HW 2

EMA 545
Mechanical Vibrations

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Contents

1 problem 1

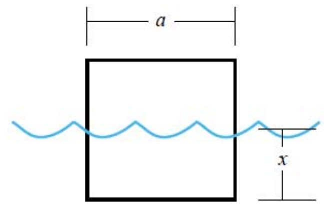
Problem 1:
A cube with density ρ and side length *a* is floating freely in a pool of water.

a.) Find the equation of motion of the cube when it is displaced in the vertical direction. (Recall that the buoyant force on a floating object is equal to the weight of the water displaced.) If necessary, re-define your vertical coordinate to eliminate any static forces.

b.) Derive an expression for the natural frequency of the cube.

c.) If the block is pine (ρ=400 kg/m^3) with a side length of 10cm, what is the natural frequency in Hz?

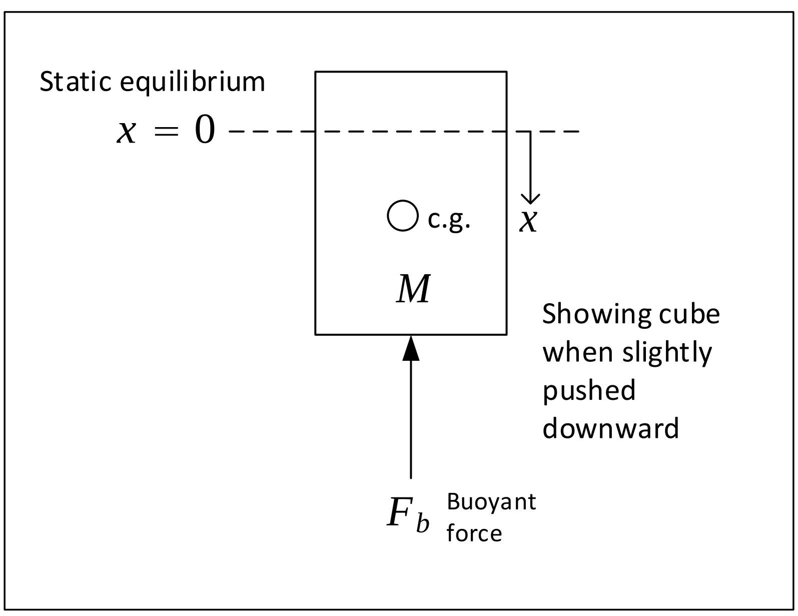
(Note that, while an analysis like this would be important when designing a boat or ocean vessel, in reality the rotational motions of the vessel would usually be more important and those require a more complicated analysis.)



1.1 Part(a)

We assume the cube is displaced downwards from its static equilibrium position and it is currently at distance x below the static position.

The buoyant force F_b will push the cube upwards. This force will equal the weight of water displaced which is $xa^2\rho_w g$ where ρ_w is density of water and g is the gravitational constant. The free body diagram is



Applying $F = mx''$ we obtain equation of motion

$$Mx'' = -F_b \tag{1}$$
$$Mx'' + F_b = 0 \tag{2}$$

$M = a^3\rho$ where ρ is density of pine. The above can be simplified to

$$a^3\rho x'' + xa^2\rho_w g = 0 \tag{3}$$
$$x'' + \frac{\rho_w g}{a\rho} x = 0 \tag{4}$$
$$x'' + \omega_n^2 x = 0 \tag{5}$$

1.2 Part(b)

Hence from the above equation

$$\omega_n = \sqrt{\frac{\rho_w g}{a\rho}}$$

1.3 Part(c)

Given $\rho = 400\ kg/m^3$ and $\rho_w = 1000\ kg/m^3$ and $a = 0.1m$ then

$$\omega_n = \sqrt{\frac{\rho_w g}{a\rho}} = \sqrt{\frac{1000 \times 9.81}{0.1 \times 400}} = 15.66\ \frac{rad}{\sec^2}$$

Hence frequency in Hz is

$$f = \frac{\omega_n}{2\pi} = \frac{15.66}{2\pi} = 2.492\ hz$$

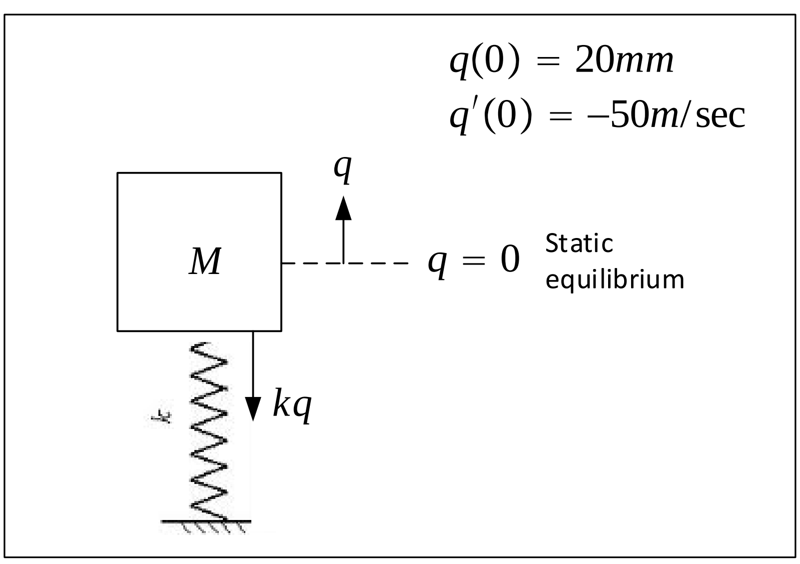
2 Problem 2

2.17 An undamped one-degree-of-freedom system has a mass coefficient of 50 kg and a natural frequency of 80 Hz. At $t = 0$ it is released from $q = 20$ mm with $\dot{q} = -50$ m/s.

