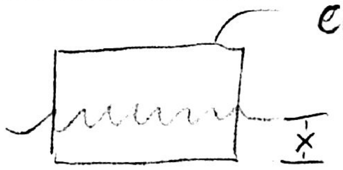
HW#2 → Floating Cube

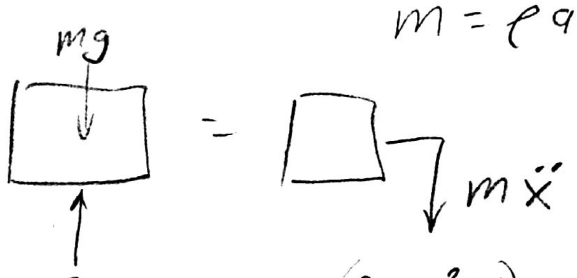


cube, side = a , ρ

Find EOM, nat. freq.
(submerged a distance x)

FBD:

$m = \rho a^3$



$F_b = m_w g = (\rho_w a^2 x) g$

$$F_b - mg = -m\ddot{x}$$

$$g(\rho_w a^2 x) - mg = -m\ddot{x}$$

$\longrightarrow$ static equilibrium $\longrightarrow \ddot{x} = 0$

$$x = \frac{mg}{\rho_w a^2 g} = \frac{\rho a^{\cancel{3}} \cancel{g}}{\rho_w \cancel{a^2} \cancel{g}} = \frac{\rho}{\rho_w} a = x_s$$

define new x from static eq.

$$\hat{x} = x - x_s \longrightarrow x = \hat{x} + x_s \qquad \ddot{\hat{x}} = \ddot{x}$$

$$\rho_w a^2 g(\hat{x} + x_s) - mg = -m\ddot{\hat{x}}$$

$$m\ddot{\hat{x}} + \rho_w a^2 g\hat{x} + \underbrace{\frac{\rho_w a^2 g \cdot \rho a}{\cancel{\rho_w}} - mg}_{mg - mg = 0} = 0$$

$$\boxed{m\ddot{\hat{x}} + \rho_w a^2 g\hat{x} = 0}$$

$$\omega_n^2 = \frac{\rho_w \cancel{a^2} g}{\rho a^{\cancel{3}}}$$

$$\boxed{\omega_n = \sqrt{\frac{\rho_w g}{\rho a}}}$$ units? $\sqrt{\frac{m/s^2}{m}} = 1/s = \text{rad}/s$ ✓

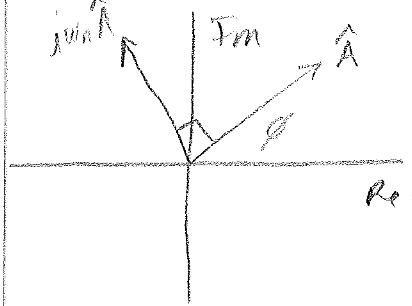
For a pine cube a = 10 cm, $\rho = 400\ kg/m^3$ $\rho_w = 1000\ kg/m^3$

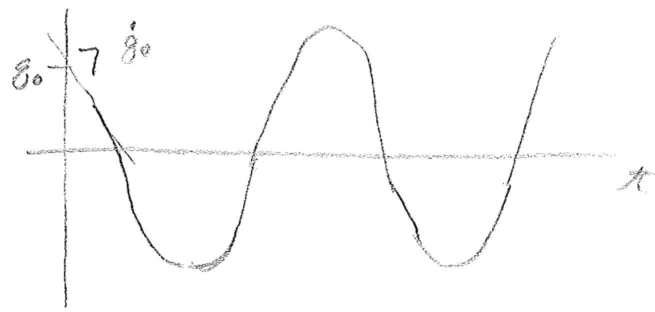
$$\boxed{\omega_n = 15.7\ \text{rad}/s \longrightarrow 2.5\ Hz}$$

