Homework #10
EMA 545, Spring 2013

For all of these problems you may use Matlab or some other package to find the natural frequencies and mode vectors and to mass normalize the mode vectors (if needed).

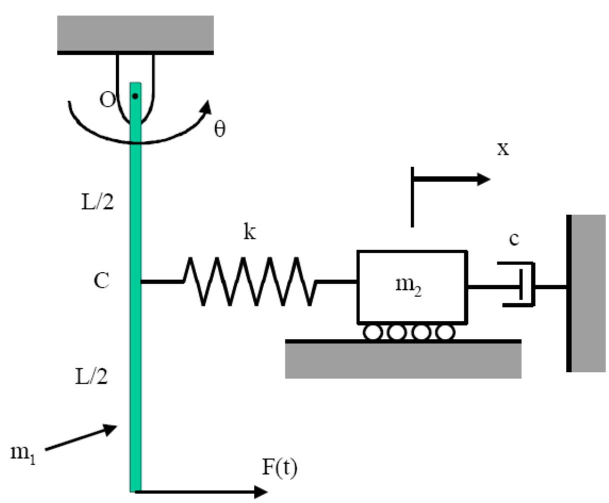
1.) Exercise 4.8 from Ginsberg. (Note: the spring constants are defined such that the frequencies given are the natural frequencies that each spring-mass system would have if it were attached to a rigid base. Notice that there is not a simple relationship between those frequencies and the natural frequencies of the system as a whole.)

2.) Exercise 4.30 as given in the text. Repeat the analysis for $k=2mg/L$ and graph that response as well. (Questions to consider: What do you notice about the natural frequencies of this system in each case? How does that affect the way the response looks? Why?)

3.) Exercise 4.43 from Ginsberg. How does the time required to reach steady state compare with $\tau_r=1/(\zeta_r\omega_r)$ for each mode, r=1,2,3?

4.) Exercise 4.47 from Ginsberg.

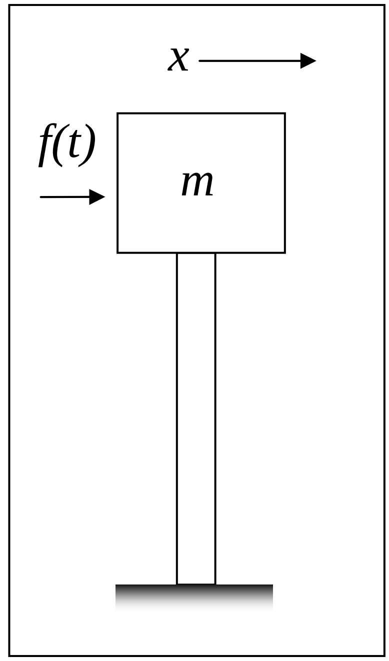
5.) (Covering material from Chapter 5)
A uniform rod of length L and mass m_1 is attached to a cart having mass m_2 by means of a spring k. A viscous damper c resists the motion of the cart.



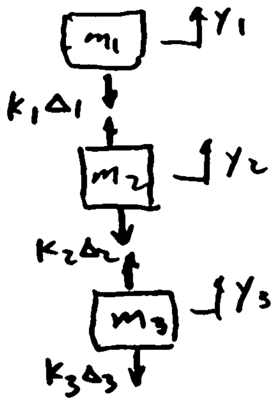
a.) Let $F(t)=\mathrm{Re}[F\exp(i\omega t)]$, $x(t) = \mathrm{Re}[X\exp(i\omega t)]$ and $\theta(t)=\mathrm{Re}[Y\exp(i\omega t)]$. Find analytical expressions for the complex transfer functions X/F and Y/F.
b.) Find the magnitude and phase of the response of x and θ when the system is forced at its natural frequencies $\omega=\omega_1$ and $\omega=\omega_2$. Compare these values to the eigenvectors for modes 1 and 2. Use the following numerical values: $m_1=m_2=1$ kg, k=3 N/m, L=1 m, g=9.81 m/s^2, and c=0.1 N-s/m.

c.) Plot the transfer functions Y/F and X/F over a range of frequencies encompassing both modes of vibration. Use the plot to determine at what frequency m_2 acts as a vibration absorber for the rod. How does that frequency compare with the natural frequency that the system would have if the rod were held fixed: $\omega_{cart} = (k/m_2)^{1/2}$?