(a) Determine the maximum positive value of q that occurs in the ensuing vibration, and the earliest instant at which it occurs.

(b) Determine the maximum positive value of $\dot{q}$ that occurs in the ensuing vibration, and the earliest instant at which it occurs.

2.1 Part(a)



applying $F = mq''$, we obtain equation of motion

$$\begin{aligned} Mq'' &= -kq && (6)\\ Mq'' + kq &= 0 && (7)\\ q'' + \frac{k}{M}q &= 0 && (8)\\ q'' + \omega_n^2 q &= 0 && (9)\end{aligned}$$

Let solution be $q(t) = \mathrm{Re}\left(\hat{A}e^{i\omega_n t}\right)$ where $\hat{A}$ is the complex amplitude, which is a complex number that can be written as $\hat{A} = a + ib$. We use initial conditions to determine $\hat{A}$. At

$t = 0$, let $q(0) = q_0$

$$q_0 = \text{Re}\left(\hat{A}e^{i\omega_n t}\right) \tag{10}$$
$$= \text{Re}\left(\hat{A}\right) \tag{11}$$
$$= a \tag{12}$$

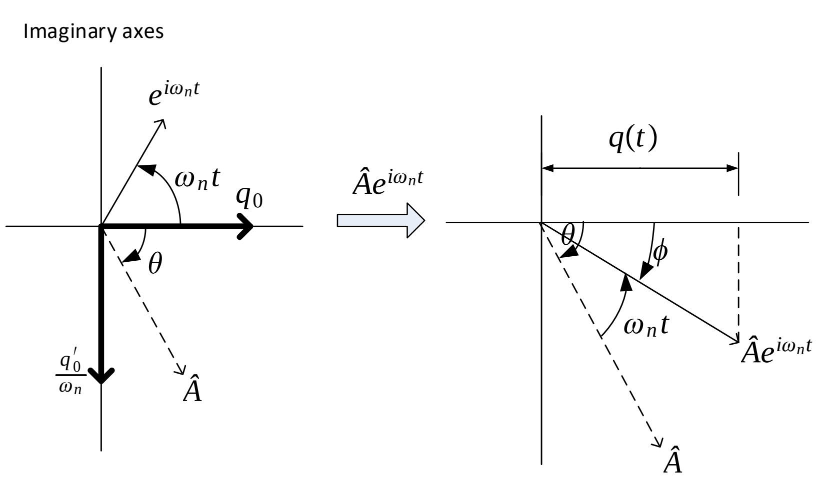
Hence $a = q_0$ And since $q'(t) = \text{Re}\left(i\omega_n \hat{A}e^{i\omega_n t}\right)$, then $t = 0$ we have

$$q_0' = \text{Re}\left(i\omega_n \hat{A}\right) \tag{13}$$
$$= \text{Re}(i\omega_n(a + ib)) \tag{14}$$
$$= \text{Re}(i\omega_n a - \omega_n b) \tag{15}$$
$$= -\omega_n b \tag{16}$$

Hence $b = -\frac{q_0'}{\omega_n}$ therefore the general solution is

$$q(t) = \text{Re}\left(\hat{A}e^{i\omega_n t}\right) \tag{17}$$
$$= \text{Re}\left((a + ib)e^{i\omega_n t}\right) \tag{18}$$
$$= \text{Re}\left(\overbrace{\left(q_0 - i\frac{q_0'}{\omega_n}\right)}^{\hat{A}} e^{i\omega_n t}\right) \tag{19}$$

Hence $\left|\hat{A}\right| = \sqrt{q_0^2 + \left(\frac{q_0'}{\omega_n}\right)^2}$ and $\arg\left(\hat{A}\right) = \theta = \tan^{-1}\left(\frac{\frac{q_0'}{\omega_n}}{q_0}\right)$. We have 2 complex quantities above being multiplied. The first is $\hat{A}$ and the second is $e^{i\omega_n t}$, therefore the result is obtained by adding the angles and by multiplied the magnitudes. The magnitude of $e^{i\omega_n t}$ is one. Hence on the complex plan, the above expression for $q(t)$ is represented as vector of length $\left|\hat{A}\right|$ and phase $\phi = \theta + \omega_n t$



From the above diagram we see that the maximum value of

$$q_{\max}(t) = \left|\hat{A}\right|$$

which occurs when

$$\phi = \theta + \omega_n t = 0$$

solving for t gives

$$t = \frac{-\theta}{\omega_n}$$

Notice that θ is negative, hence we will get positive value for t. Substituting the numerical values given we find thatAnd the earliest time this occurs is

