

**Homework #11**  
EMA 545, Spring 2013

**Problem 1.)**

Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + [K]\{q\} = \{Q\}$$

with [C] matrix given by:

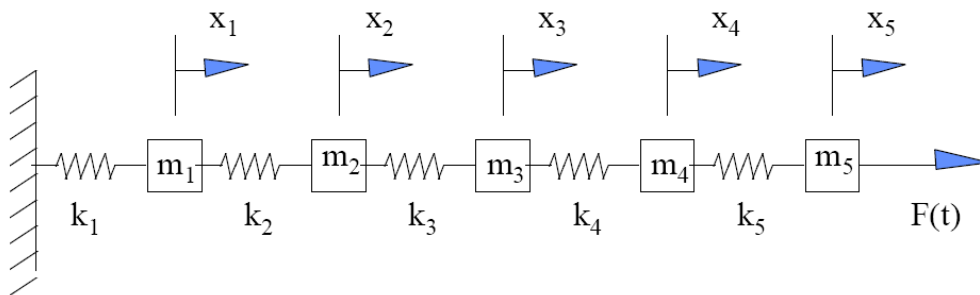
$$[C] = [M][\Phi] \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\zeta_N\omega_N \end{bmatrix} [\Phi]^T [M].$$

If we transform to normal coordinates using  $\{q\} = [\Phi]\{\eta\}$ , show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j = \{\Phi_j\}^T \{Q\} \quad j = 1, N$$

**Problem 2.)**

The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs  $k_i$  which are drawn, there are viscous dampers  $c_i$  which are not shown. The equations of motion for this system are easily found to be:

$$[m]\ddot{\underline{x}} + [c]\dot{\underline{x}} + [k]\underline{x} = \underline{F}$$

where  $[m]$  is a diagonal matrix having entries  $m_i$ , and  $[k]$  is the banded matrix:

$$\begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & -k_4 & 0 \\ 0 & 0 & -k_4 & (k_4+k_5) & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix}$$

The damping matrix has the same form as  $[k]$ , but with viscous damping coefficients  $c_i$  in place of stiffness coefficients  $k_i$ . The forcing vector  $\underline{F}$  is a 5 by 1 vector of zeros, except for the last entry:  $\underline{F}=[0 \ 0 \ 0 \ 0 \ F(t)]^T$ .

The numerical value for  $k_2$  through  $k_5$  is 1 N/m, the numerical value of  $k_1 = 2$  N/m, the numerical value for  $m_1$  through  $m_5$  is 1 kg, and the numerical values for the viscous damping coefficients  $c_i = 0.1 * k_i$ ,  $i=1,5$  (in units of N-s/m).

- (a) Find the natural frequencies and mass-normalized modes of the system.
- (b) Find the magnitude and phase of the steady-state response  $x_5(t)$  assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to  $1.2 * \omega_5$ . Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
- (c) Repeat the analysis in (b), but use the strategy described in Problem 1 to create a  $[C]$  matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
- (d) Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of  $\gamma=0.04$ .

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters  $EA$  and  $L$  are as follows:  $k_i = N * EA / L$ ,  $i=2,N$  where  $N$  is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high,  $k_1 = 2N * EA / L$ . The lumped masses are equal to the total bar mass divided by  $N$ :  $m_i = \rho AL / N$ , where  $\rho$  is the mass density of the bar.

**Problem 1)** (Creating a damping matrix with any desired modal damping ratios.)

The goal is to show that one can use the modal transformation together with the given damping matrix to obtain uncoupled equations of motion.

We need to show that  $[\Phi]^T [C][\Phi]$  is diagonal:

$$[\Phi]^T [C][\Phi] = [\Phi]^T [M][\Phi] \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\zeta_N\omega_N \end{bmatrix} [\Phi]^T [M][\Phi]$$

Since  $[\Phi]^T [M][\Phi] = [I]$ , we have the desired result:

$$[\Phi]^T [C][\Phi] = \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\zeta_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\zeta_N\omega_N \end{bmatrix}$$

Now, starting with the coupled equations of motion:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{Q\}$$