6.) Consider **Exercise 3.45 and 3.46** in the text (you solved this in problem #3 in HW#6). Use the steady-state displacement that you computed using FFT techniques for $\tau = 3\pi/\omega_n$ to compute the maximum stress in the spring. Assume that the spring is a cantilever beam (in bending) modeled after one of the pillars supporting the ERB, which have length L=40m, rectangular cross section with height h, equal to the width b=h=0.6m, and is constructed from a material with modulus E=30 GPa and ultimate tensile strength σ=40 MPa. (The mass of the beam is assumed to be included in *m*, so its density is not needed.) Let the mass *m* be such that the natural frequency of the mass-spring system is ω_n=0.2Hz. What is the amplitude of the force, P, such that the beam fails due to the dynamic load? Compare that to the static load required to cause the beam to fail (also in bending).



Exercise 4.8



$m_1 = 100,\ m_2 = 200,\ m_3 = 300$ kg

$k_j = \omega_j^2 m_j$, $\omega_1 = 40,\ \omega_2 = 50,\ \omega_3 = 60$ rad/s

$q_1 = Y_1,\ q_2 = Y_2,\ q_3 = Y_3$

$$[M] = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2+k_3 \end{bmatrix} = \begin{bmatrix} 1.6 & -1.6 & 0 \\ -1.6 & 6.6 & -5 \\ 0 & -5 & 15.8 \end{bmatrix} \times 10^5$$

Divide all equations by 100:

$$\left[[K] - \omega^2[M]\right]\{\phi\} = \begin{bmatrix} 1600-\omega^2 & -1600 & 0 \\ -1600 & 6600-2\omega^2 & -5000 \\ 0 & -5000 & 15800-3\omega^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Char eq: $(1600-\omega^2)(6600-2\omega^2)(15800-3\omega^2)$

$-(1600)^2(15800-3\omega^2) - (5000)^2(1600-\omega^2) = 0$

$$6\omega^6 - 61(10^3)\omega^4 + 153.84(10^6)\omega^2 - 86.4(10^9) = 0 \quad \Longleftarrow$$

$\omega^2 = 789.7,\ 2752.8,\ 6624.2$

Set $\phi_{1j} = 1$ & use first two eqs

$$\begin{Bmatrix} 1600-\omega_j^2 \\ -1600 \end{Bmatrix} + \begin{bmatrix} -1600 & 0 \\ 6600-2\omega_j^2 & -5000 \end{bmatrix} \begin{Bmatrix} \phi_{2j} \\ \phi_{3j} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\phi_{2j} = \frac{1600-\omega_j^2}{1600} \quad \& \quad \phi_{3j} = \frac{-1600 + (6600-2\omega_j^2)\phi_{2j}}{5000}$$

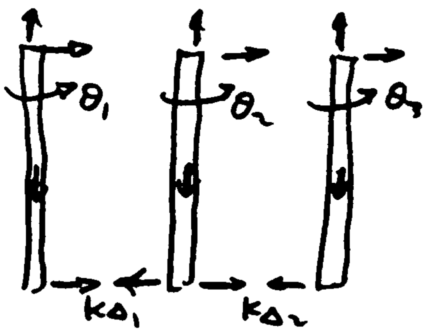
Thus

$\omega_1 = 28.10$ rad/s, $\{\phi_1\} = \begin{Bmatrix} 1 \\ 0.5065 \\ 0.1885 \end{Bmatrix}$

$\omega_2 = 52.47$ rad/s, $\{\phi_2\} = \begin{Bmatrix} 1 \\ -0.7205 \\ -0.4777 \end{Bmatrix} \quad \Longleftarrow$