$$t = \frac{1.3724}{2\pi(80)} = 2.7303 \times 10^{-3} = 2.73 \text{ ms}$$

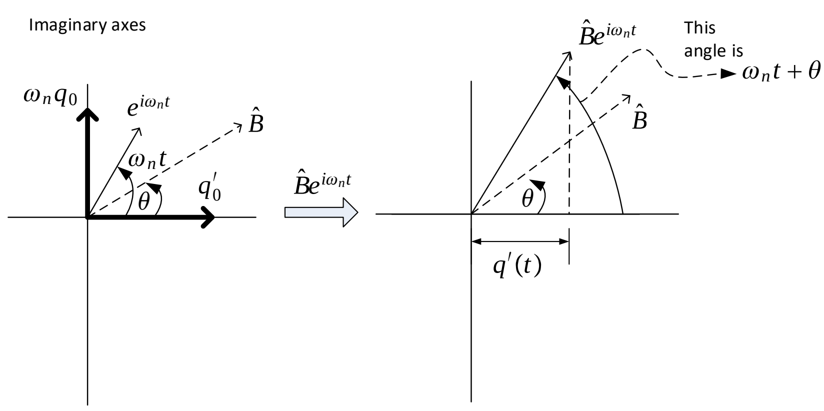
We confirm this by noticing that the initial position vector was at about $\frac{1}{4}cycle$ away from the positive x-axis (we found the phase of $\hat{A}$ above to be about −80 degrees), and the rotational speed is given as 80 cycles per second. Hence it takes 12.5 ms to make one cycle and $\frac{1}{4}$ of this is about $3ms$.

2.2 Part(b)

Since we found $q(t) = \operatorname{Re}\left(\left(q_0 - i\frac{q_0'}{\omega_n}\right)e^{i\omega_n t}\right)$, then

$$q'(t) = \operatorname{Re}\left(i\omega_n\left(q_0 - i\frac{q_0'}{\omega_n}\right)e^{i\omega_n t}\right) \tag{20}$$
$$= \operatorname{Re}\left(\left(\omega_n q_0 - e^{i\frac{\pi}{2}}q_0'\right)e^{i\frac{\pi}{2}}e^{i\omega_n t}\right) \tag{21}$$
$$= \operatorname{Re}\left(\left(\omega_n q_0 e^{i\frac{\pi}{2}} - e^{i\pi}q_0'\right)e^{i\omega_n t}\right) \tag{22}$$
$$= \operatorname{Re}\left(\overbrace{\left(\omega_n q_0 e^{i\frac{\pi}{2}} + q_0'\right)}^{\hat{B}}e^{i\omega_n t}\right) \tag{23}$$

Where now $\hat{B}$ is the complex amplitude of $q'(t)$. Hence $\left|\hat{B}\right| = \sqrt{\left(\omega_n q_0\right)^2 + \left(q_0'\right)^2}$ and its phase is $\arg\left(\hat{B}\right) = \tan^{-1}\frac{\omega_n q_0}{q_0'}$. The complex plane representation of $q'(t)$ is



From the above diagram we see that maximum magnitude of $q'(t)$ is $\left|\hat{B}\right|$ given by

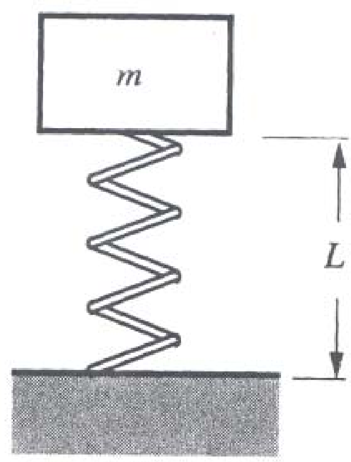
$$\left|\hat{B}\right| = \sqrt{\left(\omega_n q_0\right)^2 + \left(q_0'\right)^2} \tag{24}$$
$$= \sqrt{\left(2\pi(80)\left(20\times10^{-3}\right)\right)^2 + (-50)^2} \tag{25}$$
$$= 51.001\text{m/s} \tag{26}$$

The earliest time it occurs is found by solving for t in

$$\omega_n t + \theta = 2\pi \tag{27}$$
$$t = \frac{2\pi - \theta}{\omega_n} \tag{28}$$
$$= \frac{2\pi - \tan^{-1}\frac{\omega_n q_0}{q_0'}}{2\pi(80)} = \frac{2\pi - \tan^{-1}\frac{2\pi(80)\left(20\times10^{-3}\right)}{-50}}{2\pi(80)} \tag{29}$$
$$= \frac{2\pi - \tan^{-1}(-0.20106)}{2\pi(80)} = 1.2895\times10^{-2} \tag{30}$$
$$= 0.129 \text{ ms} \tag{31}$$

3 Problem 3

2.19 A block of mass m is mounted on a spring having stiffness k. The block moves in the vertical direction. When the system is at rest, a 2 kg block is placed gently on the original block. It is observed that the static length of the spring after insertion of the additional block is 50 mm less than it was prior to the addition. It also is observed that the natural frequency with the additional mass is 5 Hz less than it was originally. Determine k and m.



EXERCISE 2.19

Adding 2 kg caused deflection of 50 mm, hence from $F = k\Delta$ we can find k as follows

$$k = \frac{F}{\Delta} = \frac{2g}{0.05} = \frac{2(9.81)}{0.05} = 392 \text{ N/m} \tag{32}$$

where g is the gravitational constant. We also told that $f_2 = f_1 - 5$ where f_2 is the natural frequency after adding the second mass and where $f_1 = \frac{1}{2\pi}\omega_1$ and $f_2 = \frac{1}{2\pi}\omega_2$, hence

