HW 11

EMA 545
Mechanical Vibrations

Spring 2013
University of Wisconsin, Madison

Nasser M. Abbasi

Spring 2013 Compiled on April 19, 2021 at 3:03am [public]

Contents

1 problem 1

Problem 1.)

Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{Q\}$$

with [C] matrix given by:

$$[C]=[M][\Phi]\begin{bmatrix}2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0\\ 0 & 2\varsigma_2\omega_2 & & & \vdots\\ 0 & 0 & \ddots & & \vdots\\ \vdots & & & \ddots & 0\\ 0 & & & 0 & 2\varsigma_N\omega_N\end{bmatrix}[\Phi]^T[M].$$

If we transform to normal coordinates using $\{q\}=[\Phi]\{\eta\}$, show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

The columns of matrix [Φ] are orthogonal w.r.t to the mass matrix. Hence the following two relations will be assumed as given in the derivation that follows

$$[\Phi]^T[M][\Phi]=\begin{bmatrix}1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & 1\end{bmatrix} \tag{1}$$

$$[\Phi]^T[K][\Phi]=\begin{bmatrix}\omega_1^2 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \omega_N^2\end{bmatrix}$$

Starting with the coupled EOM given, which is

$$[M]\{q''\}+[C]\{q'\}+[K]\{q\}=\{Q\}$$

Since $\{q\}=[\Phi]\{\eta\}$, then $\{q''\}=[\Phi]\{\eta''\}$ and $\{q'\}=[\Phi]\{\eta'\}$. Substituting these in the above EOM gives

$$[M][\Phi]\{\eta''\}+[C][\Phi]\{\eta'\}+[K][\Phi]\{\eta\}=\{Q\}$$

premultiplying by $[\Phi]^T$ both the LHS and RHS results in

$$[\Phi]^T[M][\Phi]\left\{\eta''\right\} + [\Phi]^T[C][\Phi]\left\{\eta'\right\} + [\Phi]^T[K][\Phi]\left\{\eta\right\} = [\Phi]^T\{Q\}$$

Using Eq 1 the above simplifies to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\eta''\right\} + [\Phi]^T[C][\Phi]\left\{\eta'\right\} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

Replacing $[C]$ by the expression given in the problem description, the above becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\eta''\right\} + [\Phi]^T\left([M][\Phi]\begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}[\Phi]^T[M]\right)[\Phi]\left\{\eta'\right\} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\eta''\right\} + \overbrace{[\Phi]^T[M][\Phi]}^{I}\begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 2\zeta_N\omega_N \end{bmatrix}\overbrace{[\Phi]^T[M][\Phi]}^{I}\left\{\eta'\right\} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

Since $[\Phi]^T[M][\Phi]$ is the identity matrix, then the above reduces to

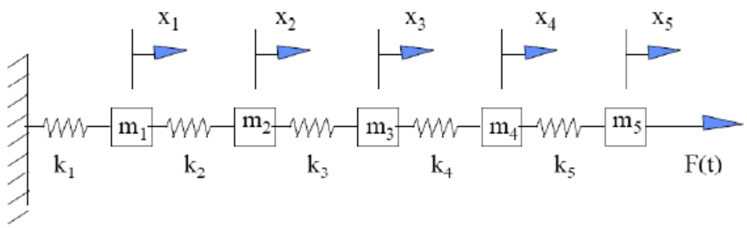
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}\left\{\eta''\right\} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 2\zeta_N\omega_N \end{bmatrix}\left\{\eta'\right\} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \omega_N^2 \end{bmatrix}\left\{\eta\right\} = [\Phi]^T\{Q\}$$

This is decoupled OEM since there is no coupling in the mass matrix, and no coupling in the damping matrix and no coupling in the stiffness matrix.

QED

2 Problem 2

Problem 2.)
The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs k_i which are drawn, there are viscous dampers c_i which are not shown. The equations of motion for this system are easily found to be:

$$[m]\ddot{\underline{x}} + [c]\dot{\underline{x}} + [k]\underline{x} = \underline{F}$$

where [m] is a diagonal matrix having entries m_i, and [k] is the banded matrix:

$$\begin{bmatrix} (k1+k2) & -k2 & 0 & 0 & 0 \\ -k2 & (k2+k3) & -k3 & 0 & 0 \\ 0 & -k3 & (k3+k4) & -k4 & 0 \\ 0 & 0 & -k4 & (k4+k5) & -k5 \\ 0 & 0 & 0 & -k5 & k5 \end{bmatrix}$$

The damping matrix has the same form as [k], but with viscous damping coefficients c_i in place of stiffness coefficients k_i. The forcing vector $\underline{F}$ is a 5 by 1 vector of zeros, except for the last entry: $\underline{F}=[0\ \ 0\ \ 0\ \ 0\ \ F(t)]^T$.

The numerical value for k_2 through k_5 is 1 N/m, the numerical value of k_1 = 2 N/m, the numerical value for m_1 through m_5 is 1 kg, and the numerical values for the viscous damping coefficients $c_i = 0.1*k_i$, i=1,5 (in units of N-s/m).

- **(a)** Find the natural frequencies and mass-normalized modes of the system.
- **(b)** Find the magnitude and phase of the steady-state response $x_5(t)$ assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to $1.2*\omega_5$. Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
- **(c)** Repeat the analysis in (b), but use the strategy described in Problem 1 to create a [C] matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
- **(d)** Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of γ=0.04.

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters EA and L are as follows: k_i = N*EA/L, i=2,N where N is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high, k_1 = 2N*EA/L. The lumped masses are equal to the total bar mass divided by N: $m_i = \rho AL/N$, where ρ is the mass density of the bar.

EOM is

$$\begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 & 0 \\ 0 & 0 & -c_4 & c_4+c_5 & -c_5 \\ 0 & 0 & 0 & -c_5 & c_5 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{Bmatrix} +$$

$$\begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 & 0 \\ 0 & 0 & -k_4 & k_4+k_5 & -k_5 \\ 0 & 0 & 0 & -k_5 & k_5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

substituting the numerical values gives $c_1 = 0.2, c_i = 0.1, i = 2, 5$, hence EOM becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \\ x_4'' \\ x_5'' \end{Bmatrix} + \frac{1}{10} \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{Bmatrix} + \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

2.1 part(a)

Natural frequency and mass normalized modes are found by solving the eigenvalue problem to find the natural frequencies and the mass normalized modes.

```
K=[3 -1 0 0 0;-1 2 -1 0 0;0 -1 2 -1 0;0 0 -1 2 -1;0 0 0 -1 1]
M=diag(ones(5,1));
[phi,omega]=eig(K,M);
omega = sqrt(diag(omega));
```

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 & -0.5635 & -0.6247 \\ -0.2871 & 0.6247 & -0.4472 & 0.0989 & 0.5635 \\ -0.4472 & 0.4472 & 0.4472 & 0.4472 & -0.4472 \\ -0.5635 & -0.0989 & 0.4472 & -0.6247 & 0.2871 \\ -0.6247 & -0.5635 & -0.4472 & 0.2871 & -0.0989 \end{bmatrix}$$

$$\begin{aligned} \omega &= \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\} \text{ rad/sec} \\ &= \{0.0498, 0.1445, 0.225, 0.284, 0.314\} \text{ hz} \end{aligned}$$