$\delta(0) = 0.02\,m \qquad \dot{\delta}(0) = -50\,m/s$

$\omega_{nat} = 80 \cdot 2\pi$ rad/s

From IC's, $\delta_0 > 0$, $\dot{\delta}_0 < 0$





$\delta(t) = Re(\hat{A}e^{i\omega_n t})$

$\delta(0) = Re(\hat{A}) = 0.02\,m \quad \rightarrow \hat{A} = a + ib$

$a = 0.02\,m$

$\dot{\delta}(0) = Re(i\omega_n \hat{A} e^{i\omega_n \cdot 0}) = Re(i\omega_n(a+ib))$

$-b\omega_n = -50\,m/s \longrightarrow b = (50\,m/s / \omega_n)$

$b = \frac{5\cancel{0}\,m/s}{16\cancel{0}\pi} = \frac{5}{16\pi}$

$\hat{A} = 0.02\,m + i\frac{5}{16\pi} \longrightarrow \phi = \tan^{-1}\left(\frac{5/(16\pi)}{0.02}\right) = 1.37$

from graph, <u>max(δ)</u> when $\omega_n t = \pi - \phi$ (neg. extreme)

$\omega_n t = 2\pi - \phi$ (pos. maximum)

$t = \frac{n\pi}{\omega_n} - \frac{\phi}{\omega_n} = \frac{n\cancel{\pi}}{160\cancel{\pi}} - \frac{\phi}{160\pi} \rightarrow t = 9.77\,ms$

from graph, <u>max($\dot{\delta}$)</u> when $(i\omega_n \hat{A} e^{i\omega_n t}) \rightarrow$ Real axis

$\rightarrow \omega_n t = \pi - \pi/2 - \phi$ Negative

or

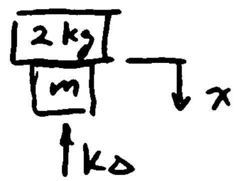
$2\pi - \pi/2 - \phi$ positive max

...

$\boxed{t = 6.64\,ms}$

✓ Plot works

Exercise 2.19



Given static $x = 0.05$ meter due to insertion of 2 kg block, ω_{nat} with 2 kg is 10π rad/s less than ω_{nat} without 2 kg. Find k & m.

Solution: Static displacement $= F/k \Rightarrow \dfrac{2(9.807)}{k} = 0.05$

Before 2 kg $(\omega_{nat})_0 = \left(\frac{k}{m}\right)^{1/2}$

After 2 kg $\omega_{nat} = \left(\frac{k}{m+2}\right)^{1/2} = (\omega_{nat})_0 - 10\pi$

or $\left(\frac{k}{m+2}\right)^{1/2} = \left(\frac{k}{m}\right)^{1/2} - 10\pi$

From static displacement $k = \dfrac{2(9.807)}{0.05} = 392.3$ N/m $\Leftarrow$

Solve $\left(\frac{392.3}{m+2}\right)^{1/2} = \left(\frac{392.3}{m}\right)^{1/2} - 10\pi \Rightarrow m = 0.1956$ kg $\Leftarrow$

HW#5 Extra Credit

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \qquad x(t) = Ae^{\lambda t}$$

Plug into ODE

$$(\lambda^2 A + 2\zeta\omega_n\lambda A + \omega_n^2 A)e^{\lambda t} = 0$$

want sol. valid for all time, and not $A = 0$ (trivial) so

$$(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2) = 0 \quad \text{— Two possible roots}$$

Quadratic formula:

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(1)(\omega_n^2)}}{2(1)} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2}$$

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$

Underdamped →

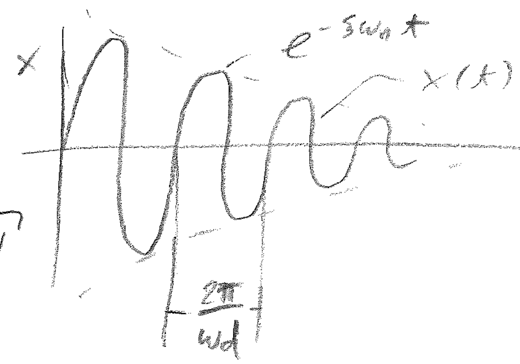
$$\boxed{\lambda = -\zeta\omega_n \pm i\omega_d \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}} \rightarrow \text{if underdamped i.e. } \zeta < 1$$