$$f_2 = f_1 - 5 \tag{33}$$

$$\frac{1}{2\pi}\omega_2 = \frac{1}{2\pi}\omega_1 - 5 \tag{34}$$

$$\omega_2 = \omega_1 - 10\pi \tag{35}$$

But $\omega_1 = \sqrt{\frac{k}{m}}$ and $\omega_2 = \sqrt{\frac{k}{m+2}}$, hence

$$\sqrt{\frac{k}{m+2}} = \sqrt{\frac{k}{m}} - 10\pi$$

From Eq 32 the above becomes

$$\sqrt{\frac{392}{m+2}} = \sqrt{\frac{392}{m}} - 10\pi$$

Solving numerically gives $m = 0.1955$kg

4 Problem 4

Problem 4: Show that $x(t)=Be^{\lambda t}$ is a solution to $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=0$ and find λ for the following cases: 1.) Underdamped system, 2.) Overdamped system. Write the solution $x(t)$ for both cases for an arbitrary set of initial conditions and draw a sketch to illustrate how each response $x(t)$ would look. Show that $x(t)$ can be written as $x(t)=\text{Re}\left(Ae^{-\zeta\omega_n t}e^{i\omega_d t}\right)$ in case (1).

To show that $x(t) = Be^{\lambda t}$ is solution to the differential equation, we substitute this solution into the LHS of the differential equation and see if we obtain zero.

$$x'(t) = \lambda Be^{\lambda t} = \lambda x(t) \tag{36}$$
$$x''(t) = \lambda^2 Be^{\lambda t} = \lambda^2 x(t) \tag{37}$$

Then

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0 \tag{38}$$
$$\lambda^2 x(t) + 2\zeta\omega_n\lambda x(t) + \omega_n^2 x(t) = 0 \tag{39}$$
$$\left(\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\right)x(t) = 0 \tag{40}$$

Hence $x(t) = Be^{\lambda t}$ is a non-trivial solution to the differential equation provided $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ since then we obtain $0 = 0$.

Now we find λ for the different cases.

4.1 case 1

The roots of $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$ are

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

For underdamped $\zeta < 1$, hence $\sqrt{\zeta^2 - 1} < 0$ and we write the above as

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1 - \zeta^2} \tag{41}$$
$$= -\zeta\omega_n \pm i\omega_d \tag{42}$$

where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Let $\lambda = -\zeta\omega_n + i\omega_d$ and its complex conjugate $\lambda^* = -\zeta\omega_n - i\omega_d$, hence the solution is

$$x(t) = B_1 e^{\lambda t} + B_2 e^{\lambda^* t}$$

To obtain a real solution we must have B_1 be complex say $\hat{B}$ and $B_2 = \hat{B}^*$. Hence the above can be written as

$$x(t) = \hat{B}e^{\lambda t} + \hat{B}^* e^{\lambda^* t} \tag{43}$$
$$= 2\,\text{Re}\left(\hat{B}e^{\lambda t}\right) \tag{44}$$
$$= \text{Re}\left(2\hat{B}e^{(-\zeta\omega_n + i\omega_d)t}\right) \tag{45}$$

Therefore

$$x(t) = \text{Re}\left(\hat{A}e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{46}$$

Where $\hat{A} = 2\hat{B} = a + ib$. Hence

$$x(t) = \text{Re}\left((a + ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right)$$

To find a, b we need to use initial conditions. Assuming $x(0) = x_0$ and $x'(0) = x_0'$ then from Eq 46 we obtain

$$x_0 = \text{Re}(a + ib) = a$$

Hence

$$a = x_0$$

and taking derivative of 46

$$x(t) = \text{Re}\left((a+ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{47}$$
$$x'(t) = \text{Re}\left(-\zeta\omega_n(a+ib)e^{-\zeta\omega_n t}e^{i\omega_d t} + i\omega_d(a+ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{48}$$
$$x'(0) = \text{Re}(-\zeta\omega_n(a+ib) + i\omega_d(a+ib)) \tag{49}$$
$$= -\zeta\omega_n a - \omega_d b \tag{50}$$