2.2 Part(b)

in modal coordinates, EOM is decoupled to become

$$I\left\{\eta''\right\} + [\Phi]^T[C][\Phi]\left\{\eta'\right\} + [\Phi]^T[k][\Phi]\left\{\eta\right\} = [\Phi]^T \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F(t) \end{Bmatrix}$$

```
EDU>> C  = 0.1*K;
C  = phi.'*C*phi
K  = phi.'*K*phi
syms f(t);
F  = zeros(5,1); F(5)=1;
F  = phi.'*F
C =
    0.0098    0.0000   -0.0000    0.0000    0.0000
    0.0000    0.0824   -0.0000    0.0000   -0.0000
   -0.0000   -0.0000    0.2000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    0.3176   -0.0000
    0.0000   -0.0000    0.0000   -0.0000    0.3902
K =
    0.0979    0.0000   -0.0000    0.0000    0.0000
    0.0000    0.8244   -0.0000    0.0000   -0.0000
   -0.0000   -0.0000    2.0000   -0.0000    0.0000
    0.0000    0.0000   -0.0000    3.1756   -0.0000
    0.0000         0    0.0000   -0.0000    3.9021
F =
   -0.6247
   -0.5635
```

```
-0.4472
 0.2871
-0.0989
```

Hence EOM in modal coordinates is

$$[I]\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\\ \eta_4''\\ \eta_5''\end{Bmatrix}+\begin{bmatrix}0.0098 & 0 & 0 & 0 & 0\\ 0 & 0.0824 & 0 & 0 & 0\\ 0 & 0 & 0.2 & 0 & 0\\ 0 & 0 & 0 & 0.3176 & 0\\ 0 & 0 & 0 & 0 & 0.3902\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\\ \eta_3'\\ \eta_4'\\ \eta_5'\end{Bmatrix}+$$

$$\begin{bmatrix}0.0979 & 0 & 0 & 0 & 0\\ 0 & 0.8244 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 3.1756 & 0\\ 0 & 0 & 0 & 0 & 3.9021\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix}=\begin{Bmatrix}-0.6247F(t)\\ -0.5635F(t)\\ -0.4472F(t)\\ 0.2871F(t)\\ -0.0989F(t)\end{Bmatrix}$$

Where in the above $F(t) = \cos(\varpi t)$ with ϖ being the forcing frequency in the range 0 to $1.2\omega_5$ where $\omega_5 = 1.9754$ rad/sec.

Since the equations are now decoupled, the 5^{th} equation can solved on its own

$$\eta_5'' + 0.3902\eta_5' + 3.9021\eta_5 = \text{Re}\left\{-0.0989e^{i\varpi t}\right\}$$

Assuming $\eta_5(t) = \text{Re}\left\{Xe^{i\varpi t}\right\}$ and substituting in the above and simplifying gives

$$\left(-\varpi^2 + i\varpi 0.3902 + 3.9021\right)X = -0.0989$$

$$X = \frac{-0.0989}{-\varpi^2 + i\varpi 0.3902 + 3.9021}$$

Hence

$$\eta_5(t) = \text{Re}\left\{\frac{-0.0989}{-\varpi^2 + i\varpi 0.3902 + 3.9021}e^{i\varpi t}\right\}$$

Similarly, all other $\eta_i, i = 1,5$ are found. Hence

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \operatorname{Re}\left(\begin{Bmatrix} \frac{-0.6247}{-\varpi^2+i\varpi 0.0098+0.0979} \\ \frac{-0.5635}{-\varpi^2+i\varpi 0.0824+0.8244} \\ \frac{-0.4472}{-\varpi^2+i\varpi 0.2+2} \\ \frac{0.2871}{-\varpi^2+i\varpi 0.3176+3.176} \\ \frac{-0.0989}{-\varpi^2+i\varpi 0.3902+3.9021} \end{Bmatrix} e^{i\varpi t} \right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\{\eta\}$. Hence

$$\begin{aligned} x_5 &= \sum_{j=1}^{5} \Phi(5,j)\eta(j) \\ &= \sum_{j=1}^{5} \Phi(5,j) \operatorname{Re}\left\{X(j)e^{i\varpi t}\right\} \\ &= \operatorname{Re}\left(\sum_{j=1}^{5} \Phi(5,j) X(j) e^{i\varpi t} \right) \\ &= \operatorname{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\varpi t} \right] \\ &= \operatorname{Re}\left[\left(\begin{array}{c} \frac{(-0.6247)(-0.6247)}{-\varpi^2+i\varpi 0.0098+0.0979} + \frac{(-0.5635)(-0.5635)}{-\varpi^2+i\varpi 0.0824+0.8244} + \frac{(-0.4472)(-0.4472)}{-\varpi^2+i\varpi 0.2+2} + \\ \frac{(0.2871)0.2871}{-\varpi^2+i\varpi 0.3176+3.176} + \frac{(-0.0989)(-0.0989)}{-\varpi^2+i\varpi 0.3902+3.9021} \end{array} \right) e^{i\varpi t} \right] \\ &= \operatorname{Re}\left[\left(\begin{array}{c} \frac{0.390\,25}{-\varpi^2+0.0098i\varpi+0.098} + \frac{0.08243}{-\varpi^2+0.3178i\varpi+3.176} + \frac{0.31753}{-\varpi^2+0.0824i\varpi+0.8244} + \\ \frac{0.00978}{-\varpi^2+0.3902i\varpi+3.9021} + \frac{0.19999}{-\varpi^2+0.2i\varpi+2} \end{array} \right) e^{i\varpi t} \right] \end{aligned}$$

Therefore

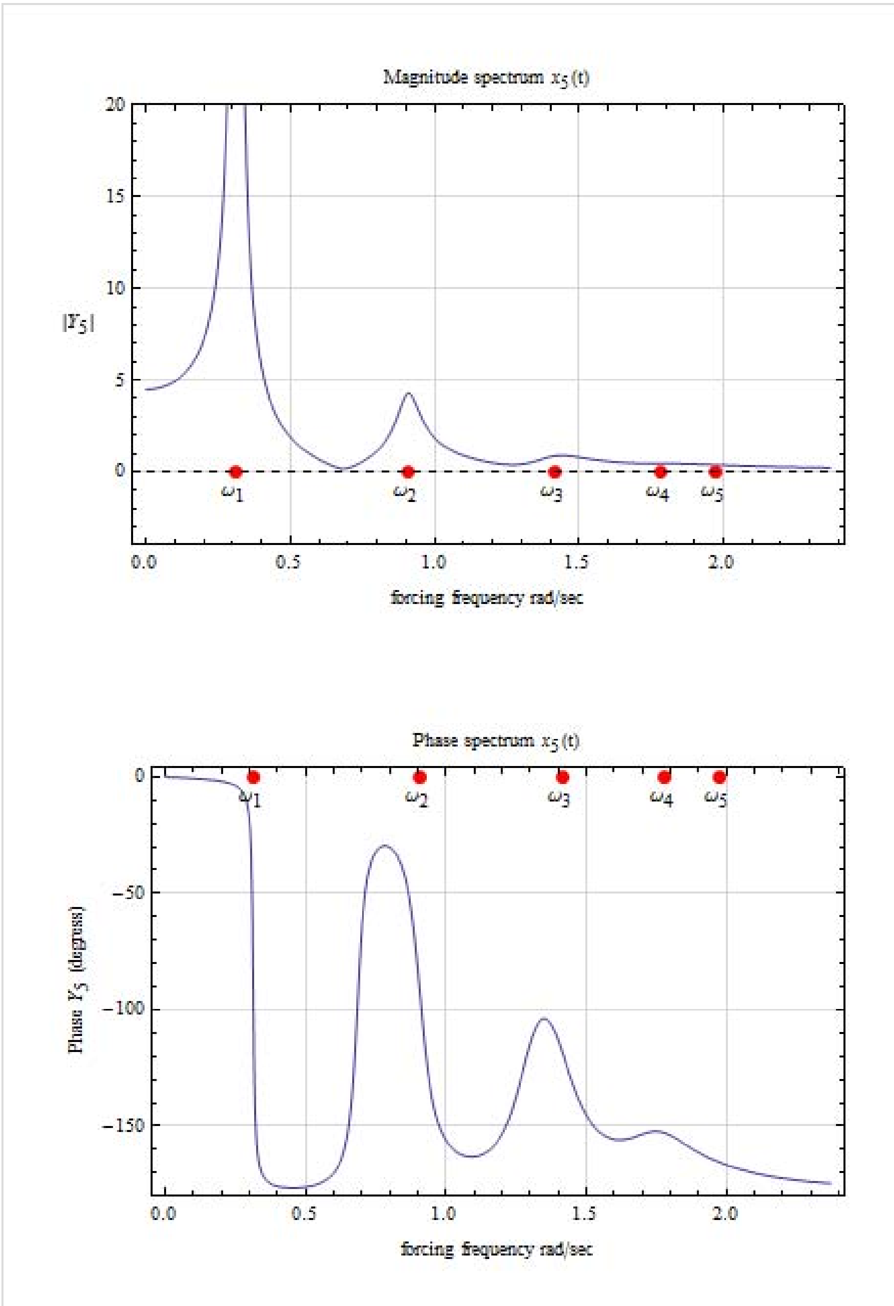
$$x_5 = \operatorname{Re}\left(Y_5 e^{i\varpi t}\right)$$

