$, then

$$\sin\left(\omega_n n\tilde{T}\right) = \sin\left(\frac{2\pi}{\tilde{T}} n\tilde{T}\right) = \sin(n2\pi) = 0$$

Also

$$\cos\left(\omega_n n\tilde{T}\right) = \cos\left(\frac{2\pi}{\tilde{T}} n\tilde{T}\right) = \cos(n2\pi) = 1$$

Hence the response given by equation 5 becomes

$$\begin{aligned}
\theta(\tilde{t}) &= -\frac{3F_0/L^2}{M\omega_n^2} \cos \omega_n \tilde{t} + \frac{3F_0/L^2}{M\omega_n^2} \cos \omega_n \tilde{t} \\
&= 0 \qquad (6)
\end{aligned}$$

But if $T$ occurs at multiple of halves of the period of the system (for example, $T = 0.5\tilde{T}, 1.5\tilde{T}, 2.5\tilde{T}$, etc...) then now

$$\sin\left(\omega_n\left(n\frac{\tilde{T}}{2}\right)\right) \rightarrow \sin\left(\frac{2\pi}{\tilde{T}}\left(n\frac{\tilde{T}}{2}\right)\right) \rightarrow \sin(n\pi) \rightarrow 0$$

However

$$\cos\left(\omega_n n\frac{\tilde{T}}{2}\right) \rightarrow \cos\left(\frac{2\pi}{\tilde{T}} n\frac{\tilde{T}}{2}\right) \rightarrow \cos(n\pi) \rightarrow -1$$

We notice that the sign is now negative. This means equation 5 becomes

$$\begin{aligned}\theta(\tilde{t}) &= -\frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} - \frac{3F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \\ &= -\frac{6F_0/L^2}{M\omega_n^2}\cos\omega_n\tilde{t} \qquad (7)\end{aligned}$$

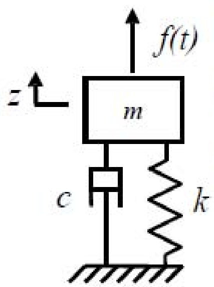
Comparing Eqs 6 and 7 we see that $\boxed{\text{when } T \text{ is an integer multiple of the period of the system}}$, then the response after $T$ is minimal (zero for the case on undamped)

While $\boxed{\text{when } T \text{ occurs at multiple of halves of the period of the system}}$,the response is large beyond the time $T$.

The above analysis was done for undamped system, but the same idea carries to the underdamped case. This explains why the response dies out quickly when $T = 3T_d$ while it was large when $T = 2.5T_d$

# 3 Problem 3

**Problem 3**: A SDOF system modeling a car bouncing on its suspension has $m$=1000kg, $k$=11 kN/m and $c$=660 N-s/m. The car is released from rest at t=0 with z(0)= –0.10m. It is possible to bring the car exactly to rest by exerting an impulsive force $f(t)$=F$_0\delta(t-T)$ at some instant $t=T$. (e.g. hitting it with a very large hammer at just the right instant). Find the magnitude of the impulse and the instant, $T$, at which it should be applied such the bouncing of the car stops completely after at least 2.0 seconds have elapsed but before 5.0 seconds have elapsed.




First lets look at the free vibration response (zero input response, called $u_{zi}$). The damping ratio $\zeta = \frac{c}{c_r} = \frac{c}{2\sqrt{km}} = \frac{660}{2\sqrt{11000\times 1000}} = 9.9499\times 10^{-2} \approx \boxed{0.1}$ and $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{1000}}$ hence $\omega_n = 3.162$ rad/sec, and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.1623\sqrt{1-0.1^2}$. Hence $\boxed{\omega_d = 3.146 \text{ rad/sec}}$. The damped period of the system is $T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{3.146} = \boxed{1.997}$ seconds and the natural period is $T_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.162} = \boxed{1.987}$ seconds.

Hence the system is underdamped and the solution is

$$u_{zi} = \text{Re}\left(\hat{A}e^{(i\omega_d - \zeta\omega_n)t}\right)$$

Where $\hat{A} = a + ib$ is the complex amplitude. At $t = 0$ we have

$$a = u_{zi}(0) = -0.1$$

and $u'_{zi}(0) \equiv u'_0 = \text{Re}((i\omega_d - \zeta\omega_n)(a+ib)) = -b\omega_d - a\zeta\omega_n$ therefore $b = \frac{-u'_0 - a\zeta\omega_n}{\omega_d}$. Since car was dropped from rest, then we take $u'_0 = 0$ which leads to $b = -\frac{(-0.1)(0.1)3.162}{3.146} = \boxed{0.1}$

Hence, since $a = u'_0(0) \equiv u_0$ and

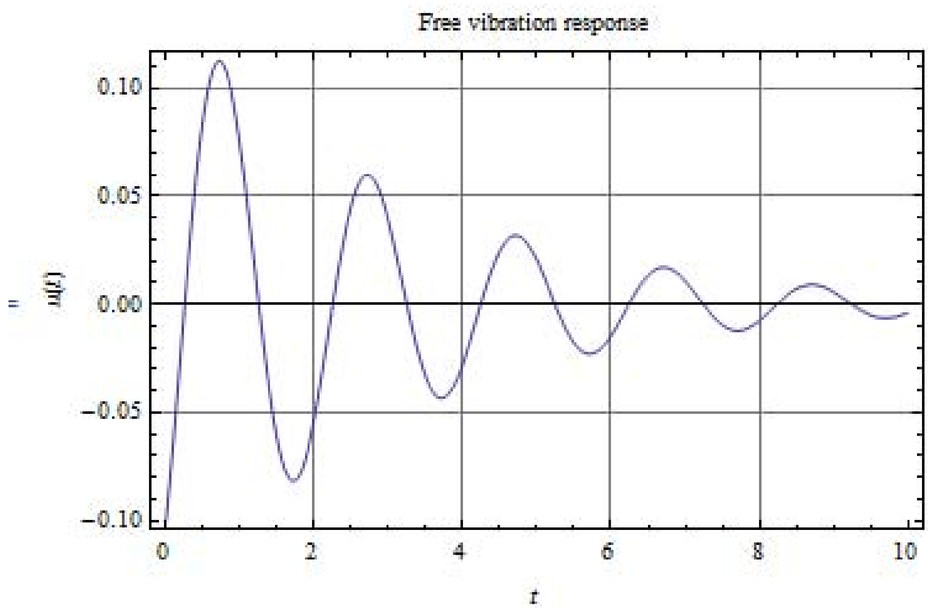
$$b = \frac{-u'_0 - a\zeta\omega_n}{\omega_d} = 0.1$$