Hence

$$b = \frac{x_0' + \zeta\omega_n a}{\omega_d}$$

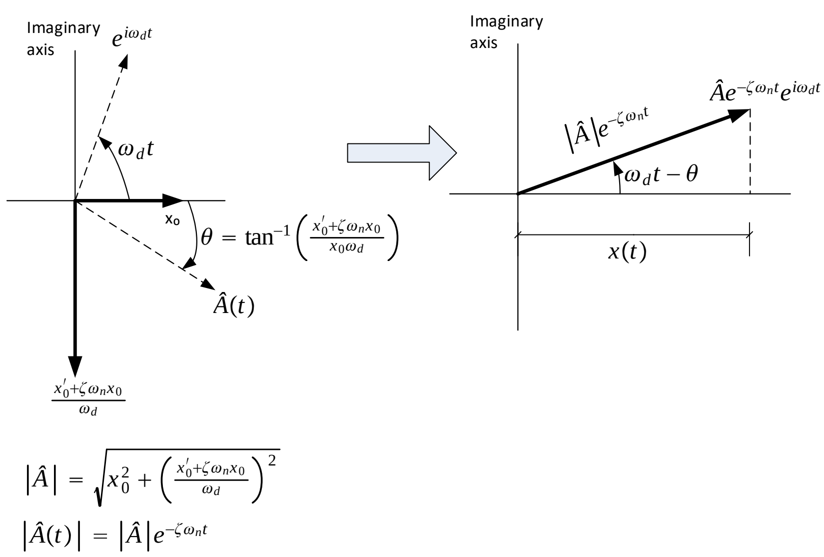
But $a = x_0$, hence

$$b = \frac{x_0' + \zeta\omega_n x_0}{\omega_d}$$

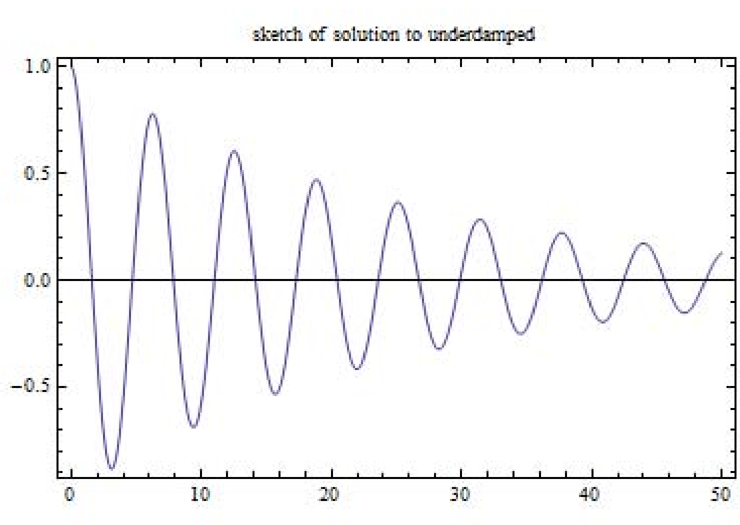
Hence 46 becomes

$$x(t) = \text{Re}\left((a+ib)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{51}$$
$$= \text{Re}\left(\left(x_0 + i\frac{x_0' + \zeta\omega_n x_0}{\omega_d}\right)e^{-\zeta\omega_n t}e^{i\omega_d t}\right) \tag{52}$$

And this is the general solution. In complex plan it is



Hence the rotating vector will have its length become smaller with time since $|\hat{A}|$ is multiplied by $e^{-\zeta\omega_n t}$. The real part, which is the solution will eventually damp down to zero. Hence it is a damped sinusoid oscillation as follows



4.2 case 2

From

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

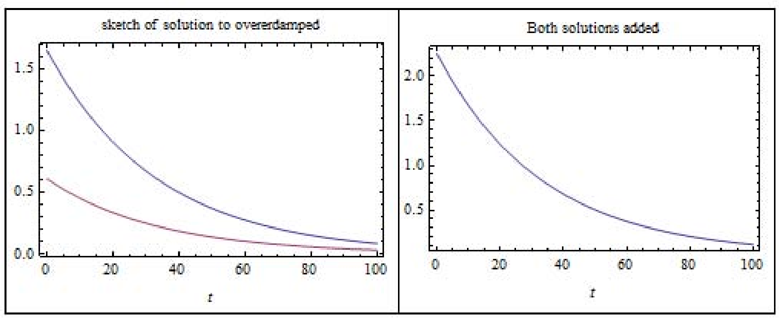
For overdamped $\zeta > 1$, hence $\sqrt{\zeta^2 - 1} > 0$ and we write the above as

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1 - \zeta^2}$$

Hence the solution is

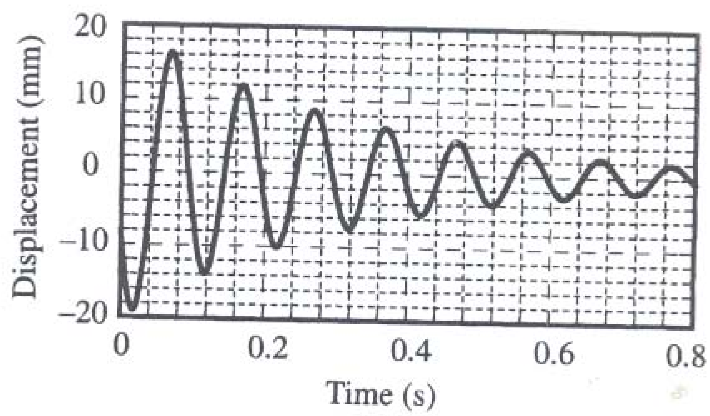
$$x(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t}$$

where $\lambda_1 = -\zeta\omega_n + \omega_n\sqrt{1 - \zeta^2}$ and $\lambda_2 = -\zeta\omega_n - \omega_n\sqrt{1 - \zeta^2}$. We see that both roots are negative always, hence we have 2 exponentially damped solution being added with no oscillation. A sketch of the solution is



5 Problem 5

2.29 The measured free vibration response of a one-degree-of-freedom system is as shown in the graph.



EXERCISE 2.29

(a) Deduce from this measurement the log decrement, the natural frequency, and the critical damping ratio of the system.

(b) Estimate the value of t beyond which the displacement magnitude $|q|$ will not exceed 0.01 mm.

(c) If the damping constant C is held fixed, while the system is modified by doubling the stiffness K and halving the generalized mass M, how would that alter the answer to part (b)?

(d) The initial displacement, at $t = 0$, is $q_0 = -10$ mm. What is the initial velocity?