Substituting  $\{x\} = [\Phi]\{\eta\}$  and pre-multiplying by  $[\Phi]^T$  yields

$$[\Phi]^T [M][\Phi]\{\ddot{\eta}\} + [\Phi]^T [C][\Phi]\{\dot{\eta}\} + [\Phi]^T [K][\Phi]\{\eta\} = [\Phi]^T \{Q\} = \{\hat{Q}\}$$

Using the fact that  $[\Phi]^T [M][\Phi] = [I]$  and  $[\Phi]^T [K][\Phi]$  is a diagonal matrix containing the square of the natural frequencies, we now have N uncoupled equations of motion.

$$\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j = \{\Phi_j\}^T \{Q\} \quad j = 1, N$$

Note that we have also used the fact that each row of  $[\Phi]^T \{Q\}$  is equal to the  $j$ th mode vector transposed times  $\{Q\}$ .

## HW 12, PROBLEM 2: LUMPED MASS APPROXIMATION FOR A BAR IN EXTENSION

### Part (a):

The mass and stiffness matrices are given, so it is easy to find the natural frequencies and mode shapes in Matlab. Here is the result: (Matlab code given at the end.)

```
wns =
    0.31287
    0.90798
    1.4142
    1.782
    1.9754
PHI =
   -0.098938    0.28713   -0.44721   -0.56352   -0.62467
   -0.28713    0.62467   -0.44721    0.098938    0.56352
   -0.44721    0.44721    0.44721    0.44721   -0.44721
   -0.56352   -0.098938    0.44721   -0.62467    0.28713
   -0.62467   -0.56352   -0.44721    0.28713   -0.098938
```

### Part (b):

The input is harmonic, so the steady-state response of each mass will also be harmonic. Their complex amplitudes can be found with the equation:  $\text{inv}([K+i\omega C-\omega^2 M])$ . See the Matlab code for details. The complex amplitude of the 5<sup>th</sup> DOF is plotted below. Note that this stiffness proportional damping approach gives the following modal damping ratios, which are clearly different than those for part (c) below:

```
zts =
    0.015643
    0.045399
    0.070711
    0.089101
    0.098769
```

### Part (c):

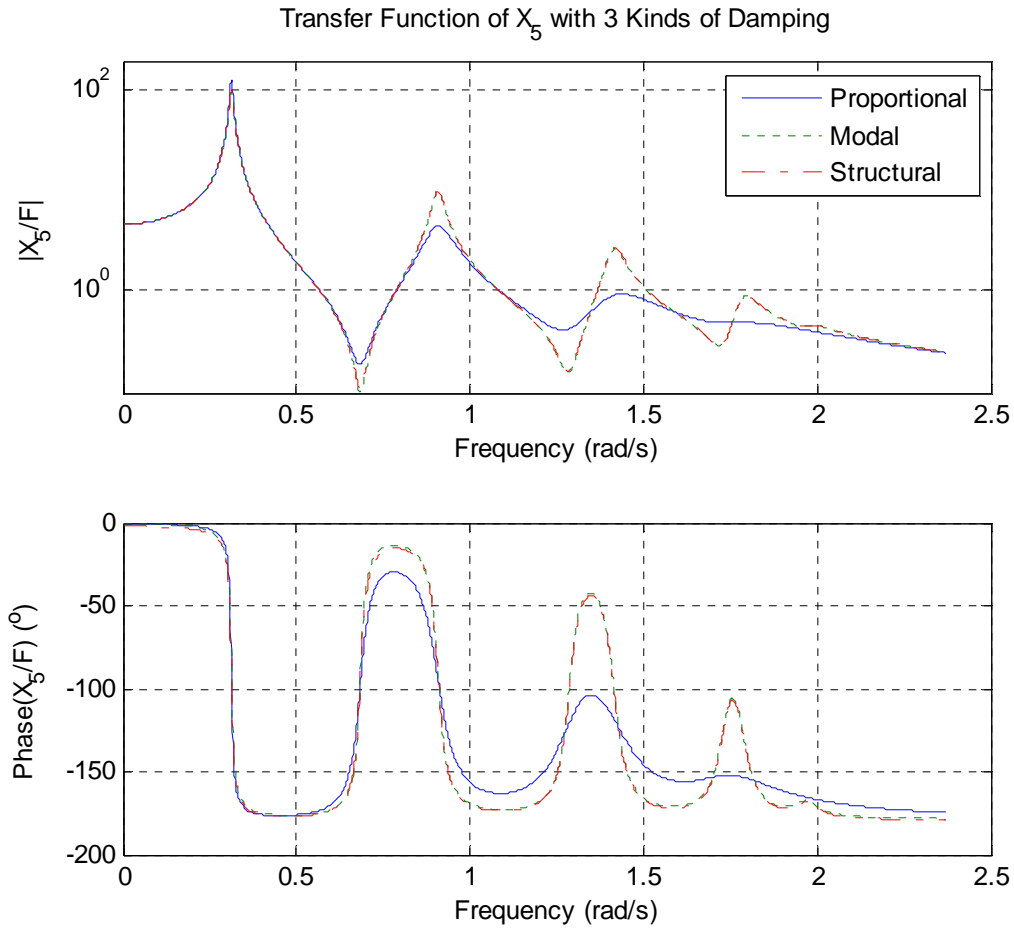
The solution here is the same as for (b), only now we use  $C = M*PHI*2*0.02*diag(wns)*PHI.'*M$