then

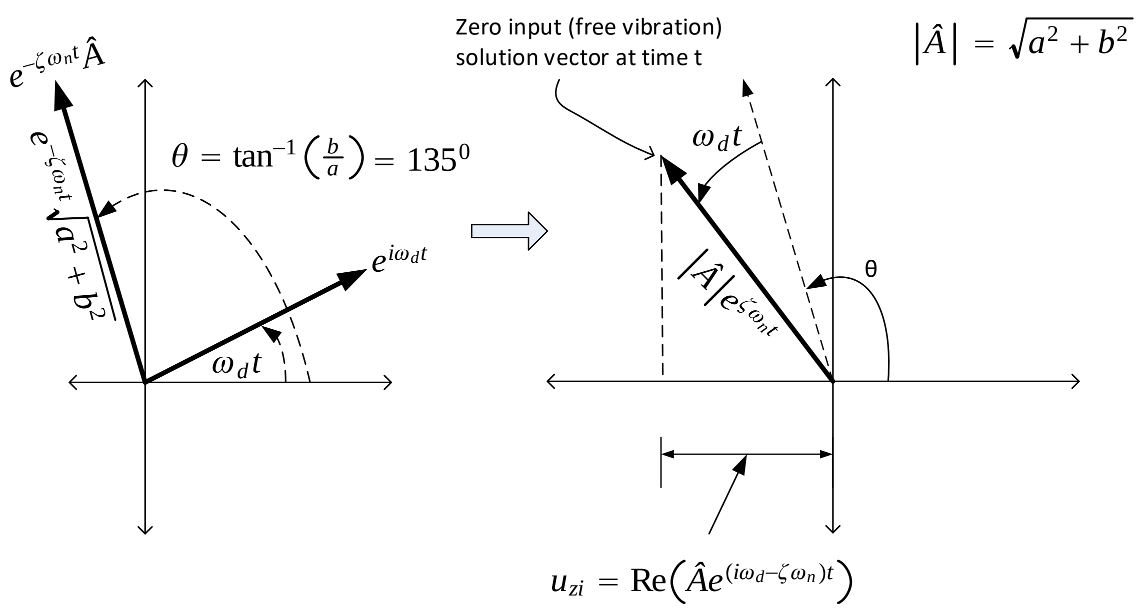
$$\begin{aligned} u_{zi}(t) &= \text{Re}\left((a - ib)e^{(i\omega_d - \zeta\omega_n)t}\right) \\ &= \text{Re}\left(e^{-\zeta\omega_n t}\left(u_0 - i\left(\frac{u'_0 + a\zeta\omega_n}{\omega_d}\right)\right)e^{i\omega_d t}\right) \\ &= e^{-\zeta\omega_n t}\left(u_0(0)\cos\omega_d t + \left(\frac{u'_0 + a\zeta\omega_n}{\omega_d}\right)\sin\omega_d t\right) \end{aligned} \tag{8}$$

For the numerical values gives, we now can plot this solution

$$u_{zi}(t) = e^{-0.1(3.162)t}(-0.1\cos 3.146t + 0.1\sin 3.146t)$$




The phase is given by $\tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{0.1}{-0.1}\right) = 2.356\ rad = 135^0$, In complex plane, $u_h(t)$ is




Now we add the zero initial conditions response, also called zero state response $u_{zs}$ for an input which is an impulse using appendix B.

$$u_{zs}(t) = e^{-\zeta\omega_n t}\left(\frac{F_0}{m\omega_d}\sin\omega_d t\right)h(t)$$