5.1 Part(a)

From looking at the plot above, here are the values estimated for displacement positive peaks and time they occur

t	$y(t)$
0.07	16
0.17	12
0.27	9
0.37	6

From the above we estimate the natural period $T \approx 0.1$ sec hence $f = 10$ hz hence $\omega_n = 2\pi f = 60.3$ rad/sec The log decrement is

$$\delta = \ln \frac{y_i}{y_{i+N}}$$

Select $i = 1$ and $N = 3$ gives

$$\delta = \ln \frac{16}{6} \tag{53}$$
$$= 0.981 \tag{54}$$

To find ζ we use the log decrement method

$$\delta = 2\pi N\zeta$$

Hence

$$\zeta = \frac{\delta}{2\pi N} = \frac{0.98083}{2\pi(3)} \tag{55}$$

$$\zeta = 0.052 \tag{56}$$

Hence

$$\zeta = 5.2\%$$

5.2 Part(b)

$$\ln\left(\frac{y_1}{y_{1+N}}\right) = 2\pi N\zeta$$

Where now we write $y_1 = 16$ and $y_{N+1} = 0.01$, and hence we need to find N the only unknown in the equation above

$$\ln\left(\frac{16}{0.01}\right) = 2\pi N(0.052)$$

Hence

$$N = \frac{\ln\left(\frac{16}{0.01}\right)}{2\pi(0.052)} = 22.581$$

We take $N = 23$. What this says is that after 23 periods beyond the first peak, we will satisfy the requirement. But $T = 0.1$ sec, and the first peak was at $t = 0.05$ sec, therefore

$$t = 0.07 + 23(0.1) \tag{57}$$

$$= 2.37\text{sec} \tag{58}$$

5.3 Part c

Since $\delta_N = 2\pi N\zeta$ and $\zeta = \frac{c}{c_r} = \frac{c}{2\sqrt{km}}$, then if we double k and half the mass m, then ζ would remain the same since c is held constant. Therefore the answer in part b would not change.

5.4 Part d

Since this is an underdamped system, the solution is

$$q(t) = \text{Re}\left(\hat{A}e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

Where $\hat{A}$ is the complex amplitude, say $(a + ib)$, hence

$$q(t) = \text{Re}\left((a + ib)e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

At $t = 0$ we find that

$$a = q(0) = q_0$$

Hence

$$a = -0.01$$

and the general solution is

$$q(t) = \text{Re}\left(\left(q_0 + ib\right)e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

Now taking derivatives of the above gives

$$q'(t) = \text{Re}\left((-\omega_n\zeta + i\omega_d)\left(q_0 + ib\right)e^{(-\omega_n\zeta+i\omega_d)t}\right)$$

At $t=0$ then, assuming q_0' is the initial velocity

$$q_0' = \operatorname{Re}\Big((-\omega_n\zeta + i\omega_d)\big(q_0 + ib\big)\Big) \tag{59}$$
$$= -\omega_n\zeta q_0 - \omega_d b \tag{60}$$

Hence

$$b = -\frac{q_0' + \omega_n\zeta q_0}{\omega_d}$$

Therefore the general solution is

$$q(t) = \operatorname{Re}\left(\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta + i\omega_d)t}\right)$$

and

$$q'(t) = \operatorname{Re}\left((-\omega_n\zeta + i\omega_d)\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta + i\omega_d)t}\right)$$

Now at $t_0 = 0.07$ sec the velocity is zero, since this is where the displacement is maximum (first peak). Hence now we have one equation with one unknown q_0'that we can solve for from the above

$$0 = \operatorname{Re}\left((-\omega_n\zeta + i\omega_d)\left(q_0 - i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\right)e^{(-\omega_n\zeta + i\omega_d)t_0}\right) \tag{61}$$
$$= e^{-\omega_n\zeta t_0}\operatorname{Re}\left(\left(-\omega_n\zeta q_0 + i\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + iq_0\omega_d + \frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_d\right)e^{i\omega_d t_0}\right) \tag{62}$$
$$= e^{-\omega_n\zeta t_0}\operatorname{Re}\left(\left(i\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right) + q_0'\right)e^{i\omega_d t_0}\right) \tag{63}$$
$$= e^{-\omega_n\zeta t_0}\operatorname{Re}\left(i\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i\omega_d t_0} + q_0' e^{i\omega_d t_0}\right) \tag{64}$$
$$= e^{-\omega_n\zeta t_0}\operatorname{Re}\left(\frac{-1}{i}\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i(\omega_d t_0)} + q_0' e^{i\omega_d t_0}\right) \tag{65}$$
$$= e^{-\omega_n\zeta t_0}\left[\operatorname{Re}\left(\frac{-1}{i}\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)e^{i(\omega_d t_0)}\right) + \operatorname{Re}\left(q_0' e^{i\omega_d t_0}\right)\right] \tag{66}$$
$$= e^{-\omega_n\zeta t_0}\left[-\left(\frac{q_0' + \omega_n\zeta q_0}{\omega_d}\omega_n\zeta + q_0\omega_d\right)\sin(\omega_d t_0) + q_0'\cos(\omega_d t_0)\right] \tag{67}$$

But $q_0 = -0.01$ m/sec, $\zeta = 0.052$, $\omega_n = 60.3$ rad/sec and $\omega_d = \omega_n\sqrt{1-\zeta^2} = 60.3\sqrt{1-0.052^2} = 60.218$, therefore $\omega_d t_0 = 60.218\times 0.07 = 4.2153$ and $q_0\omega_d = (60.218)(-0.01) = -0.60218$ hence the above equation becomes

$$0 = e^{-(60.3)(0.052)(0.07)}\left(-\left(\frac{q_0' + (60.3)(0.052)(-0.01)}{60.218}60.3\times 0.052 - 0.602\right)\sin(4.215) + q_0'\cos(4.215)\right) \tag{68}$$
$$= 0.80293\left(-\left(\frac{q_0' - 3.136\times 10^{-2}}{60.218}(3.1356) - 0.602\right)(-0.879) + q_0'(-0.477)\right) \tag{69}$$

Solving for q_0' gives

$$q_0' = -1.231 \text{ m/sec}$$

Now that we q_0',

Now that we q_0', we can find the numerical value for b and write the general solution again.