Both solutions are plotted below.

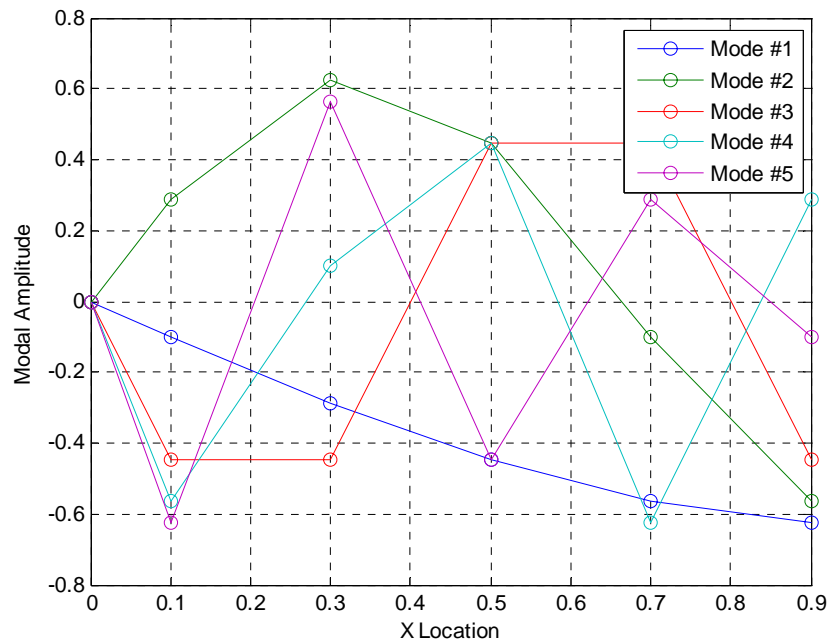
### Part(d): (not included in Spring 2011)

For the structural damping case,  $K = K*(1+i\gamma)$ .

The plot shows that the stiffness proportional damping approach gives heavier damping for the higher frequency modes.



The mode shapes were requested as part of the next problem. They are plotted below:



## Matlab Code:

```
% ME 6442 Homework #10, MDOF Systems-LUMPED MASS APPROX TO BAR IN EXTENSION
clear all; close all

M = eye(5); % Identity matrix since each mj=1;
K = eye(5)*2+diag([-1 -1 -1 -1],1)+diag([-1 -1 -1 -1],-1);
    % Above is a fancy way to make the banded stiffness matrix, although
    % the two terms below must be fixed manually.
K(1,1) = 1+2; K(5,5) = 1;
C = 0.1*K;

[phi,lam] = eig(K,M);
wns = sqrt(diag(lam))

% Normalize Eigenvectors
[natfreqs,Isort]=sort(sqrt(diag(lam))); % sort by nat freq
phi=phi(:,Isort); % sort eigenvectors
scale= phi.'*M*phi;
PHI=real(phi*inv(sqrt(scale))) % normalize e.vectors
zts = (diag(PHI.'*C*PHI)/2)./wns

% Check Orthogonality
check_m = norm(PHI.'*M*PHI-eye(size(PHI)))
check_k = norm(PHI.'*K*PHI-lam)

% Plot Mode Shapes
% Create x-vector. Remember that the masses are at the center of each
% element, so the first node is at 0.5*(L/N)
xs = [0, 0.5:1:4.5]/5;
    % Will have to stack zeros above PHI below for the displacement at x=0.
figure(1)
plot(xs.',[0 0 0 0 0; PHI],'o-'); grid on;
xlabel('X Location'); ylabel('Modal Amplitude');
legend('Mode #1', 'Mode #2', 'Mode #3', 'Mode #4', 'Mode #5');

ws = [0:max(wns)*1.2/2000:max(wns)*1.2];
% TF Using Proportional C-Matrix
for ii = 1:1:length(ws);
    Gn(:,ii) = [K+i*ws(ii)*C-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gb = Gn(5,:);

% TF Using modal damping ratios
Cc = M*PHI*2*0.02*diag(wns)*PHI.'*M
for ii = 1:1:length(ws);
    Gn(:,ii) = [K+i*ws(ii)*Cc-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gc = Gn(5,:);

% TF Using Structural Damping
Kd = K*(1+i*0.04);
for ii = 1:1:length(ws);
    Gn(:,ii) = [Kd-ws(ii)^2*M]\([0 0 0 0 1].');
end
Gd = Gn(5,:);

figure(2)
subplot(211);
semilogy(ws,abs(Gb),ws,abs(Gc),':',ws,abs(Gd),'-'); grid on;
xlabel('Frequency (rad/s)'); ylabel('|X_5/F|');
legend('Proportional', 'Modal', 'Structural');
title('Transfer Function of X_5 with 3 Kinds of Damping');
axis([0 2.5 0.09 200]);
```

```
subplot(212);  
plot(ws,angle(Gb)*180/pi,ws,angle(Gc)*180/pi,':',ws,angle(Gd)*180/pi,'-.');  
grid on;  
xlabel('Frequency (rad/s)'); ylabel('Phase(X_5/F) (^o)');
```

**Parts (a & b):**

First, note that since  $k_2$  in Problem 3 is 1 N/m, this suggests  $EA/L = 1/5$  N/m. Also, since  $m_1 = 1$  kg,  $\rho AL = 5$  kg. The kinetic energy of the rod is given by:

$$T = \frac{1}{2} \int_0^L \rho A \left( \frac{\partial u}{\partial t} \right)^2 dx \quad (4.1)$$

We use the following Ritz series to approximate the displacement field for the rod:

$$u(x, t) = \sum_{i=1}^3 q_i(t) \psi_i(x) \quad (4.2)$$

Substitution of (4.2) into (4.1) yields:

$$T = \frac{1}{2} \int_0^L \rho A \left( \sum_{i=1}^3 \dot{q}_i(t) \psi_i(x) \right) \left( \sum_{j=1}^3 \dot{q}_j(t) \psi_j(x) \right) dx \quad (4.3)$$

$$T = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \dot{q}_i \dot{q}_j \int_0^L \rho A \psi_i \psi_j dx \equiv \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \dot{q}_i \dot{q}_j m_{ij} \quad (4.4)$$

where the elements of the mass matrix are identified to be:

$$m_{ij} = \int_0^L \rho A \left( \frac{x}{L} \right)^i \left( \frac{x}{L} \right)^j dx = \frac{\rho A}{L^{i+j}} \int_0^L x^{i+j} dx = \rho AL \frac{1}{i+j+1} \quad (4.5)$$

The potential energy of the rod is given by

$$V = \frac{1}{2} \int_0^L EA \left( \frac{\partial u}{\partial x} \right)^2 dx \quad (4.6)$$

The derivative of the displacement field (4.2) with respect to  $x$  is given by:

$$\frac{\partial u}{\partial x}(x, t) = \sum_{i=1}^3 q_i(t) \frac{\partial \psi_i(x)}{\partial x} \quad (4.7)$$