where

$$Y_5 = \frac{0.390\,25}{-\varpi^2 + 0.0098i\varpi + 0.098} + \frac{0.08243}{-\varpi^2 + 0.3178i\varpi + 3.176} + \frac{0.31753}{-\varpi^2 + 0.0824i\varpi + 0.8244} + \frac{0.00978}{-\varpi^2 + 0.3902i\varpi + 3.9021} + \frac{0.19999}{-\varpi^2 + 0.2i\varpi + 2}$$

Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$. This shows that $x_5(t)$ response will have the largest magnitude when the forcing frequency coincides with the first natural frequency (the fundamental frequency). In otherwords when $\varpi = \omega_1$.

The amplitude of $x_5(t)$ at resonance is smaller for the remaining 4 natural frequencies. For higher order natural frequencies, resonances at those frequencies produces lower amplitudes than lower order natural frequencies.



2.3 part(c)

Using $\zeta_i = \zeta = 0.02$ for $i = 1,5$ the EOM is

$$[I]\begin{Bmatrix}\eta_1''\\ \eta_2''\\ \eta_3''\\ \eta_4''\\ \eta_5''\end{Bmatrix} + 2\zeta\begin{bmatrix}\omega_1 & 0 & 0 & 0 & 0\\ 0 & \omega_2 & 0 & 0 & 0\\ 0 & 0 & \omega_3 & 0 & 0\\ 0 & 0 & 0 & \omega_4 & 0\\ 0 & 0 & 0 & 0 & \omega_5\end{bmatrix}\begin{Bmatrix}\eta_1'\\ \eta_2'\\ \eta_3'\\ \eta_4'\\ \eta_5'\end{Bmatrix} + \begin{bmatrix}\omega_1^2 & 0 & 0 & 0 & 0\\ 0 & \omega_2^2 & 0 & 0 & 0\\ 0 & 0 & \omega_3^2 & 0 & 0\\ 0 & 0 & 0 & \omega_4^2 & 0\\ 0 & 0 & 0 & 0 & \omega_5^2\end{bmatrix}\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix} = \begin{Bmatrix}-0.6247F(t)\\ -0.5635F(t)\\ -0.4472F(t)\\ 0.2871F(t)\\ -0.0989F(t)\end{Bmatrix}$$

Hence the solution

$$\eta_j = \mathrm{Re}\left\{X_j e^{i\varpi t}\right\}$$

where now

$$X_j = \frac{F_j}{-\varpi^2 + 2i\zeta_j\varpi\omega_j + \omega_j^2}$$

Hence, since $\omega = \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\}$ the solutions in modal coordinates is

$$\begin{Bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\end{Bmatrix} = \mathrm{Re}\left(\begin{Bmatrix}\frac{-0.6247}{-\varpi^2+2i\varpi(0.02)(0.3129)+0.0979}\\ \frac{-0.5635}{-\varpi^2+2i\varpi(0.02)(0.9080)+0.8244}\\ \frac{-0.4472}{-\varpi^2+2i\varpi(0.02)(1.4142)+2}\\ \frac{0.2871}{-\varpi^2+2i\varpi(0.02)(1.7820)+3.176}\\ \frac{-0.0989}{-\varpi^2+2i\varpi(0.02)(1.9754)+3.9021}\end{Bmatrix} e^{i\varpi t}\right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\{\eta\}$. Hence

$$\begin{aligned}
x_5 &= \sum_{j=1}^{5} \Phi(5,j)\eta(j) \\
&= \sum_{j=1}^{5} \Phi(5,j)\,\mathrm{Re}\left\{X(j)e^{i\varpi t}\right\} \\
&= \mathrm{Re}\left(\sum_{j=1}^{5} \Phi(5,j)X(j)e^{i\varpi t}\right) \\
&= \mathrm{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\varpi t}\right] \\
&= \mathrm{Re}\left[\left(\begin{array}{c} \frac{(-0.6247)(-0.6247)}{-\varpi^2+2i\varpi(0.02)(0.3129)+0.0979} + \frac{(-0.5635)(-0.5635)}{-\varpi^2+2i\varpi(0.02)(0.9080)+0.8244} + \frac{(-0.4472)(-0.4472)}{-\varpi^2+2i\varpi(0.02)(1.4142)+2} + \\ \frac{(0.2871)0.2871}{-\varpi^2+2i\varpi(0.02)(1.7820)+3.176} + \frac{(-0.0989)(-0.0989)}{-\varpi^2+2i\varpi(0.02)(1.9754)+3.9021} \end{array}\right)e^{i\varpi t}\right] \\
&= \mathrm{Re}\left[\left(\begin{array}{c} \frac{0.390\,25}{-\varpi^2+1.251\,6\times10^{-2}i\varpi+0.097\,9} + \frac{0.317\,53}{-\varpi^2+0.036\,32i\varpi+0.824\,4} + \frac{0.19999}{-\varpi^2+5.656\,8\times10^{-2}i\varpi+2.0} + \\ \frac{8.242\,6\times10^{-2}}{-\varpi^2+0.071\,28i\varpi+3.176} + \frac{9.781\,2\times10^{-3}}{-\varpi^2+7.901\,6\times10^{-2}i\varpi+3.902\,1} \end{array}\right)e^{i\varpi t}\right]
\end{aligned}$$