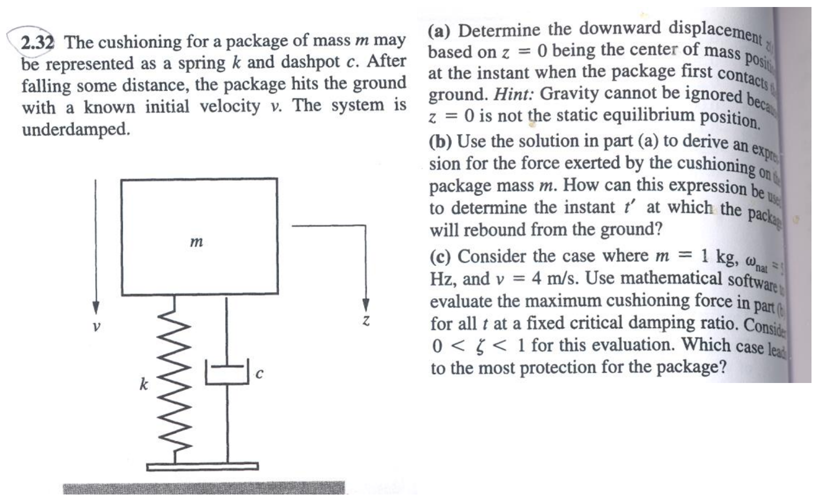
$$b = -\frac{q_0' + \omega_n\zeta q_0}{\omega_d} \tag{70}$$
$$= -\frac{-1.231 + 60.3(0.052)(-0.01)}{60.218} \tag{71}$$
$$= 2.0963\times 10^{-2} \tag{72}$$

Hence from

$$q(t) = \mathrm{Re}\Big(\big(q_0 + ib\big)e^{(-\omega_n\zeta + i\omega_d)t}\Big) \tag{73}$$
$$= \mathrm{Re}\Big((-0.01 + i0.0209)e^{(-\omega_n\zeta + i\omega_d)t}\Big) \tag{74}$$

giving$|\hat{A}| = \sqrt{0.01^2 + 0.0209^2} = 0.023$

6 Problem 6



2.32 The cushioning for a package of mass m may be represented as a spring k and dashpot c. After falling some distance, the package hits the ground with a known initial velocity v. The system is underdamped.

(a) Determine the downward displacement based on $z = 0$ being the center of mass at the instant when the package first contacts the ground. *Hint:* Gravity cannot be ignored $z = 0$ is not the static equilibrium position.

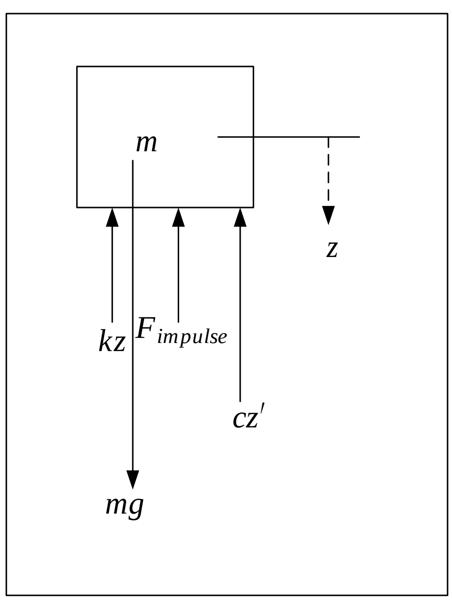
(b) Use the solution in part (a) to derive an expression for the force exerted by the cushioning on the package mass m. How can this expression be to determine the instant t' at which the will rebound from the ground?

(c) Consider the case where $m = 1$ kg, ω_{nat} = Hz, and $v = 4$ m/s. Use mathematical software evaluate the maximum cushioning force in part for all t at a fixed critical damping ratio. Conside $0 < \zeta < 1$ for this evaluation. Which case lea to the most protection for the package?

6.1 Part a

Assume the system is underdamped.

When the package hits the ground, its speed becomes zero. Therefore the impulse generated on it is the change of linear momentum. Since it speed was v just before impact, then impulse= mv.



Hence the EQM is

$$mz'' = -cz' - kz - F_{impulse} + mg \tag{75}$$
$$mz'' + cz' + kz = mg - F_{impulse} \tag{76}$$

With the initial conditions now being $z = 0$ and $z' = 0$.