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

λ_1 and λ_2 are cc's, A_1 and A_2 must be conjugates as well to obtain a real $x(t)$

$$x(t) = A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_1^* e^{(-\zeta\omega_n - i\omega_d)t}$$

$$\boxed{x(t) = \mathrm{Re}(2A_1 e^{-\zeta\omega_n t} e^{i\omega_d t})}$$



Overdamped

$$\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}, \quad \lambda_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

Both λ_1 and λ_2 will be negative real #'s for $\zeta > 1$ since $\sqrt{\zeta^2-1} < \zeta$ for $\zeta > 1$.

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

more negative, faster decay to zero.

Exercise 2.29

Given q vs t

8 peak values $\Rightarrow$ 7 cycles

$$\delta = \frac{1}{7}\ln\left(\frac{x_1}{x_8}\right) \approx \frac{1}{7}\ln\left(\frac{16\,mm}{2\,mm}\right) = 0.297 \quad \Leftarrow$$

$$\zeta = \frac{\delta}{(4\pi^2+\delta^2)^{1/2}} = 0.04723 \quad \Leftarrow$$

16 zeros $\Rightarrow$ 15 half periods $\Rightarrow 15\left(\frac{\pi}{\omega_d}\right) = (0.795 - 0.045)$
estimates

so $\omega_d = \frac{15\pi}{0.75} = 20\pi = \omega_{nat}(1-\zeta^2)^{1/2}$

$\omega_{nat} = 62.90$ rad/s $\quad \Leftarrow$

Want max or min $|q| < 0.01$

$$\delta = \frac{1}{N}\ln\left(\frac{x_1}{x_{N+1}}\right) \Rightarrow 0.297 = \frac{1}{N}\ln\left(\frac{16}{0.01}\right)$$

$N = 24.8$ round up to nearest 0.5 $\Rightarrow N = 25$

Thus 25 damped periods beyond first peak

$$t > 0.007 + 25\left(\frac{2\pi}{\omega_d}\right) = 2.507 \text{ sec} \quad \Leftarrow$$

The number of cycles N for specified x_1 & x_{N+1} depends on $\delta = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}$, $\zeta = \frac{c}{2(KM)^{1/2}}$

If $c' = c$, $K' = 2K$ & $M' = \frac{1}{2}M \Rightarrow \zeta' = \frac{c'}{2(K'M')^{1/2}} = \zeta$

Thus $N' = N$.

However $\omega'_{nat} = \left(\frac{K'}{M'}\right)^{1/2} = 2\left(\frac{K}{M}\right)^{1/2} = 2\omega_{nat}$

Because $\zeta' = \zeta \Rightarrow \omega'_d = 2\omega_d$, so

$$t' > 0.007 + 25\left(\frac{2\pi}{\omega_d'}\right) = 1.257 \text{ sec} \quad \Leftarrow$$

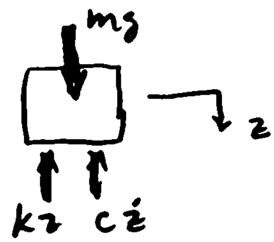
i.e. half the time from the first peak

To find $\dot{q}_0$ use eq. (2.2.24). Set $q = 0$ @ $t = 0.795$:

$$q_0 \cos[\omega_d(0.795)] + \frac{\dot{q}_0 + \zeta\omega_{nat} q_0}{\omega_d}\sin[\omega_d(0.795)] = 0$$

$$\dot{q}_0 = -\zeta\omega_{nat} q_0 - \omega_d q_0 \cot[\omega_d(0.795)] = -1.904 \text{ m/s} \quad \Leftarrow$$