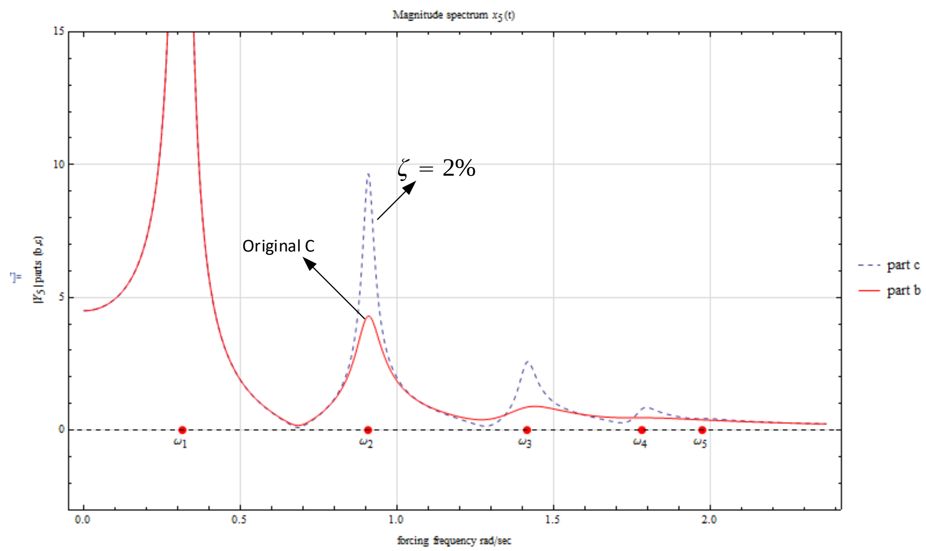
Therefore

$$x_5 = \mathrm{Re}\left(Y_5 e^{i\varpi t}\right)$$

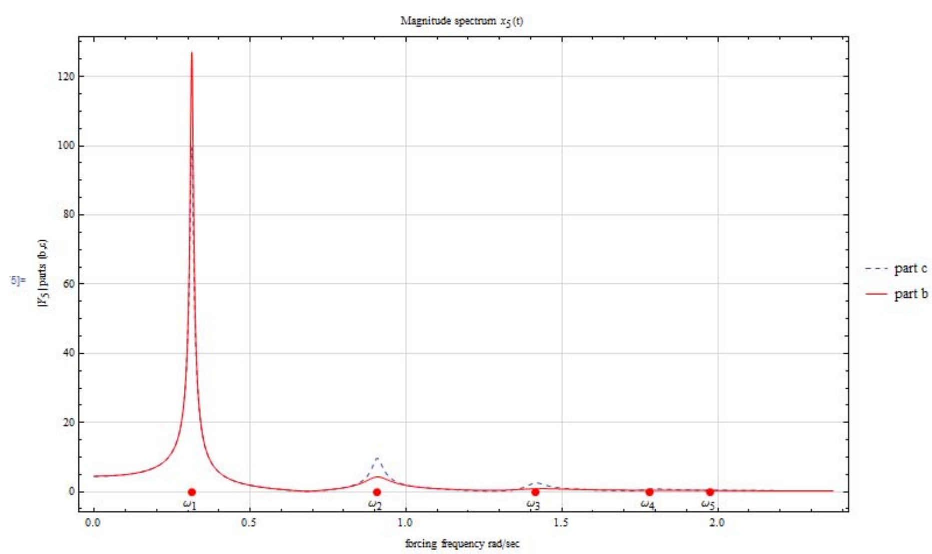
where

$$Y_5 = \frac{0.390\,25}{-\varpi^2 + 1.2516\times10^{-2}i\varpi + 0.0979} + \frac{0.317\,53}{-\varpi^2 + 0.036\,32i\varpi + 0.8244} + \frac{0.19999}{-\varpi^2 + 5.656\,8\times10^{-2}i\varpi + 2.0} + \frac{8.2426\times10^{-2}}{-\varpi^2 + 0.07128i\varpi + 3.176} + \frac{9.7812\times10^{-3}}{-\varpi^2 + 7.9016\times10^{-2}i\varpi + 3.902\,1}$$

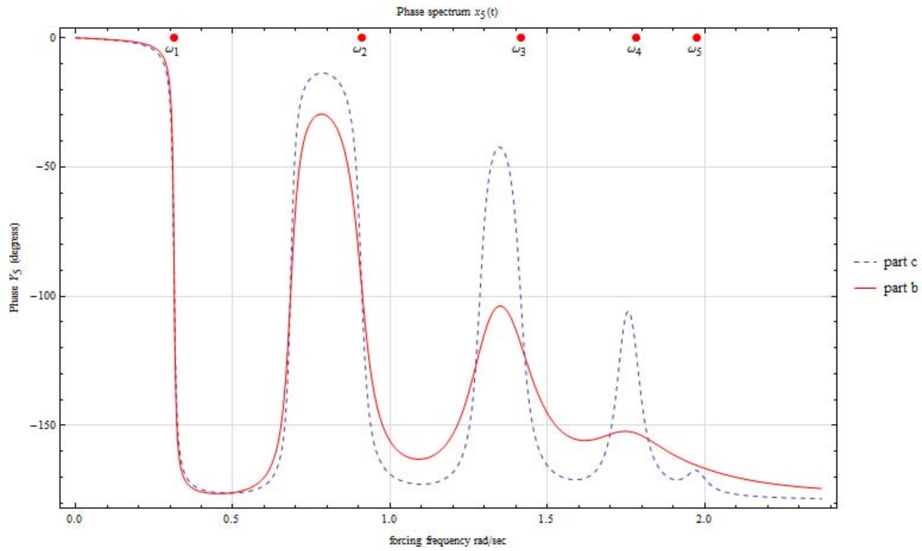
Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$ for both part(b) and (c) on the same plot



When $\zeta = 2\%$ was used, the resonance is seen to be higher (part c) compared to part (b). Here is a full range plot of the above.



Comparing the phase between part(b) and (c) gives



Which shows the effect on the phase spectrum.

2.4 Part (d)

In structural damping, the damping force is proportional to the elastic force. For example given an EOM $my'' + cy' + ky = f$, and converting to frequency domain to obtain transfer function

$$Y = \frac{F}{-\varpi^2 m + ic\varpi + k}$$

Then structural damping implies replacing $c\varpi$ with γk in the above, giving

$$Y = \frac{F}{-\varpi^2 m + i\gamma k + k} = \frac{F}{-\varpi^2 m + \left(1 + i\gamma\right)k}$$

The above method is now applied to the EOM given, and the resulting transfer function for x_5 is compared to the last results in order to see the effect of using structural damping on the response. The eigenvalue problem was solved in part (a) where the result was

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 & -0.5635 & -0.6247 \\ -0.2871 & 0.6247 & -0.4472 & 0.0989 & 0.5635 \\ -0.4472 & 0.4472 & 0.4472 & 0.4472 & -0.4472 \\ -0.5635 & -0.0989 & 0.4472 & -0.6247 & 0.2871 \\ -0.6247 & -0.5635 & -0.4472 & 0.2871 & -0.0989 \end{bmatrix}$$

$$\omega = \{0.3129, 0.9080, 1.4142, 1.7820, 1.9754\}$$

Hence the modal EOM is now

$$[I]\begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \\ \eta_4'' \\ \eta_5'' \end{Bmatrix} + (1 + i\gamma)\begin{bmatrix} 0.0979 & 0 & 0 & 0 & 0 \\ 0 & 0.8244 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3.1756 & 0 \\ 0 & 0 & 0 & 0 & 3.9021 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \begin{Bmatrix} -0.6247F(t) \\ -0.5635F(t) \\ -0.4472F(t) \\ 0.2871F(t) \\ -0.0989F(t) \end{Bmatrix}$$