The response due to the force mg can be found from the response to a unit step of amplitude mg Hence the response due to the force mg is

$$u(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right]\right)$$

The response due to the impulse is the response of a free system with zero initial position but with initial velocity $\frac{impulse}{m}$ in the upward (negative) direction. Hence the response due to the impulse only is

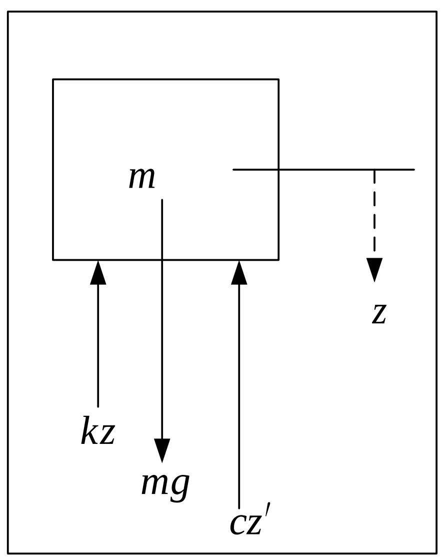
$$g(t) = \frac{mv}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{77}$$
$$= \frac{v}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{78}$$

Hence the downward displacement is given by

$$z(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t \tag{79}$$

6.2 Part b

Now that the impulse have taken place and we have accounted for it in the $z(t)$ solution, then we can use this expression to find the spring force since $F_{spring} = kz(t)$ and the damping force on the mass $F_{damper} = cz'(t)$. When resultant net force F is negative then the mass will rebound from the ground.



$$F = mg - kz(t) - cz'(t)$$

But

$$z(t) = \frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t \tag{80}$$

Hence

$$z'(t) = \frac{mg}{k}\left(\zeta\omega_n e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t\right] - e^{-\zeta\omega_n t}[-\omega_d \sin \omega_d t + \zeta\omega_n \cos \omega_d t]\right) - \frac{\zeta\omega_n ve^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t - \tag{81}$$

or

$$z'(t) = \frac{e^{-\zeta\omega_n t}}{k\omega_d}\left(\left(kv\omega_n\zeta + gm\left(\omega_d^2 + \omega_n^2\zeta^2\right)\right) \sin \omega_d t - kv\omega_d \cos(\omega_d t)\right) \tag{82}$$

Hence

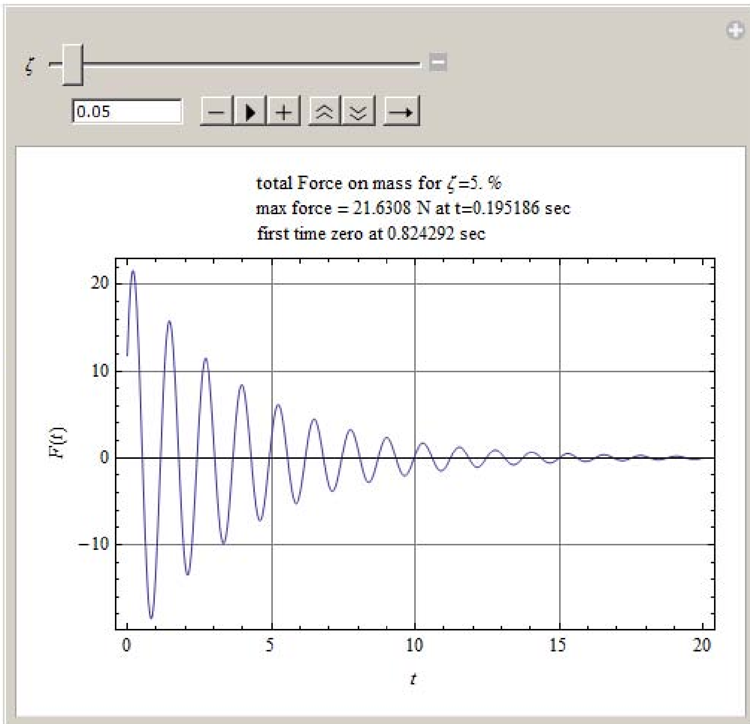
$$F = mg - kz(t) - cz'(t) \tag{83}$$
$$= mg - k\left[\frac{mg}{k}\left(1 - e^{-\zeta\omega_n t}\left[\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t\right]\right) - \frac{ve^{-\zeta\omega_n t}}{\omega_d} \sin \omega_d t\right] - c\frac{e^{-\zeta\omega_n t}}{k\omega_d}\left(\left(kv\omega_n\zeta + gm\left(\omega_d^2 + \omega_n^2\zeta^2\right)\right)\right. \tag{84}$$

To find when this force will turn negative first time, we can take the derivative with respect to time and set it to zero and solve for first $t = t'$ that will make it zero. Since the force was positive first, then it has to become zero before turning negative.

6.3 Part(c)

Let $m = 1$ kg, $\omega_n = 5$ rad/sec[1], $v = 4$ m/s. Hence $\omega_d = 5\sqrt{1-\zeta^2}$. Since $\omega_n^2 = \frac{k}{m}$, hence $k = 25$ N/m. Also $c = \zeta c_{cr} = \zeta 2m\omega_n = 10\zeta$

Using these values, the force in part(b) is plotted for different values of ζ. For example, setting $\zeta = 5\%$ gives this plot of $F(t)$ for $t = 0$ to 20 seconds.



The maximum force is seen as little over 20 N.Therefore, to find which ζ gives the smallest value of maximum force, we can try different values of ζ and see how the maximum force changes as a function of ζ. Using software the following values of maximum for for different ζ are generated along with $t = t_{\max}$ when this maximum occurs and with the time $t = t'$ when the mass rebounds first time from $z = 0$

maximum force (N)	ζ%	$t_{\max}$(sec)	t'(sec)
22.1	1	0.21	0.84
21.85	3	0.206	0.834
21.45	7	0.184	0.81
21.27	10	0.16	0.779
21.4	20	0.11	0.75
25.8	40	0.01	0.68
30	50	0.001	0.64

Most protection when damping ratio is below 10%

[1] typo in book. `hz` is assumed to mean `rad/sec`