Hence $u_{zs}$ for an impulse that occurs at time $T$ is

$$u_{zs}(t) = e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T) \tag{9}$$

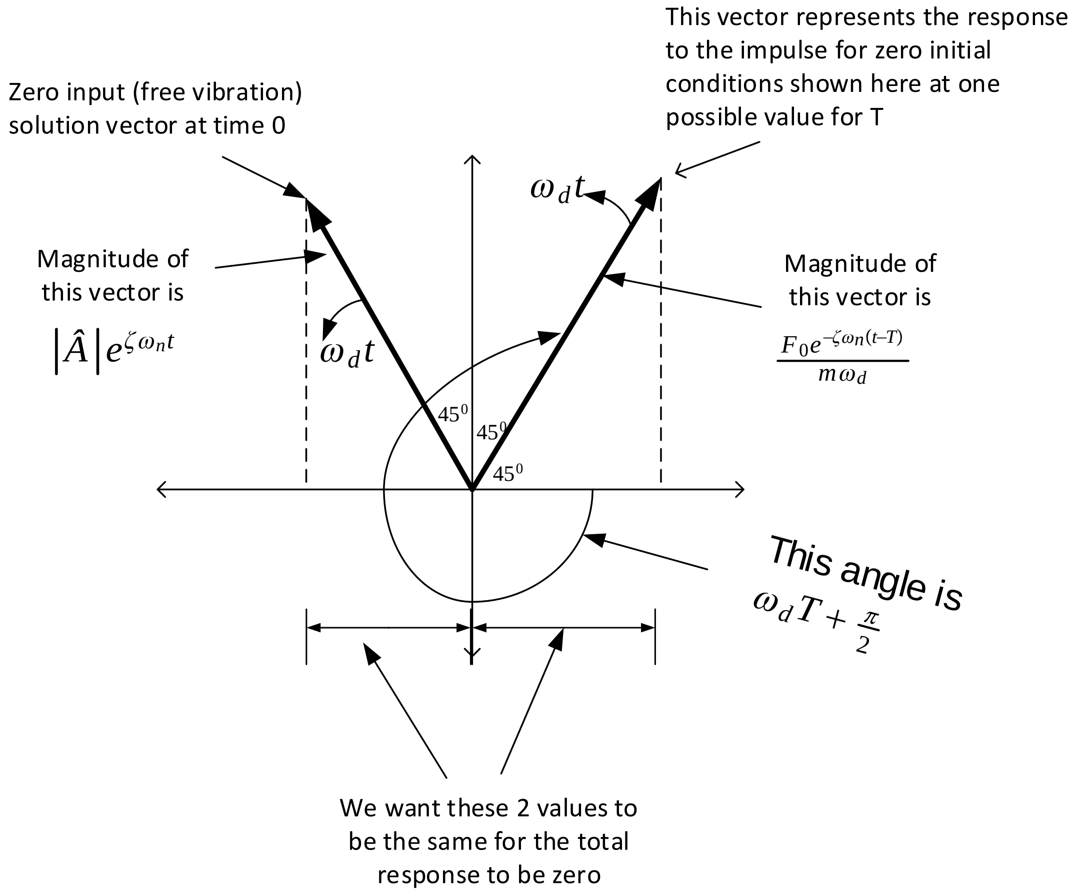
Hence the solution is found by combining Eq. 8 and Eq 9

$$\begin{aligned}u(t) &= u_{zi} + u_{zs}\\ &= e^{-\zeta\omega_n t}\left(u_0(0)\cos\omega_d t + \left(\frac{u_0' + a\zeta\omega_n}{\omega_d}\right)\sin\omega_d t\right)h(t) + e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T)\end{aligned}$$

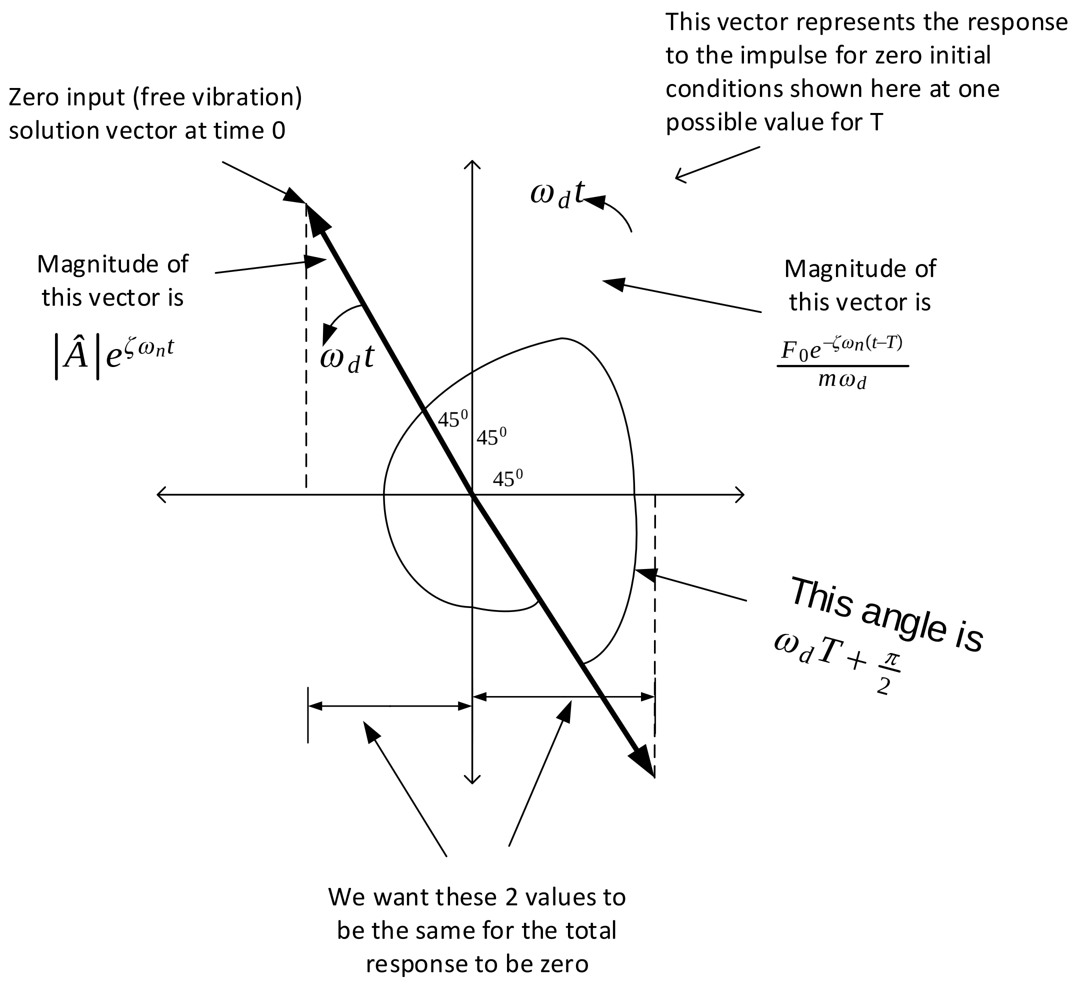
We need now to solve for $T$ and $F_0$ in order to meet the requirements that $u(t)$ should become zero between for $2 < t < 5$. To do this in the complex plane, we draw the zero state response as a vector