Exercise 2.32



$$\Sigma F_z = mg - kz - c\dot{z} = m\ddot{z}$$

$$\ddot{z} + 2\zeta\omega_{nat}\dot{z} + \omega_{nat}^2 z = g$$

$$z = 0 \ \& \ \dot{z} = v \ @ \ t = 0$$

$$z = z_c + z_p$$

$$z = \exp(-\zeta\omega_{nat} t)\left[c_1\cos(\omega_d t) + c_2\sin(\omega_d t)\right] + \underbrace{\frac{g}{\omega_{nat}^2}}_{=\frac{mg}{k}}$$

Satisfy initial conditions

$$c_1 + \frac{mg}{k} = 0 \quad \& \quad -\zeta\omega_{nat} c_1 + \omega_d c_2 = v$$

$$c_1 = -\frac{mg}{k} \quad \& \quad c_2 = \frac{\zeta\omega_{nat}}{\omega_d}\left(-\frac{mg}{k}\right) + \frac{v}{\omega_d}$$

$$= \frac{v}{\omega_d} - \frac{\zeta}{(1-\zeta^2)^{1/2}}\frac{mg}{k}$$

$$z = \exp(-\zeta\omega_{nat} t)\left\{-\frac{mg}{k}\cos(\omega_d t) + \left[\frac{v}{\omega_d} - \frac{\zeta}{(1-\zeta^2)^{1/2}}\frac{mg}{k}\right]\sin(\omega_d t)\right\} + \frac{mg}{k} \quad \Leftarrow$$

$$F_{susp} = kz + c\dot{z} = m\left(\omega_{nat}^2 z + 2\zeta\omega_{nat}\dot{z}\right) \quad \Leftarrow$$

where $\dot{z} = \exp(-\zeta\omega_{nat} t)\left[(\omega_d c_2 - \zeta\omega_{nat} c_1)\cos(\omega_d t) - (\omega_d c_1 + \zeta\omega_{nat} c_2)\sin(\omega_d t)\right]$

Rebound if $F_{susp} < 0$ $\Leftarrow$

If $m = 1$ kg, $\omega_{nat} = 10\pi$ rad/s $\Rightarrow k = m\omega_{nat}^2 = 100\pi^2$ N/m

For $v = 4$ m/s, $c_1 = -9.937(10^{-3})$

$$c_2(\zeta) = \frac{1}{(1-\zeta^2)^{1/2}}\left[\frac{4}{10\pi} - \zeta(9.937)(10^{-3})\right]$$

Pick a ζ in the range $0 < \zeta < 1$, evaluate $F_{susp}(t, \zeta)$ for $0 < t < T_{nat}$ (maximum force will occur in first cycle, where $z(t)$ is largest). Identify max F_{susp} for each ζ value in the range of t.

$v := 4 \qquad \omega_{nat} := 10\cdot\pi \qquad m := 1 \qquad g := 9.807 \qquad \omega_d(\zeta) := \sqrt{1-\zeta^2}\cdot\omega_{nat}$

$$c_1 := \frac{-g}{\omega_{nat}^{\,2}} \qquad c_2(\zeta) := \frac{v}{\omega_d(\zeta)} - \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\frac{g}{\omega_{nat}^{\,2}}$$