Substitution of (4.7) into (4.6) yields:

$$V = \frac{1}{2} \int_0^L EA \left( \sum_{i=1}^3 q_i(t) \frac{\partial \psi_i}{\partial x} \right) \left( \sum_{j=1}^3 q_j(t) \frac{\partial \psi_j}{\partial x} \right) dx \quad (4.8)$$



$$V = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 q_i q_j \int_0^L EA \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} dx \equiv \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 q_i q_j k_{ij} \quad (4.9)$$

where the elements of the stiffness matrix are identified to be:

$$k_{ij} = \int_0^L EA \left( \frac{i}{L^i} \right) \left( \frac{j}{L^j} \right) x^{i+j-2} dx = \frac{EA i j}{L^{i+j}} \int_0^L x^{i+j-2} dx = \frac{EA}{L} \frac{i j}{i+j-1} \quad (4.10)$$

The three-term Ritz approximation for the rod is given by the following 3DOF system:

$$\rho AL \begin{bmatrix} 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \\ 1/5 & 1/6 & 1/7 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \frac{EA}{L} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4/3 & 6/4 \\ 1 & 6/4 & 9/5 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (4.11)$$

Note that the dimensional factors can be pulled out so that the eigenvalue problem can be solved in Matlab (for part (d)). To do so, we define:

$$\lambda^2 = \omega^2 \frac{\rho L^2}{E}$$

**Part (c):**

The integrals are not convenient to evaluate analytically using these basis functions, so the solution is carried out only in Matlab. The resulting matrices are:

```
>> M
M =
    0.5         0         0
         0     0.5         0
         0         0     0.5

>> K
K =
    1.2337         0         0
         0    11.103         0
         0         0    30.843
```

Notice that the matrices are diagonal. This occurs because the chosen basis functions happen to be the eigenfunctions for this continuous system. So, the coordinate governing each basis function is independent of all of the others and the natural frequencies can be found by inspection, for example:  $\omega_1 = \sqrt{1.2337/0.5} = 1.5708$ , etc...

**Part (d):**

**COMPARISON OF NATURAL FREQUENCIES:**

First we observe that since  $\rho AL/(EA/L) = 25$ , we must multiply the natural frequencies found in Problem 2 by  $(25)^{(1/2)}$  to compare with the results from the nondimensional Ritz analysis. The results are summarized in the table below, where the natural frequencies for other Ritz Series lengths are also shown FYI. Notice that as the series length increases, new modes appear at higher frequencies, and the lower natural frequencies decrease slightly, converging towards the true values. The lumped parameter method in Problem 2 gives similar results although the frequencies are sometimes lower than the true values, while the Ritz method always over predicts the natural frequencies.

Ritz Series Length	Natural Frequencies (nondimensional)				
	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$
2	1.5767	5.6728	-	-	-
3	1.5709	4.8365	10.4471	-	-
4	1.5708	4.7246	8.3309	16.3036	-
5	1.5708	4.7132	7.9390	12.1739	23.3614
Results for Lumped Parameter Approximation with N=5 below					
N=5	1.5643	4.539	7.0711	8.9101	9.8769

With the second set of basis functions from part (c), one obtains the true analytical natural frequency for any length Ritz Series.

Ritz Series Length	Natural Frequencies (nondimensional)				
	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$
2	1.5708	4.7124	-	-	-
3	1.5708	4.7124	7.8540	-	-
4	1.5708	4.7124	7.8540	10.9956	-
5	1.5708	4.7124	7.8540	10.9956	14.1372

## MODE SHAPES

Using the polynomials in part (b):

PHI =

$$\begin{array}{ccc} -2.2642 & -11.2099 & -13.0082 \\ 0.2314 & 25.3536 & 47.2984 \\ 0.6181 & -12.7003 & -37.5941 \end{array}$$

Each column gives the proportion of each Ritz vector in the approximation for the eigenfunction. For example, the approximation for the first modeshape is given by:

$$-2.2642\left(\frac{x}{L}\right) + 0.2314\left(\frac{x}{L}\right)^2 + 0.6181\left(\frac{x}{L}\right)^3$$

The second eigenfunction is approximated by:

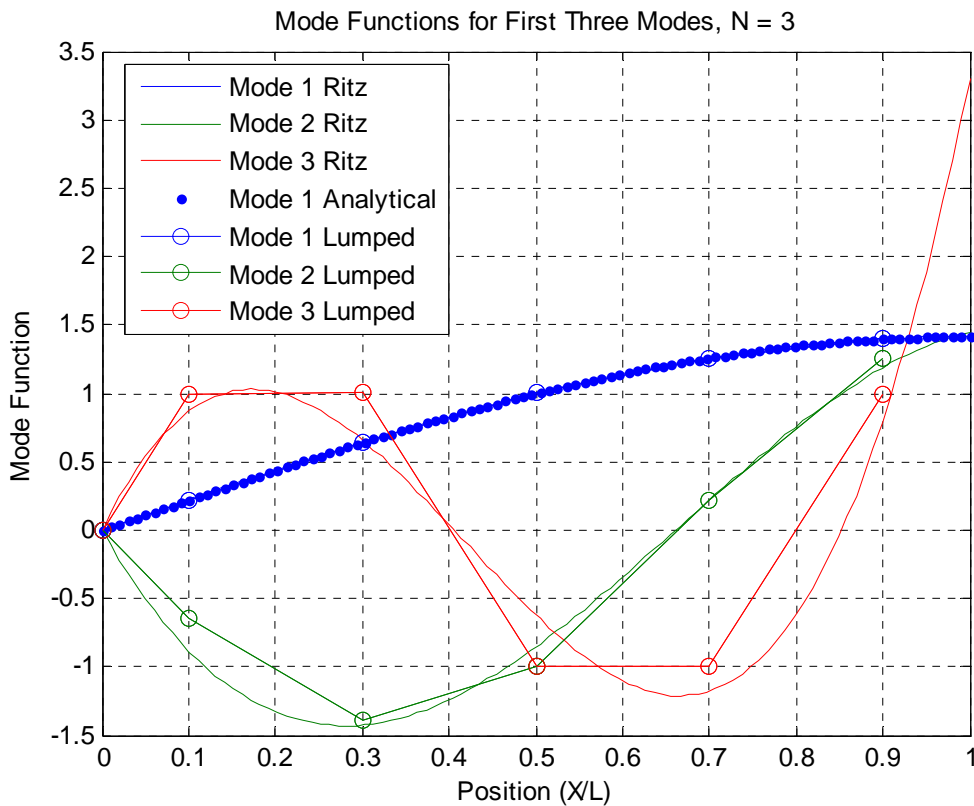
$$-11.2099\left(\frac{x}{L}\right) + 25.3536\left(\frac{x}{L}\right)^2 - 12.7003\left(\frac{x}{L}\right)^3$$

The third eigenfunction is approximated by:

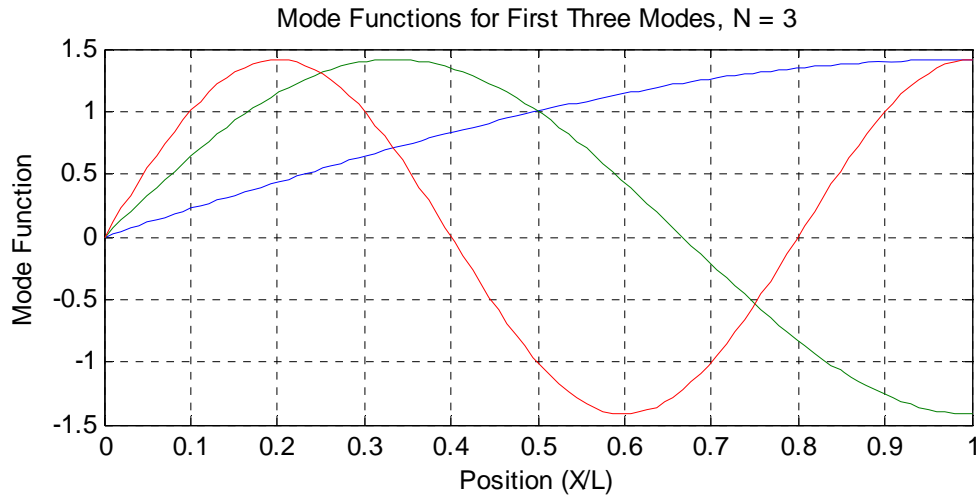
$$-13.0082\left(\frac{x}{L}\right) + 47.2984\left(\frac{x}{L}\right)^2 - 37.5941\left(\frac{x}{L}\right)^3$$