$$\begin{aligned}u_{zs} &= e^{-\zeta\omega_n(t-T)}\left(\frac{F_0}{m\omega_d}\sin\omega_d(t-T)\right)h(t-T)\\ &= \operatorname{Re}\left(\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}\frac{1}{i}e^{i\omega_d(t-T)}\right)h(t-T)\\ &= \operatorname{Re}\left(\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}e^{i\left(\omega_d(t-T)-\frac{\pi}{2}\right)}\right)h(t-T)\end{aligned}$$

Hence $u_{zs}$ vector has $\boxed{\text{magnitude } \frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d}}$ and phase $\boxed{\omega_d t - \omega_d T - \frac{\pi}{2}}$ Now to solve the problem of finding $T$ and $F_0$: To make the response become *zero* we need the magnitude of $u_{zs}$ to be equal but opposite in sign to the magnitude of $u_{zi}$ so that the projection on the x-axis cancel out (the projection on the x-axis of the vector is the real part which is the solution). Therefore, for the projection of $u_{zs}$ to be the same as the projection of $u_{zi}$ but of different sign, the following diagram shows all the possible $T$ values that allows this. We will pick the first $T$ value which is larger than 2 seconds to use.




From the above diagram, we need $\omega_d T + \frac{\pi}{2} = 2\pi - \frac{\pi}{4}$, hence $T = \frac{2\pi - \frac{\pi}{4} - \frac{\pi}{4}}{\omega_d} = \frac{\frac{3}{2}\pi}{\pi} = 1.5$ seconds. Hence this value of $T$ is not acceptable. We now look for the next possible $T$.




From the above diagram we see it will be $\omega_d T + \frac{\pi}{2} = 2\pi + \frac{\pi}{4}$ hence $T = \frac{2\pi+\frac{\pi}{4}-\frac{\pi}{2}}{\pi} = 1.75$ seconds. Hence this is still too early to apply the impulse. We look at the next possible case. We see that now we must rotate the vector all the way it was in the first diagram above to get the projection on the x-axis canceling the projection of the free vibration vector. Hence now the relation to solve for is

$$\omega_d T + \frac{\pi}{2} = 4\pi - \frac{\pi}{4}$$

Where in the above we added full $2\pi$ to the first case we considered above. This gives

$$T = \frac{4\pi - \frac{\pi}{4} - \frac{\pi}{2}}{\pi} = 3.25 \text{ sec}$$

.We have found $T$ which brings the system to halt after at least 2 seconds has elapsed. Now we find $F_0$ This is done by equating the amplitudes of the vectors as follows

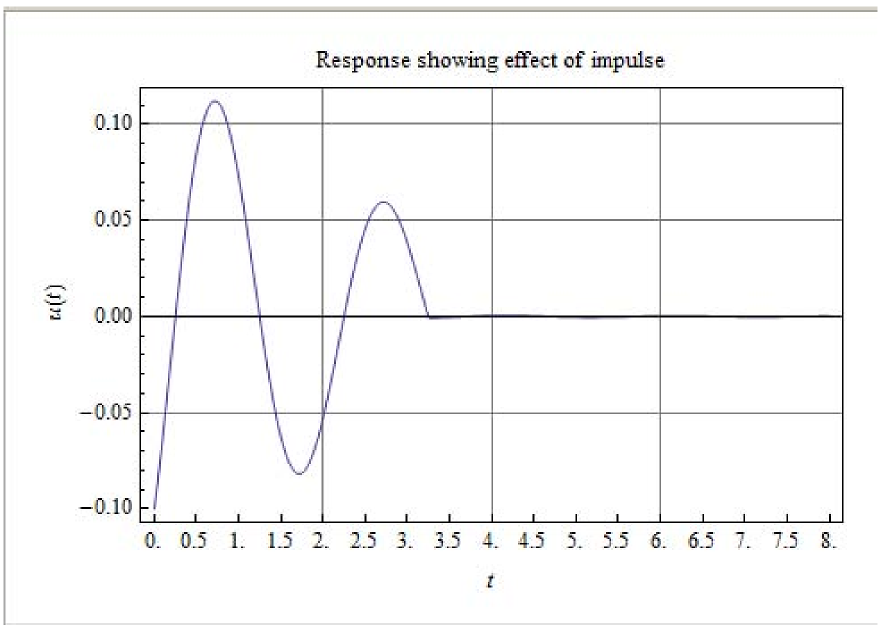
$$\frac{F_0 e^{-\zeta\omega_n(t-T)}}{m\omega_d} = e^{-\zeta\omega_n t}|\hat{A}|$$

Now for$t = T = 3.25$ second, plug-in numerical values

$$
\begin{aligned}
\frac{F_0}{1000(3.146)} &= e^{-(0.1)(3.162)(3.25)}\sqrt{0.1^2 + 0.1^2} \\
\frac{F_0}{3146.0} &= 5.0607 \times 10^{-2} \\
F_0 &= 159.21
\end{aligned}
$$