Hence the steady state solution now in modal coordinates is Hence the solution

$$\eta_j = \mathrm{Re}\left\{X_j e^{i\varpi t}\right\}$$

where now

$$X_j = \frac{F_j}{-\varpi^2 + (1 + i\gamma)\omega_j^2}$$

The solutions in modal coordinates are (where $\gamma = 0.04$)

$$\begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{Bmatrix} = \mathrm{Re}\left(\begin{Bmatrix} \frac{-0.6247}{-\varpi^2+(1+i\gamma)0.0979} \\ \frac{-0.5635}{-\varpi^2+(1+i\gamma)0.8244} \\ \frac{-0.4472}{-\varpi^2+(1+i\gamma)2} \\ \frac{0.2871}{-\varpi^2+(1+i\gamma)3.176} \\ \frac{-0.0989}{-\varpi^2+(1+i\gamma)3.9021} \end{Bmatrix} e^{i\varpi t}\right)$$

and the solution in physical coordinates is now found from $\{x\} = [\Phi]\left\{\eta\right\}$. Hence

$$\begin{aligned}
x_5 &= \sum_{j=1}^{5}\Phi\left(5,j\right)\eta(j)\\
&= \sum_{j=1}^{5}\Phi\left(5,j\right)\operatorname{Re}\left\{X(j)e^{i\varpi t}\right\}\\
&= \operatorname{Re}\left(\sum_{j=1}^{5}\Phi\left(5,j\right)X(j)e^{i\varpi t}\right)\\
&= \operatorname{Re}\left[(-0.6247X_1(t) - 0.5635X_2(t) - 0.4472X_3(t) + 0.2871X_4(t) - 0.0989X_5(t))e^{i\varpi t}\right]\\
&= \operatorname{Re}\left[\left(\begin{array}{c}\frac{0.390\,25}{-\varpi^2+\left(1+i\gamma\right)0.0979} + \frac{0.317\,53}{-\varpi^2+\left(1+i\gamma\right)0.8244} + \frac{0.199\,99}{-\varpi^2+\left(1+i\gamma\right)2} + \\ \frac{8.242\,6\times10^{-2}}{-\varpi^2+\left(1+i\gamma\right)3.176} + \frac{9.781\,2\times10^{-3}}{-\varpi^2+\left(1+i\gamma\right)3.9021}\end{array}\right)e^{i\varpi t}\right]
\end{aligned}$$

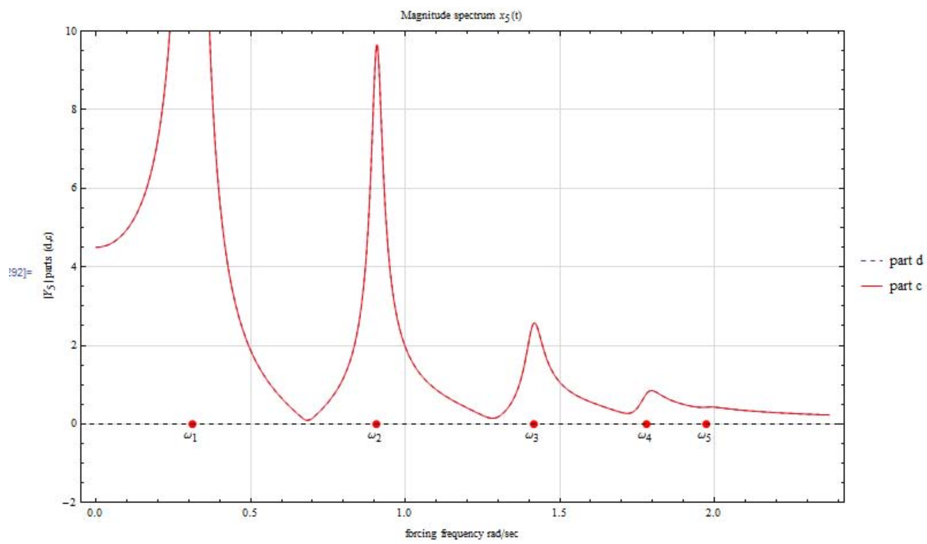
Therefore

$$x_5 = \operatorname{Re}\left(Y_5 e^{i\varpi t}\right)$$

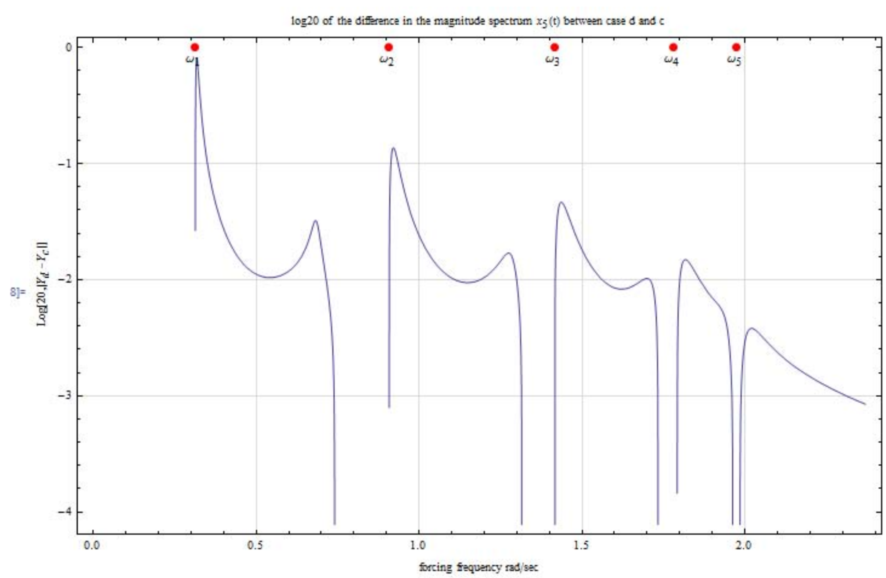
where

$$Y_5 = \frac{0.39025}{-\varpi^2 + \left(1 + i\gamma\right)0.0979} + \frac{0.31753}{-\varpi^2 + \left(1 + i\gamma\right)0.8244} + \frac{0.19999}{-\varpi^2 + \left(1 + i\gamma\right)2} + \frac{8.2426\times10^{-2}}{-\varpi^2 + \left(1 + i\gamma\right)3.176} + \frac{9.7812\times10^{-3}}{-\varpi^2 + \left(1 + i\gamma\right)3.9021}$$

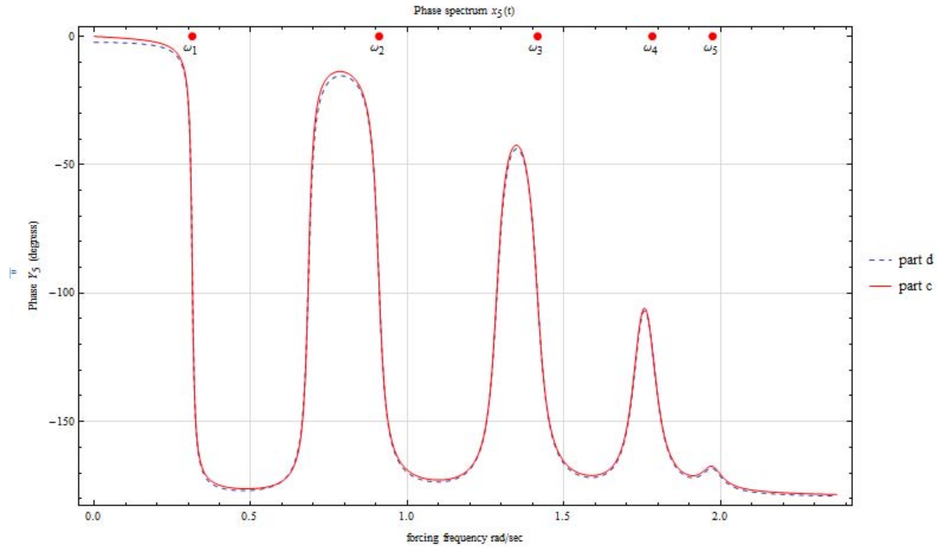
Here is a plot of the magnitude spectrum of Y_5 and the phase spectrum for the range of ϖ of 0 to $1.2\omega_5$ using the above transfer function, and superimposed on top of part (c). The magnitude spectrum is identical and no difference can be seen. Looking the phase spectrum there is very small change. Here are the plots. In the following plot, part(d) and (c) can not be distinguished. (the x-axis is drawn using dashed as well, not to be confused with the actual response curve).