The three modeshapes obtained from the Ritz method are shown in Figure 4.1. It is seen that the first mode closely resembles a "1/4 sine," which is the exact modeshape for a fixed-free rod. The second mode resembles a "3/4 sine." The theoretical prediction of the third mode is a "5/4 sine." It is seen that there is considerable inaccuracy in the Ritz-method prediction for this mode. The Matlab code given at the back of the solution set gives a concise way of finding these approximate modeshapes given the modal matrix PHI.

The three mode shapes obtained from the Ritz method for part (b) are shown below. The first mode closely resembles a "1/4 sine," which is the exact mode shape for a fixed-free rod. The second resembles a "3/4 sine." The theoretical prediction of the third mode is a "5/4 sine," but one can see that there is considerable error in the Ritz approximation for that mode since the basis is inadequate to describe it.



When using the basis functions from part (c), the modes are exact and are given in the figure below.



The Matlab code used for these calculations follows. Symbolic variables were used to check the answer for part c), and to compute the  $M$  and  $K$  matrices for part d).

### Part b.)

```
% EMA 545, HW12
% Ritz Series solution for Clamped-free rod in extension
%
% M.S. Allen, May 2011

clear all; close all
syms x jj kk real;
tic
N = 5;

% Create basis functions as symbolic functions in Matlab - Symbolic
% variables x and jj and kk defined above
for jj = 1:N;
    psi(jj) = x^jj;
end
% Loop to create (jj,kk) terms of mass and stiffness matrices
for jj = 1:N;
    for kk = 1:jj;
        % Usig Matlab symbolics
        MM(jj,kk) = int(psi(jj)*psi(kk),x,0,1);% *rho*A*L
        KK(jj,kk) = int((diff(psi(jj),x,1)*diff(psi(kk),x,1)),x,0,1); % *EA/L
        % Using analytically derived formulas for M and K
        M1(jj,kk) = 1/(jj+kk+1);
        K1(jj,kk) = (jj*kk)/(jj+kk-1);
        % note - matrices are symmetric, so we can fill the rest of the
        % matrix in with the same terms.
        MM(kk,jj) = MM(jj,kk); KK(kk,jj) = KK(jj,kk);
        M1(kk,jj) = M1(jj,kk); K1(kk,jj) = K1(jj,kk);
    end
end
M = double(MM); % convert symbolic expressions to numbers.
C = 0; K = double(KK);
toc
% display the difference, which is on the order of numerical round off
% error.
M-M1
K-K1
```

```

% Use Embedding Property to find Eigensolutions for 2 <= N <= 5. For
% example, the EVP for N = 3 uses the 3x3 matrix in the upper left quadrant
% of M and K.
for p = 2:1:N;
    [phi,lambda] = eig(K(1:p,1:p),M(1:p,1:p));
    wnd(p-1,1:p) = sqrt(diag(lambda))'
end

% Finding Mode Functions
Ns = [3]; % number of basis functions to use for mode shape plots.
delta = 1/(100-1); % spacing for y-axis on mode shapes.
yd = [0:delta:1]';
for q = 1:length(Ns);
    psi_vals = 0; phi = 0; lambda = 0;
    [phi lambda] = eig(K(1:Ns(q),1:Ns(q)),M(1:Ns(q),1:Ns(q)));
    wns = sqrt(diag(lambda));
    for p = 1:1:100;
        for n = 1:Ns(q);
            % evaluate each basis function 'n' at each point 'p'. (or use
            yd(p)^n)
            psi_vals(p,n) = subs(psi(n),'x',yd(p)); % Matlab symbolics - sub
            yd(p) for 'y'
        end
    end
    % mode shapes for plotting are psi_vals*phi
    psi_c(1:100,1:Ns(q),q) = psi_vals(:,1:Ns(q))*phi;
end

% Analytical Mode Shape
psi_an = sin((2*1-1)/2*pi*yd);
psi_an = psi_an/max(psi_an)*max(psi_c(:,1,1)); % scale to same amplitude as
psi_c

% Plotting
figure(3)
plot(yd, psi_c(:,1,1),yd, psi_c(:,2,1), yd, psi_c(:,3,1),...
    yd, psi_an,'b. '); grid on; %, yd, psi_c(:,1,2), '.: ', yd,
    psi_c(:,3,2),'.: '); grid;
legend('Mode 1 Ritz','Mode 2 Ritz','Mode 3 Ritz','Mode 1 Analytical');
xlabel('Position (X/L)'); ylabel('Mode Function');
title(['Mode Functions for First Three Modes, N = ',num2str(Ns)]);

return

%% Plot the solution to Problem 2 on top:
M = eye(5); % Identity matrix since each mj=1;
K = eye(5)*2+diag([-1 -1 -1 -1],1)+diag([-1 -1 -1 -1],-1);
    % Above is a fancy way to make the banded stiffness matrix, although
    % the two terms below must be fixed manually.
K(1,1) = 1+2; K(5,5) = 1;
M = M/5; K = K*5; % change to non-dimensional

[phi,lam] = eig(K,M);
PHI=real(phi*inv(sqrt(phi.'*M*phi)));
xs = [0, 0.5:1:4.5]/5;

% To get a good plot, have to manually adjust the sign of some of the mode
% vectors (the sign of a mode vector is arbitrary). Make all end values
% positive:
PHI = PHI*diag(sign(PHI(end,:)));
hold on; plot(xs.',[0 0 0; PHI(:,1:3)],'o-'); hold off;
legend('Mode 1 Ritz','Mode 2 Ritz','Mode 3 Ritz','Mode 1 Analytical','Mode 1
Lumped','Mode 2 Lumped','Mode 3 Lumped');