To verify, here is a plot of the response when the impulse hit with

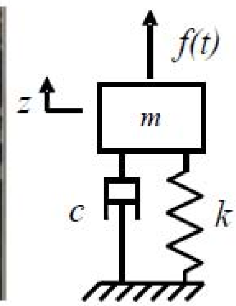
$$\boxed{F_0 = 159.21 \text{ N at } t = 3.25 \text{ seconds}}$$



# 4 Problem 4

**Problem 4**: Suppose that the bridge over University Avenue (pictured below) can be modeled as a simply-supported beam with length L=50 m. To simplify the analysis, let's assume that the beam has rectangular cross section with height 18 inches, width 4 feet and that it is constructed from steel with ρ=7800 kg/m^3 and E=210 GPa. (Note that the stiffness for various beam configurations is given in Figure 1.1 in the text.) Model this bridge as a SDOF system with an effective mass that is one third of the total mass of the beam and a stiffness equal to the stiffness of the beam when a static force is applied at its center. The damping ratio of the system is observed to be ζ=0.01.

Suppose that a single student jumping up and down on the bridge can exert a force *f*(*t*)=(1000 N)cos(ωt) where ω can be between 0 and 8π rad/s depending on how quickly he jumps up and down. How many students must jump on the bridge to cause a displacement amplitude of 50 cm? What frequency should they jump at to minimize the number of students required? (Don't worry, the actual bridge is stiffer and lighter than that given in the problem statement. Extra Credit: What would be more reasonable values for its mass and natural frequency? How does this change the solution?)


## 4.1 First part

Let $A$ be the area of the cross section and $\rho$ the mass density and $L$ the length, then actual mass is

$$\begin{aligned} m_{actual} &= LA\rho \\ &= 50(18 \times 0.0254)(4 \times 0.3048)(7800) \\ &= 217393 \text{ kg} \end{aligned}$$

Hence we will use

$$m = \frac{217393}{3} = 72464 \text{ kg}$$

The actual stiffness for a simply supported mean with loading at the center is $\frac{48EI}{L^3}$ where $I$ is the area moment of inertia. Hence

$$I = \frac{wh^3}{12} = \frac{(4 \times 0.3048)(18 \times 0.0254)^3}{12} = 0.00971 \text{ m}^4$$

Therefore the stiffness of the beam is

$$k = \frac{48EI}{L^3} = \frac{48\left(210\times10^9\right)(0.00971)}{50^3} = 783014\text{ N/m}$$

The natural frequency is

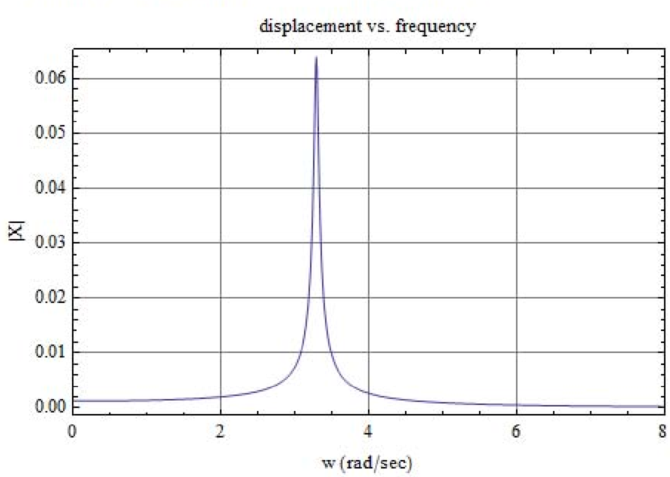
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{783014}{72464}} = 3.287\text{rad/sec}$$
$$f_n = 0.523\text{ Hz}$$

Therefore, assuming the loading is given by $F_0\cos(\bar{\omega}t)$ where $\bar{\omega}$ is the forcing frequency. The dynamic response at any time is given by

$$\left|\hat{X}\right| = \frac{F_0/k}{\sqrt{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

Where $r = \frac{\bar{\omega}}{\omega_n}$. We start by drawing $\left|\hat{X}\right|$ vs. $\bar{\omega}$ for the load of 1000 N by changing $\bar{\omega}$ from 0 to $8\pi$, Hence for a single student the displacement vs. forcing frequency is

```
parms = {f → 1000, k → 783 014, wn → 3.287, z → 0.01};
y[w_] := (f / k) / Sqrt[(1 - (w/wn)^2)^2 + (2 z w/wn)^2];
Plot[y[w] /. parms, {w, 0, 8 Pi}, PlotRange → {{0, 8}, All},
 GridLines → Automatic, Frame → True,
 FrameLabel → {{"|X|", None}, {"w (rad/sec)", "displacement vs. frequency"}}]
```



Hence we see that for one student, the maximum displacement is around 6 cm when the student is jumping at resonance frequency.

To answer the question of how many students are needed to cause $|X|$ to be 50 cm then that will depend on what forcing frequency is used. Now we will find the minimum number of students needed.

The minimum number will be when they all jump at the resonance frequency which is found from solving for $\bar{\omega}_{resonance}$ in

$$\begin{aligned}\frac{\bar{\omega}_{resonance}}{\omega_n} &= \sqrt{1-2\zeta^2}\\ \bar{\omega}_{resonance} &= \omega_n\sqrt{1-2\zeta^2}\\ &= 3.287\sqrt{1-2(0.01)^2}\\ &= 3.28667 \text{ rad/sec}\end{aligned}$$

Therefore, at this forcing frequency, we now solve for $F_0$ to determine the number of students

$$\begin{aligned}\left|\hat{X}\right| &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2}}\\ F_0 &= k\left|\hat{X}\right|\sqrt{\left(1-\left(\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}_{resonance}}{\omega_n}\right)^2}\\ &= (783014)(0.5)\sqrt{\left(1-\left(\frac{3.28667}{3.287}\right)^2\right)^2+\left(2(0.01)\frac{3.28667}{3.287}\right)^2}\\ &= 7829.75 \text{ N}\end{aligned}$$

Therefore we need at least $\boxed{\text{8 students}}$ all jumping at 3.287 rad/sec to cause a displacement of at least 50 cm.