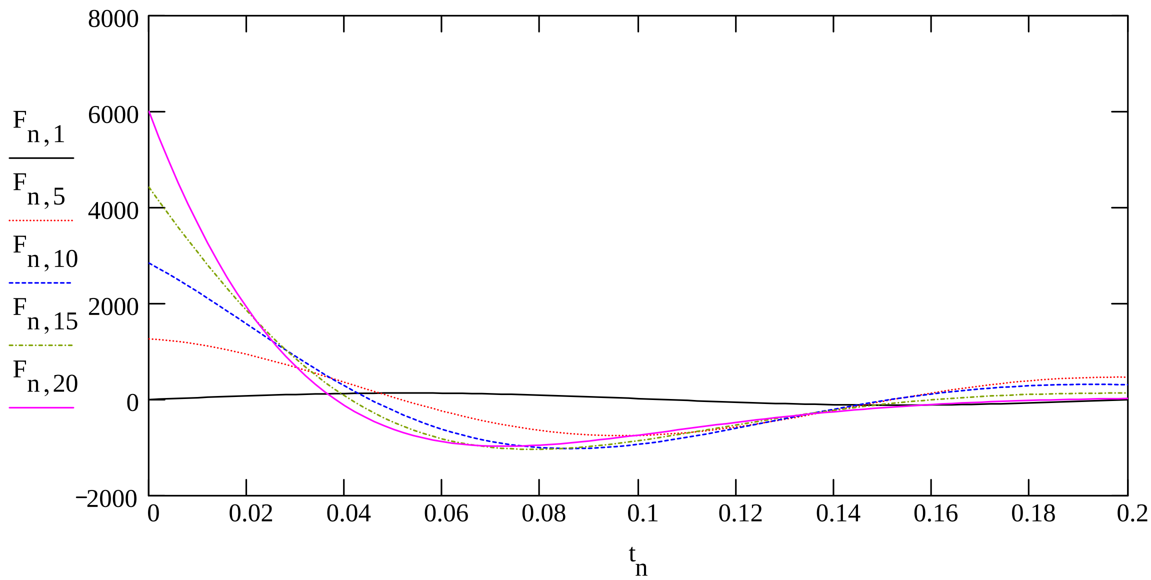
$$z(t,\zeta) := \exp(-\zeta\cdot\omega_{nat}\cdot t)\cdot\left(c_1\cdot\cos(\omega_d(\zeta)\cdot t) + c_2(\zeta)\cdot\sin(\omega_d(\zeta)\cdot t)\right) + \frac{g}{\omega_{nat}^{\,2}}$$

$$z'(t,\zeta) := \exp(-\zeta\cdot\omega_{nat}\cdot t)\cdot\left[\begin{array}{l}\left(\left(\omega_d(\zeta)\cdot c_2(\zeta) - \zeta\cdot\omega_{nat}\cdot c_1\right)\right)\cdot\cos(\omega_d(\zeta)\cdot t)\ \ldots \\ + \left(-\omega_d(\zeta)\cdot c_1 - \zeta\cdot\omega_{nat}\cdot c_2(\zeta)\right)\cdot\sin(\omega_d(\zeta)\cdot t)\end{array}\right]$$

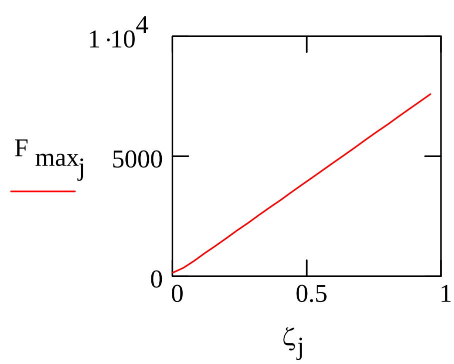
$$F_{susp}(t,\zeta) := m\cdot\left(\omega_{nat}^{\,2}\cdot z(t,\zeta) + 2\cdot\zeta\cdot\omega_{nat}^{\,2}\cdot z'(t,\zeta)\right)$$

$$j := 1..25 \qquad \zeta_j := \frac{j-1}{25} \qquad n := 1..101 \qquad t_n := \frac{n-1}{100}\cdot\frac{2\cdot\pi}{\omega_{nat}}$$

$$F_{n,j} := F_{susp}(t_n,\zeta_j)$$



$$F_{mag_{n,j}} := \left|F_{n,j}\right| \qquad F_{max_j} := \max\left(F_{mag}^{\langle j\rangle}\right)$$



$$F_{max_1} = 135.839$$