To better see the difference, the plot was reproduced by taking the difference of the absolute values from part(d) and part(c) and plotting the log to base 20 of this difference. Now the difference can be better seen as very small.



The following the phase difference between case d and c.



The above plots show that using structural damping instead of using the same value of ζ for each EOM made very little difference in the result.

3 Problem 3

3.1 part(a)

Given $u(x,t)$ equations 6.1.1 and 6.1.2 in the text are

$$T_{bar} = \frac{1}{2}\int_0^L \dot{u}^2 \rho A dx \tag{2}$$

$$V_{bar} = \frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{3}$$

To obtain the mass matrix components T_{bar} is evaluated and each set of quadratic term are used to generate M_{jn} as follows. Using Ritz method, Let $u(x,t) = \sum_{j=1}^{N} \Psi_j(x) q_j(t)$. Substituting this in Eq 2 gives

$$\begin{aligned}
T_{bar} &= \frac{1}{2}\int_0^L \left(\frac{\partial}{\partial t}\sum_{j=1}^{N}\Psi_j(x)q_j(t)\right)^2 \rho A dx = \frac{1}{2}\int_0^L \left(\sum_{j=1}^{N}\Psi_j(x)q_j'(t)\right)^2 \rho A dx \\
&= \frac{1}{2}\int_0^L \left(\sum_{j=1}^{N}\Psi_j(x)q_j'(t)\right)\left(\sum_{n=1}^{N}\Psi_n(x)q_n'(t)\right)\rho A dx \\
&= \frac{1}{2}\int_0^L \left(\sum_{j=1}^{N}\sum_{n=1}^{N}\Psi_j(x)\Psi_n(x)q_j'(t)q_n'(t)\right)\rho A dx
\end{aligned}$$

Replacing order or integration with summation (since both are linear operations) and moving $q_j'(t)q_n'(t)$ outside the integration since it does not depend on x results in

$$T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}\left(\int_0^L \Psi_j(x)\Psi_n(x)\rho A dx\right)q_j'(t)q_n'(t) \tag{4}$$

Let

$$M_{jn} = \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx$$

Then eq 4 becomes Eq 6.1.11 in the textbook

$$T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N} M_{jn} q_j'(t)q_n'(t) \tag{5}$$

Now, obtain the components of the stiffness matrix. Starting with eq 3 and replacing $u(x,t)$ in this equation gives

$$\begin{aligned}
V_{bar} &= \frac{1}{2}\int_0^L EA\left(\frac{\partial}{\partial x}\sum_{j=1}^{N}\Psi_j(x)q_j(t)\right)^2 dx = \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\frac{d\Psi_j(x)}{dx}q_j(t)\right)^2 dx \\
&= \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\frac{d\Psi_j(x)}{dx}q_j(t)\right)\left(\sum_{n=1}^{N}\frac{d\Psi_n(x)}{dx}q_n(t)\right)dx \\
&= \frac{1}{2}\int_0^L EA\left(\sum_{j=1}^{N}\sum_{n=1}^{N}\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}q_j(t)q_n(t)\right)dx
\end{aligned}$$

Replacing order of integration with summation and moving $q_j(t)q_n(t)$ outside the integration since it does not depend on x gives

$$V_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}\left(\int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx\right)q_j(t)q_n(t)$$

Let $K_{jn} = \int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$ then the above becomes

$$V_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N} K_{jn} q_j q_n$$

Which is eq 6.1.13 in the book. QED.

3.2 Part(b)

The basic function to use are $\Psi_1 = \frac{x}{L}, \Psi_2 = \left(\frac{x}{L}\right)^2, \Psi_3 = \left(\frac{x}{L}\right)^3$. Let $u(x,t) = \sum_{j=1}^{3} \Psi_j(x) q_j(t)$. Now eq 6.1.11 and eq. 6.1.13 are used to obtain the mass matrix and the stiffness matrix components based on the power balance method. $T_{bar} = \frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N} M_{jn} q_j'(t) q_n'(t)$ where $M_{jn} = \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx$ hence

$$\begin{aligned}
M_{jn} &= \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx \\
&= \int_0^L \left(\frac{x}{L}\right)^j \left(\frac{x}{L}\right)^n \rho A dx \\
&= \int_0^L \left(\frac{x}{L}\right)^{j+n} \rho A dx \\
&= \frac{\rho A}{L^{j+n}} \int_0^L x^{j+n} dx \\
&= \frac{\rho A}{L^{j+n}} \left[\frac{x^{j+n+1}}{j+n+1}\right]_0^L = \frac{\rho A}{\left(j+n+1\right) L^{j+n}} L^{j+n+1} \\
&= \frac{\rho A L}{j+n+1}
\end{aligned}$$

Therefore, the mass matrix is

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} = \rho AL \begin{bmatrix} \frac{1}{1+1+1} & \frac{1}{1+2+1} & \frac{1}{1+3+1} \\ \frac{1}{2+1+1} & \frac{1}{2+2+1} & \frac{1}{2+3+1} \\ \frac{1}{3+1+1} & \frac{1}{3+2+1} & \frac{1}{3+3+1} \end{bmatrix} = \rho AL \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

and $\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}K_{jn}q_jq_n$ where $K_{jn} = \int_0^L EA\frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$, hence

$$
\begin{aligned}
K_{jn} &= \int_0^L EA\frac{d\left(\frac{x}{L}\right)^j}{dx}\frac{d\left(\frac{x}{L}\right)^n}{dx}dx = \int_0^L EAj\left(\frac{x^{j-1}}{L^j}\right)n\left(\frac{x^{n-1}}{L^n}\right)dx \\
&= \frac{EAjn}{L^jL^n}\int_0^L x^{j-1}x^{n-1}dx \\
&= \frac{EAjn}{L^{j+n}}\int_0^L x^{j+n-2}dx \\
&= \frac{EAjn}{L^{j+n}}\left[\frac{x^{j+n-1}}{j+n-1}\right]_0^L \\
&= \frac{EAjn}{\left(j+n-1\right)L^{j+n}}\left[x^{j+n-1}\right]_0^L \\
&= \frac{EAjn}{\left(j+n-1\right)L^{j+n}}L^{j+n-1} \\
&= \frac{EA}{L}\frac{jn}{j+n-1}
\end{aligned}
$$

Hence the stiffness matrix is

$$
\begin{aligned}
K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} &= \frac{EA}{L}\begin{bmatrix} \frac{1(1)}{1+1-1} & \frac{1(2)}{1+2-1} & \frac{1(3)}{1+3-1} \\ \frac{2(1)}{2+1-1} & \frac{2(2)}{2+2-1} & \frac{2(3)}{2+3-1} \\ \frac{3(1)}{3+1-1} & \frac{3(2)}{3+2-1} & \frac{3(3)}{3+3-1} \end{bmatrix} \\
&= \frac{EA}{L}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}
\end{aligned}
$$

3.3 Part(c)

The basic function to use are $\Psi_r = \sin\left(\alpha_r\frac{x}{L}\right)$ where $\alpha_r = \left(\frac{2r-1}{2}\right)\pi$ for $r = 1, 2, 3$.