# 5 Extra part

To make the structure avoid resonance, we need to make sure the ratio $\frac{\varpi}{\omega_n}$ stays away from one. This is the ratio of the forcing frequency to the natural frequency. One way is to make $\omega_n$ much larger than any expected $\varpi$ that can occur is typical use of this structure.

But to make $\omega_n = \sqrt{\frac{k}{m}}$ large, means either making $m$ small or making $k$ large. It is hard to reduce the mass of the structure. Therefore, making the structure more stiff will be a better solution.

The bridge can be made more stiff in many ways, such as by adding additional truss structure to it (assuming this will add minimal weight). For this example, suppose we double the stiffness. Hence $\omega_n = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2(783014)}{72464}} = 4.649\text{rad/sec}$.

Therefore now $\bar{\omega}_{resonance} = \omega_n\sqrt{1-2\zeta^2} = 4.649\sqrt{1-2(0.01)^2} = 4.65$ rad/sec. Now the same number of students (8) as before, jumping at same frequency of 3.28667 will cause displacement of

$$\begin{aligned}
\left|\hat{X}\right| &= \frac{8F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}}\\
&= \frac{8000/783014}{\sqrt{\left(1-\left(\frac{3.28667}{4.649}\right)^2\right)^2+\left(2(0.01)\frac{3.28667}{4.649}\right)^2}}\\
&= 0.02 \text{ meter}
\end{aligned}$$

Therefore by making the bridge twice as stiff, now the same 8 students at $\bar{\omega} = 3.287$ will cause only 2 cm displacement instead of 50 cm.

# 6 Problem 5

A radar display is to be tested by mounting it on spring-dashpot suspension and subjecting it to harmonic force $Q = F\cos(\bar{\omega}t)$. The mounted mass is 8 kg and $\zeta = 0.25$. A free vibration shows that damped natural frequency $f_d = 5$hz.It is observed that when the force is applied at very low frequency the displacement amplitude is 2 mm. The test is to be performed at 5.2 Hz. What will be the steady state response?

We are given are the following

$$\begin{aligned}
m &= 8 \text{ kg}\\
\zeta &= 0.25\\
\omega_d &= \omega_n\sqrt{1-\zeta^2} = 2\pi(5) \text{ rad/sec}\\
F_0/k &= 0.002 \text{ meter}\\
\bar{\omega} &= 2\pi(5.2) \text{ rad/sec}
\end{aligned}$$

Hence $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{2\pi(5)}{\sqrt{1-0.25^2}} = 32.446$ rad/sec. The steady state response is given by

$$u_{ss} = \operatorname{Re}\left(\hat{X}e^{i\bar{\omega}t}\right)$$

where $\hat{X} = |\hat{X}|e^{i\theta}$. Hence

$$
\begin{aligned}
|\hat{X}| &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}} \\
&= \frac{0.002}{\sqrt{\left(1-\left(\frac{2\pi(5.2)}{32.446}\right)^2\right)^2+\left(2(0.25)\frac{2\pi(5.2)}{32.446}\right)^2}} \\
&= 0.00397
\end{aligned}
$$

and

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \\
&= \tan^{-1}\left(\frac{2(0.25)}{0}\right) \\
&\tan^{-1}(\infty)
\end{aligned}
$$

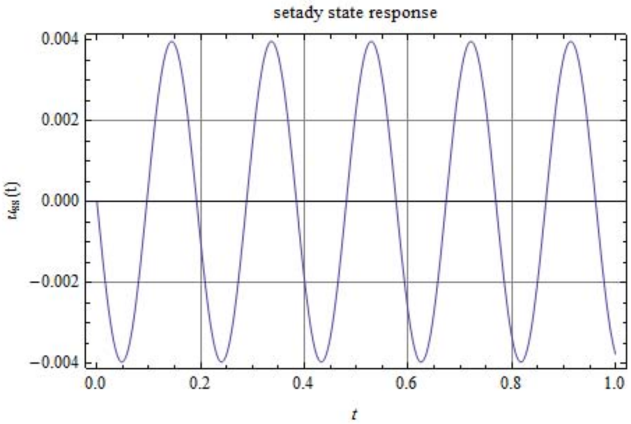
Since $0 \le \theta \le \pi$ then the phase is

$$
\theta = \frac{\pi}{2}
$$

Hence

$$
\begin{aligned}
u &= \mathrm{Re}\left(\hat{X}e^{i\bar{\omega}t}\right) \\
&= \mathrm{Re}\left(0.00397e^{i\frac{\pi}{2}}e^{i\bar{\omega}t}\right) \\
&= 0.00397\cos\left(\bar{\omega}t+\frac{\pi}{2}\right) \\
&= -0.00397\sin(\bar{\omega}t)
\end{aligned}
$$

```
Plot[-0.00397 Sin[2 * Pi * 5.2 t], {t, 0, 1}, Frame → True, GridLines → Automatic,
  FrameLabel → {{"uss(t)", None}, {t, "setady state response"}}]
```




# 7 Problem 6

A one degree of freedom system whose mass is 10 kg and whose natural frequency is 1 khz is subjected to a harmonic excitation 1.2 sin $\bar{\omega}t$ kN. The steady state amplitude when $\bar{\omega} = 1$ khz is observed to be 2.4 mm. Determine the steady state response at $\bar{\omega} = 0.95$ khz and $\bar{\omega} = 1.05$ khz.