```

**Part (c):**

```
% EMA 545, HW12
% Ritz Series solution for Clamped-free rod in extension
%
% M.S. Allen, April 2008

clear all; close all
syms x jj kk real;
tic
N = 3;

% Create basis functions as symbolic functions in Matlab - Symbolic
% variables x and jj and kk defined above
for jj = 1:N;
    psi(jj) = sin((2*jj-1)/2*pi*x);
end
% Loop to create (jj,kk) terms of mass and stiffness matrices
for jj = 1:N;
    for kk = 1:jj;
        % Usig Matlab symbolics
        MM(jj,kk) = int(psi(jj)*psi(kk),x,0,1);% *rho*A*L
        KK(jj,kk) = int((diff(psi(jj),x,1)*diff(psi(kk),x,1)),x,0,1); % *EA/L
        % note - matrices are symmetric, so we can fill the rest of the
        % matrix in with the same terms.
        MM(kk,jj) = MM(jj,kk); KK(kk,jj) = KK(jj,kk);
    end
end
M = double(MM); % convert symbolic expressions to numbers.
C = 0; K = double(KK);
toc

% Use Embedding Property to find Eigensolutions for 2 <= N <= 5. For
% example, the EVP for N = 3 uses the 3x3 matrix in the upper left quadrant
% of M and K.
for p = 2:1:N;
    [phi,lambda] = eig(K(1:p,1:p),M(1:p,1:p));
    wnd(p-1,1:p) = sqrt(diag(lambda))'
end

% Finding Mode Functions
Ns = [3]; % number of basis functions to use for mode shape plots.
delta = 1/(100-1); % spacing for y-axis on mode shapes.
yd = [0:delta:1]';
for q = 1:length(Ns);
    psi_vals = 0; phi = 0; lambda = 0;
    [phi lambda] = eig(K(1:Ns(q),1:Ns(q)),M(1:Ns(q),1:Ns(q)));
    wns = sqrt(diag(lambda));
    for p = 1:1:100;
        for n = 1:Ns(q);
            % evaluate each basis function 'n' at each point 'p'. (or use
            yd(p)^n)
            psi_vals(p,n) = subs(psi(n),'x',yd(p)); % Matlab symbolics - sub
            yd(p) for 'y'
        end
    end
end
% mode shapes for plotting are psi_vals*phi
psi_c(1:100,1:Ns(q),q) = psi_vals(:,1:Ns(q))*phi;
end
```