Let $u(x,t) = \sum_{j=1}^{3}\Psi_j(x)q_j(t)$. Now eq 6.1.11 and eq. 6.1.13 are used to obtain the mass matrix and the stiffness matrix components based on the power balance method. $T_{bar} =$

$\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N}M_{jn}q_j'(t)q_n'(t)$ where $M_{jn}=\int_0^L \Psi_j(x)\Psi_n(x)\rho A dx$ hence

$$\begin{aligned}
M_{jn} &= \int_0^L \Psi_j(x)\Psi_n(x)\rho A dx \\
&= \int_0^L \sin\left(\alpha_j\frac{x}{L}\right)\sin\left(\alpha_n\frac{x}{L}\right)\rho A dx \\
&= \int_0^L \sin\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L}\right)\sin\left(\left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right)\rho A dx
\end{aligned}$$

Using $\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$ the above can be solved.

$$\begin{aligned}
M_{jn} &= \int_0^L \frac{1}{2}\left(\cos\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L}-\left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right)-\cos\left(\left(\frac{2j-1}{2}\right)\pi\frac{x}{L}+\left(\frac{2n-1}{2}\right)\pi\frac{x}{L}\right)\right)\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos\left(\frac{(2j-1)-(2n-1)}{2}\right)\pi\frac{x}{L}-\cos\left(\frac{(2j-1)+(2n-1)}{2}\right)\pi\frac{x}{L}\right]\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos\left(\frac{2(j-n)}{2}\right)\pi\frac{x}{L}-\cos\left(\frac{2(j+n)-2}{2}\right)\pi\frac{x}{L}\right]\rho A dx \\
&= \int_0^L \frac{1}{2}\left[\cos(j-n)\pi\frac{x}{L}-\cos((j+n)-1)\pi\frac{x}{L}\right]\rho A dx
\end{aligned}$$

For $j=1,n=1$ the above gives

$$M_{jn}=\int_0^L \frac{1}{2}\left[1-\cos\pi\frac{x}{L}\right]\rho A dx=\frac{\rho A}{2}\int_0^L 1-\cos\pi\frac{x}{L}dx=\frac{\rho A}{2}\left(L-\left(\frac{\sin\pi\frac{x}{L}}{\frac{\pi}{L}}\right)_0^L\right)=\frac{\rho A}{2}L$$

For $j=1,n=2$

$$\begin{aligned}
M_{jn} &= \int_0^L \frac{1}{2}\left[\cos\left(-\pi\frac{x}{L}\right)-\cos 2\pi\frac{x}{L}\right]\rho A dx=\frac{\rho A}{2}\int_0^L \cos\left(\pi\frac{x}{L}\right)-\cos\left(2\pi\frac{x}{L}\right)dx \\
&= \frac{\rho A}{2}\left[\left(\frac{\sin\left(\pi\frac{x}{L}\right)}{\frac{\pi}{L}}\right)_0^L-\left(\frac{\sin\left(2\pi\frac{x}{L}\right)}{\frac{2\pi}{L}}\right)_0^L\right] \\
&= \frac{\rho A}{2}[0-0]=0
\end{aligned}$$

The rest of the computation is now done using a small code below to generate the final mass and stiffness matrix

- **problem 3 part c**

```
2]:= Clear[L, x, α, M, ρ , A, Ψ, k];
     Ψ[n_] := Sin[α[n] x/L];
     α[n_] := (2 n - 1)/2 π;
     M[j_, n_] := ∫_0^L Ψ[j] Ψ[n] ρ A dx
     k[j_, n_] := ∫_0^L e A D[Ψ[j], x] D[Ψ[n], x] dx

7]:= MatrixForm @ Table[M[i, j], {i, 1, 3}, {j, 1, 3}]
```

/MatrixForm=

$$\begin{pmatrix} \frac{A L \rho}{2} & 0 & 0 \\ 0 & \frac{A L \rho}{2} & 0 \\ 0 & 0 & \frac{A L \rho}{2} \end{pmatrix}$$

```
8]:= MatrixForm @ Table[k[i, j], {i, 1, 3}, {j, 1, 3}]
```

/MatrixForm=

$$\begin{pmatrix} \frac{A e \pi^2}{8 L} & 0 & 0 \\ 0 & \frac{9 A e \pi^2}{8 L} & 0 \\ 0 & 0 & \frac{25 A e \pi^2}{8 L} \end{pmatrix}$$

Therefore, the mass matrix is

$$M = \frac{AL\rho}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\frac{1}{2}\sum_{j=1}^{N}\sum_{n=1}^{N} K_{jn} q_j q_n$ where $K_{jn} = \int_0^L EA \frac{d\Psi_j(x)}{dx}\frac{d\Psi_n(x)}{dx}dx$, hence

$$K_{jn} = \int_0^L EA \frac{d\left(\Psi_j\right)}{dx}\frac{d(\Psi_n)}{dx}dx$$

From the above code, the result is

$$K = \frac{AE\pi^2}{8L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Using this set of basis functions produces mass and stiffness matrices that are already decoupled. This is good.

3.4 part (d)

The natural frequencies obtained in problem 2 were

$$problem2 \Rightarrow M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 3 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{aligned} \omega &= (0.3129, 0.9080, 1.4142, 1.7820, 1.9754) \text{ rad/sec} \\ &= (0.0498, 0.1445, 0.225, 0.284, 0.314) \text{ hz} \end{aligned}$$

Now the eigenvalue problem $\det\left([k] - \omega^2[M]\right)$ is solved again using the mass and stiffness matrices in parts b,c above and the natural frequencies are compared with the above result from problem 2. Recall, the M and K from part b were

$$part(b) \Rightarrow M = \rho AL\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, K = \frac{EA}{L}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}$$

$$part(c) \Rightarrow M = \frac{AL\rho}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, K = \frac{AE\pi^2}{8L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

First, a numerical values given at end of problem 2 are used, therefore $\rho AL = m = 1$ and

$\frac{EA}{L} = \frac{1}{2}$, hence the K and M for part(b) and c become

$$part(b) \Rightarrow M = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}, K = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{9}{5} \end{bmatrix}$$

$$part(c) \Rightarrow M = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, K = \frac{\pi^2}{16}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

The natural frequencies are found. Here is a summary table

	ω (rad/sec)	f hz
problem 2	0.3129, 0.9080, 1.4142, 1.7820, 1.9754	0.0498, 0.1445, 0.225, 0.284, 0.314
part(b)	1.1108, 3.4199, 7.3872	0.1768, 0.5443, 1.1757
part(c)	1.1107, 3.3322, 5.5536	0.1768, 0.5303, 0.8839

It can be seen that the first three natural frequencies using Ritz basic functions as given for both part b and c are higher than the natural frequencies generated by part b.

The stiffness matrix K for both parts b and c contains much smaller numerical values than the one used in problem 2. Since $\omega^2 = \frac{k}{m}$ then one expects this result.