We are given

$$
\begin{aligned}
m &= 10 \text{ kg} \\
\omega_n &= 2\pi(1000) \text{ rad/sec} \\
F_0 &= 1200 \text{ N} \\
|X| &= 2.4 \times 10^{-3} \text{ meter when } \bar{\omega} = \omega_n
\end{aligned}
$$

Since $\omega_n^2 = \frac{k}{m}$, hence $k = \omega_n^2 m = (2\pi(1000))^2(10)$, therefore

$$k = 3.949 \times 10^8 \text{ N/m}$$

Now when $\bar{\omega} = \omega_n$ we have

$$
\begin{aligned}
\left|\hat{X}\right| &= \frac{F_0/k}{\sqrt{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}} \\
2.4 \times 10^{-3} &= \frac{1200/\left(3.949 \times 10^8\right)}{\sqrt{(2\zeta)^2}} \\
&= \frac{3.039 \times 10^{-6}}{2\zeta}
\end{aligned}
$$

Hence

$$\zeta = \left(\frac{3.039\times 10^{-6}}{2\times 2.4\times 10^{-3}}\right) = 0.000633$$

## 7.1 Part (1)

when $\boxed{\bar{\omega} = 2\pi(950)}$ now $r = \frac{\bar{\omega}}{\omega_n} < 1$ hence dynamic magnification factor is positive. Therefore loading and displacement will be in phase with each others. (i.e. displacement is in same direction as force). Since the force is sin then the response will be sin with same frequency but different phase and amplitude. Hence let

$$u_{ss} = X\sin(\bar{\omega}t - \theta)$$

Where

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}} \\ &= \frac{1200/\left(3.949\times 10^8\right)}{\sqrt{\left(1-\left(\frac{2\pi(950)}{2\pi(1000)}\right)^2\right)^2 + \left(2(0.000633)\frac{2\pi(950)}{2\pi(1000)}\right)^2}} \\ &= 3.116\times 10^{-5} \text{ meter} \end{aligned}$$

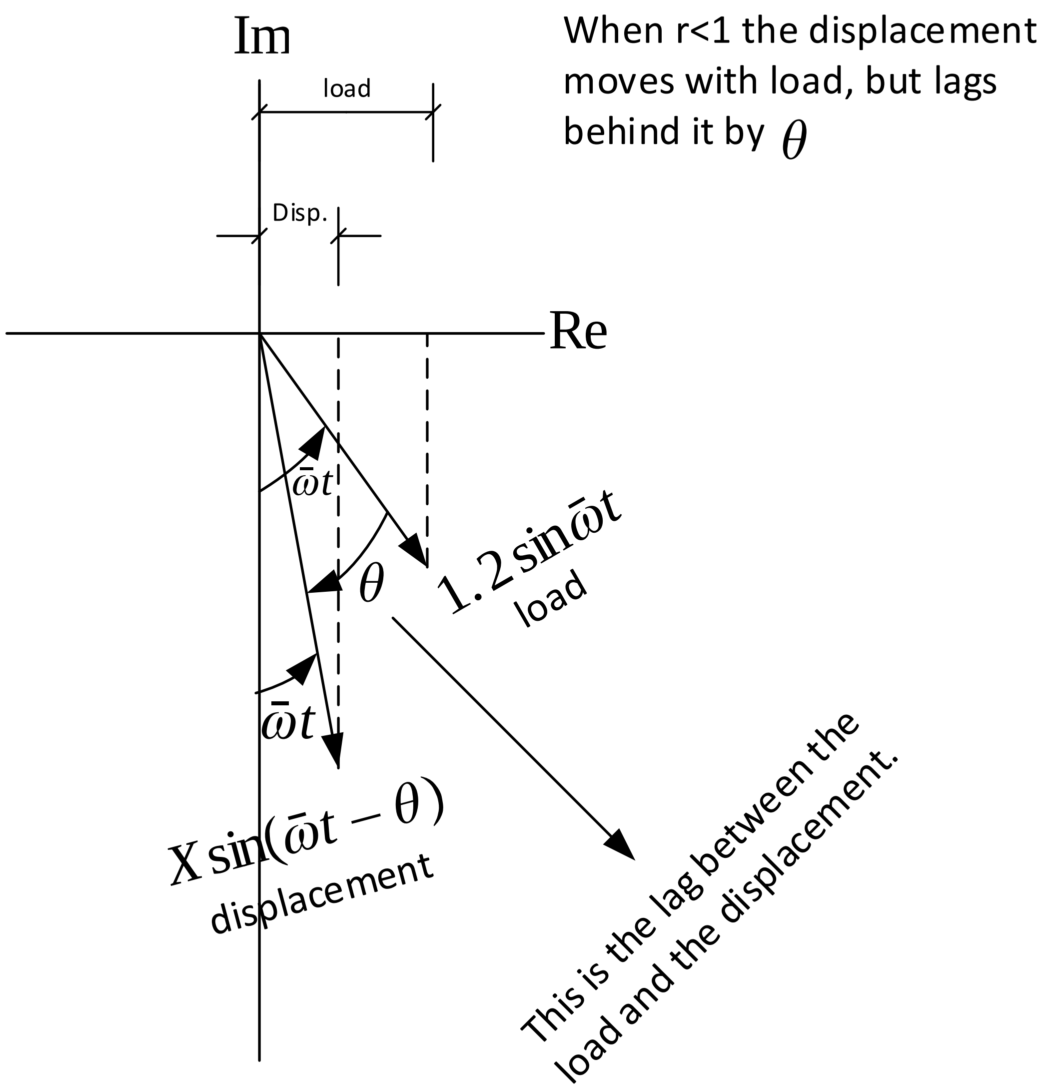
and

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \\ &= \tan^{-1}\left(\frac{2(0.000633)\frac{2\pi(950)}{2\pi(1000)}}{1-\left(\frac{2\pi(950)}{2\pi(1000)}\right)^2}\right) \\ &= \tan^{-1}\left(1.234\times 10^{-2}\right) \\ &= 0.01235 \text{ radians} \\ &= 0.71^\circ \end{aligned}$$

Hence steady state response is

$$u_{ss} = 3.116\times 10^{-5}\sin(\bar{\omega}t - 0.71^\circ)$$

Hence we see that the displacement is lagging the load by $0.71^\circ$. On complex plane it looks as follows




## 7.2 Part (2)

When $\boxed{\bar{\omega} = 2\pi(1050)}$ now $r = \frac{\bar{\omega}}{\omega_n} > 1$ hence dynamic magnification factor is negative. Therefore loading and displacement will be out of phase with loading. (i.e .displacement is in opposite direction to force). Doing the same calculations are done as above

$$u_{ss} = X\sin(\bar{\omega}t - \theta)$$

where $X$

$$
\begin{aligned}
X &= \frac{F_0/k}{\sqrt{\left(1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\bar{\omega}}{\omega_n}\right)^2}}\\
&= \frac{1200/\left(3.949\times 10^8\right)}{\sqrt{\left(1-\left(\frac{2\pi(1050)}{2\pi(1000)}\right)^2\right)^2+\left(2(0.000633)\frac{2\pi(1050)}{2\pi(1000)}\right)^2}}\\
&= 2.964\times 10^{-5}\text{ meter}
\end{aligned}
$$

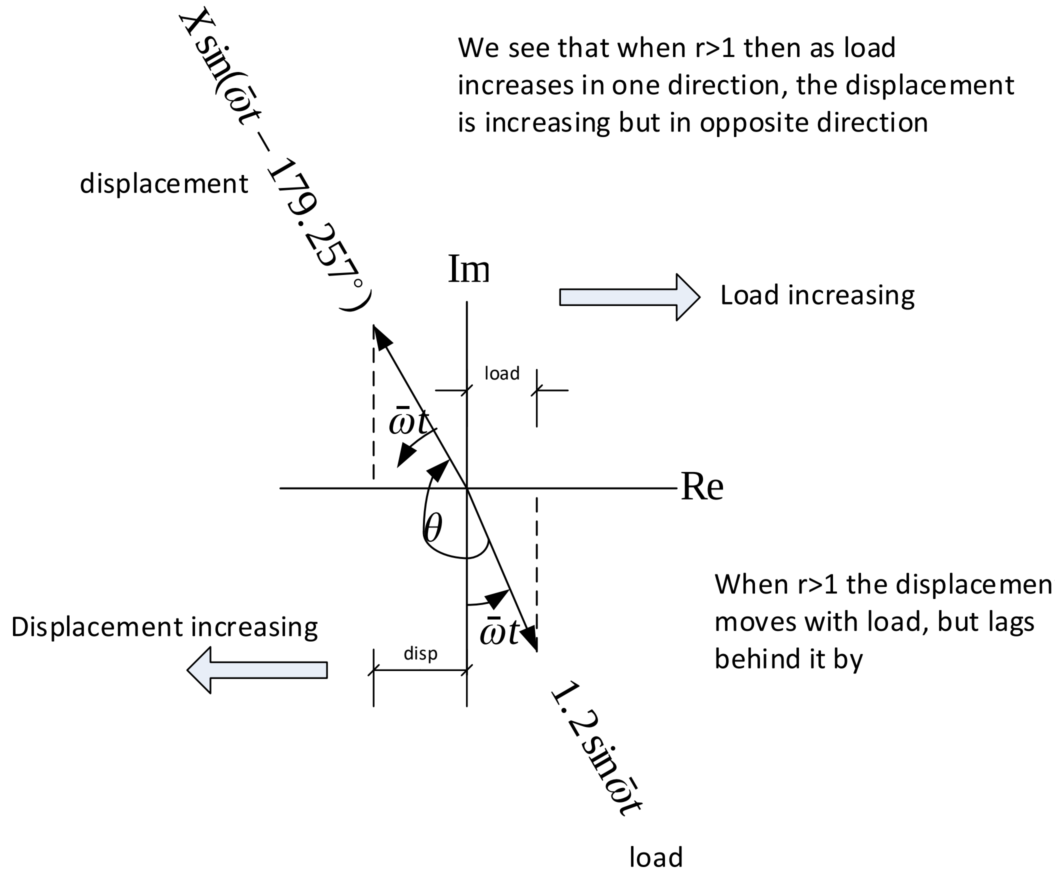
and

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)\\
&= \tan^{-1}\left(\frac{2(0.000633)\frac{2\pi(1050)}{2\pi(1000)}}{1-\left(\frac{2\pi(1050)}{2\pi(1000)}\right)^2}\right)\\
&= \tan^{-1}\left(\frac{0.0013293}{-0.1025}\right)\\
&= 3.12862\text{ radians}\\
&= 179.257^\circ
\end{aligned}
$$

Hence steady state response is

$$
u_{ss} = 2.964\times 10^{-5}\sin(\bar{\omega}t-179.257^\circ)
$$

On complex plane it looks as follows




Here is a plot by hand for the above 2 cases. First, the period that the loading is using $T = \frac{2\pi}{\varpi} = \frac{1}{950} = 1.0526 \times 10^{-3}\text{sec}$

$$T = 1.053 \text{ ms}$$