3.5 Part(e)

The first 3 mode shapes from problem 2 were

$$[\Phi] = \begin{bmatrix} -0.0989 & 0.2871 & -0.4472 \\ -0.2871 & 0.6247 & -0.4472 \\ -0.4472 & 0.4472 & 0.4472 \\ -0.5635 & -0.0989 & 0.4472 \\ -0.6247 & -0.5635 & -0.4472 \end{bmatrix}$$

The mode shapes from part(b)

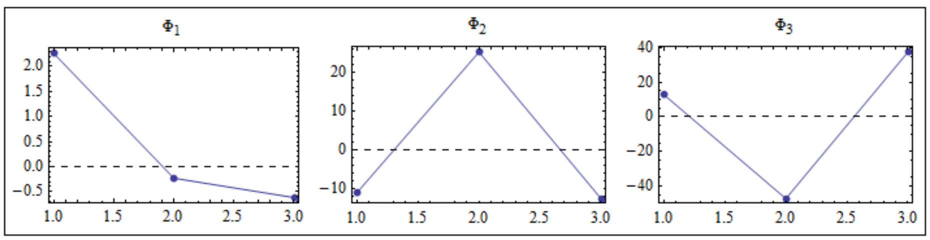
$$[\Phi] = \begin{bmatrix} 2.2642 & -11.2099 & 13.0082 \\ -0.2314 & 25.3536 & -47.2984 \\ -0.6181 & -12.7003 & 37.5941 \end{bmatrix}$$

The mode shapes from part(c)

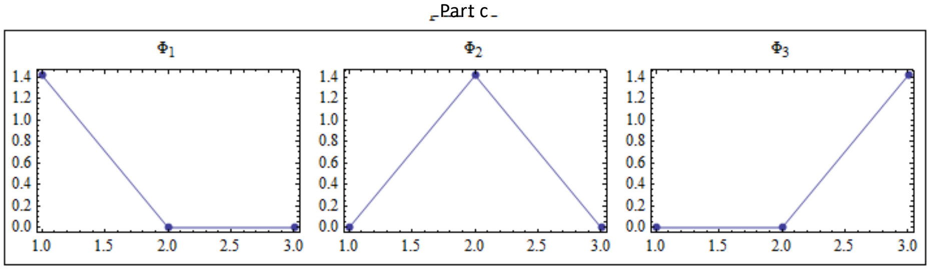
$$[\Phi] = \begin{bmatrix} 1.4142 & 0 & 0 \\ 0 & 1.4142 & 0 \\ 0 & 0 & 1.4142 \end{bmatrix}$$

Here is a plot of the above mode shapes

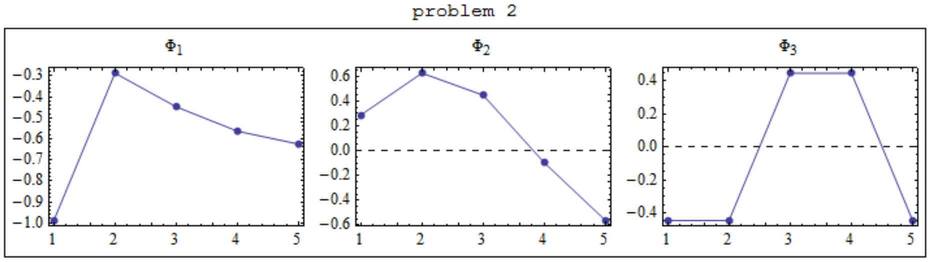
part b



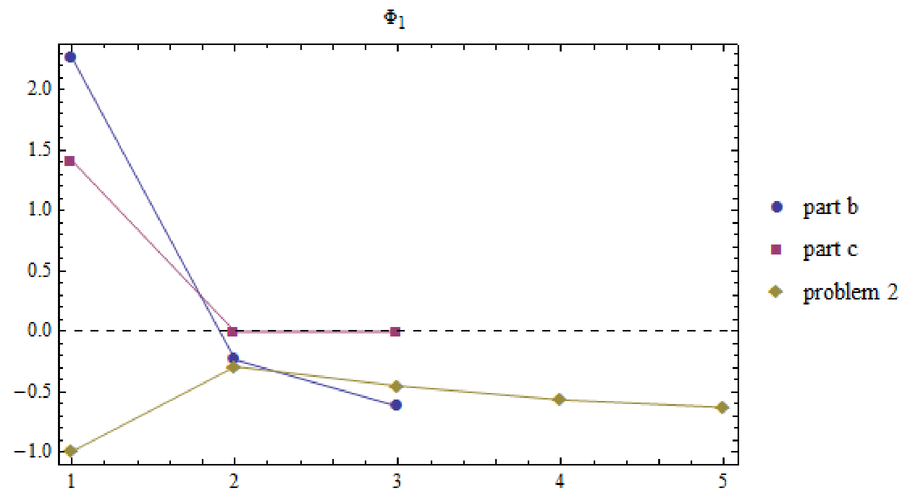
Part c



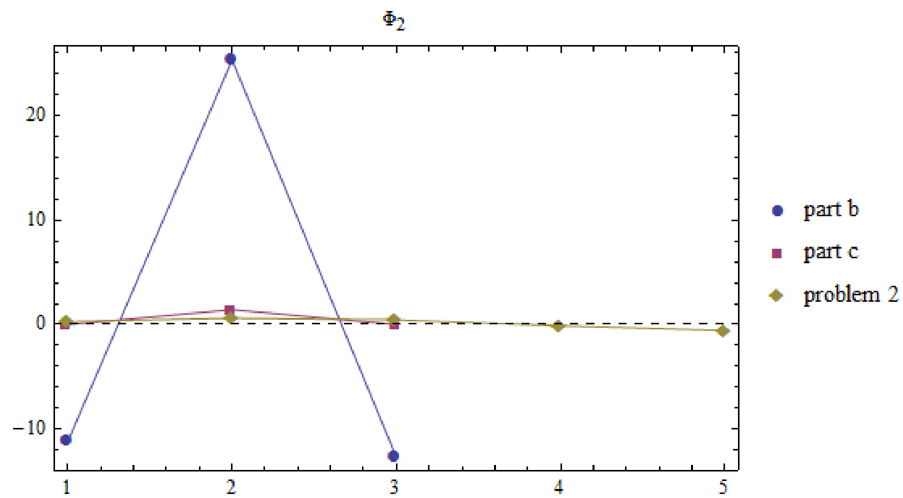
problem 2



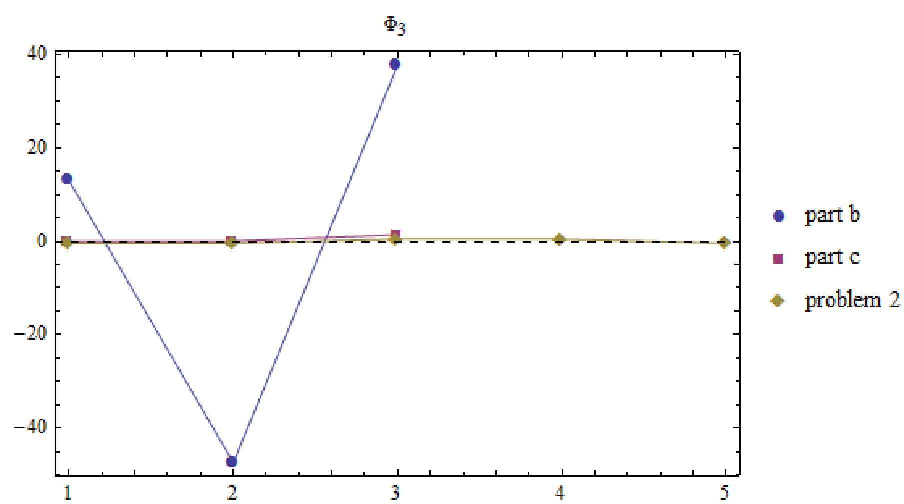
Here is a plot of the mode shapes overlay.



Φ_1
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
1
2
3
4
5
part b
part c
problem 2



Φ_2
20
10
0
−10
1
2
3
4
5
part b
part c
problem 2



Φ_3
40
20
0
−20
−40
1
2
3
4
5
part b
part c
problem 2