$\omega_3 = 81.39$ rad/s $\{\phi_3\} = \begin{Bmatrix} 1 \\ -3.140 \\ 3.855 \end{Bmatrix}$

Exercise 4.30



$q_1 = \theta_1,\ q_2 = \theta_2,\ q_3 = \theta_3$

$T = \frac{1}{2} I_0 (\dot\theta_1^2 + \dot\theta_2^2 + \dot\theta_3^2)$

$I_0 = \frac{1}{3} m L^2$

$V = V_{sp} + V_{gr}$

$V_{sp} = \frac{1}{2} k \Delta_1^2 + \frac{1}{2} k \Delta_2^2$

$\Delta_1 = L(\theta_2 - \theta_1),\ \Delta_2 = L(\theta_3 - \theta_2)$

$$V_{sp} = \frac{1}{2} k L^2 [(\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2]$$
$$= \frac{1}{2} k L^2 [\theta_1^2 + 2\theta_2^2 + \theta_3^2 - 2\theta_1\theta_2 - 2\theta_2\theta_3]$$

$(K_{11})_{sp} = kL^2,\ (K_{22})_{sp} = 2kL^2,\ (K_{33})_{sp} = kL^2$

$(K_{12})_{sp} = -kL^2,\ (K_{23})_{sp} = -KL^2,\ (K_{13})_{sp} = 0$

$V_{gr} = -mg\frac{L}{2}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$

$(K_{11})_{gr} = \left(\frac{\partial^2 V_{sp}}{\partial \theta_1^2}\right)_{\theta_j = 0} = mg\frac{L}{2}$, similarly $(K_{22})_{gr} = (K_{33})_{gr} = mg\frac{L}{2}$

$(K_{12})_{gr} = \left(\frac{\partial^2 V}{\partial\theta_1 \partial\theta_2}\right)_0 = 0 = (K_{23})_{gr} = (K_{13})_{gr}$

Set $k = 0.05 mgL$

$$[K] = [K_{sp}] + [K_{gr}] = mgL \begin{bmatrix} 0.55 & -0.05 & 0 \\ -0.05 & 0.60 & -0.05 \\ 0 & -0.05 & 0.55 \end{bmatrix}$$

Evaluate ω_j & $[\Phi]$ Scaling factor for ω_j is $\left(\frac{g}{L}\right)^{1/2}$

Set $\{q\} = [\Phi]\{\eta\} \Rightarrow \ddot\eta_j + \omega_j^2 \eta_j = 0$

Initial conditions $\{q\} = \{0\}$ & $\{\dot q\} = \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$ rad/s @ $t=0$

Thus $\{\eta\} = 0$ & $\{\dot\eta\} = [\Phi]^T [M] \begin{Bmatrix} 0 \\ 2 \\ 0 \end{Bmatrix}$ at $t=0$

Complementary solution $\eta_j = A_j \cos(\omega_j t) + B_j \sin(\omega_j t)$

Satisfy initial condition $\Rightarrow A_j = 0,\ B_j = \dot\eta_j(0)/\omega_j$

Evaluate η_j at discrete t_n covering four fundamental periods: $[\eta] = [\{\eta(t_1)\}\ \{\eta(t_2)\} \dots]$

Then $[q] = [\Phi][\eta]$; Rotations as a function of $\left(\frac{g}{L}\right)^{1/2} t$

HW 4.30 Solution

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Spring 2013

Using the equations of motion and modal responses derived on the previous page, the following Matlab code can then be used to find the transient response:

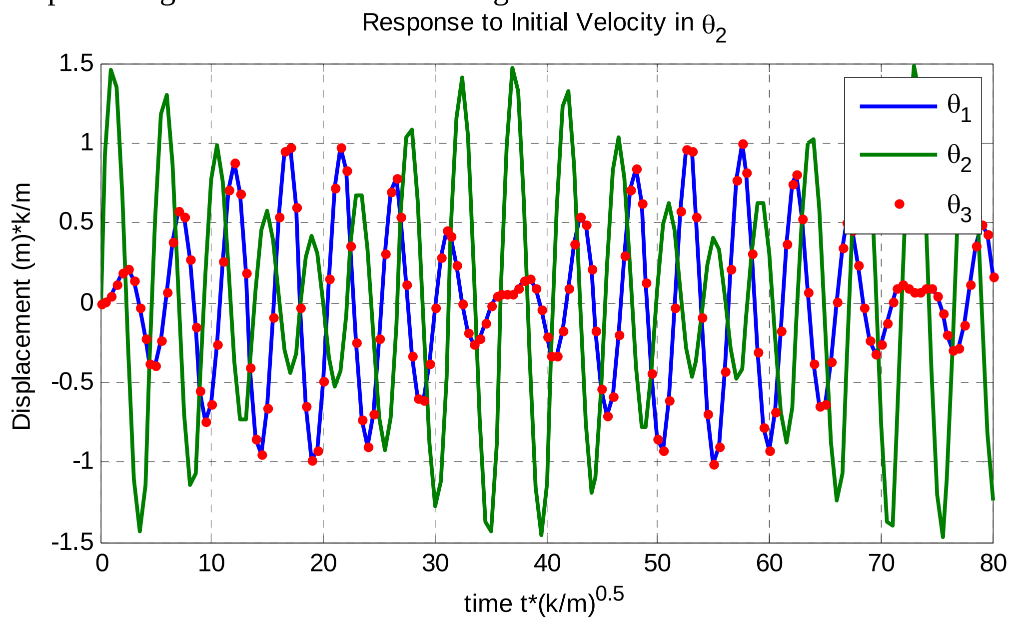
```
M = eye(3)/3; %*mL^2
 K = [1 -1 0; -1 2 -1; 0 -1 1]*0.05 + eye(3)*0.5; % Kspr + Kgrav

 [phi,lam] = eig(K,M);
 wns = sqrt(diag(lam));

 % Sort & Normalize Eigenvectors to unity modal mass and Check
Orthogonality
[lam_sort,lam_indx] = sort(diag(lam));
wns = sqrt(lam_sort) % *sqrt(k/m)

phi_sort = (phi(:,lam_indx));
mu = phi_sort.'*M*phi_sort;
PHI = real(phi_sort*sqrt(inv(mu)))

check_orth = norm(PHI.'*M*PHI-eye(size(phi)))

n_0 = [0; 0; 0];
nd_0 = PHI'*M*[0; 2; 0]./wns % *m/k
t = [0:0.5:80];
q = PHI*[nd_0(1)*sin(wns(1)*t);
        nd_0(2)*sin(wns(2)*t);
        nd_0(3)*sin(wns(3)*t)];

figure(1)
plot(t,q(1,:), t,q(2,:), t,q(3,:), '.'); grid on;
xlabel('time t*(k/m)^0^.^5'); ylabel('Displacement (m)*k/m');
legend('\theta_1', '\theta_2', '\theta_3');
title('Response to Initial Velocity in \theta_2')

% To animate the solution
%{
figure(2)
for ii = 1:1:length(t);
    sf = 50;
    plot([-0.5 0.5].', [0 0].','o:',[-0.7 0.7].', [0 0].','k');
    line([-0.5 -0.5+10*sin(q(1,ii)/sf)],[0 -
10*cos(q(1,ii)/sf)],'LineWidth', 4); grid on;
    line([0 0+10*sin(q(2,ii)/sf)],[0 -10*cos(q(2,ii)/sf)],'LineWidth',
4); grid on;
    line([0.5 0.5+10*sin(q(3,ii)/sf)],[0 -
10*cos(q(3,ii)/sf)],'LineWidth', 4); grid on;
    xlabel('X-position (*L)'); ylabel('Displacement (m)*k/m');
    title(['Time (m/k)^0.5 = ' num2str(t(ii))])
    axis([-0.7 0.7 -12 2]);
    mov1(ii) = getframe(2);
end
```

```
movie(mov1,2,20)
%}
```

The natural frequencies and mode shapes are (only the first and third modes are excited):

```
wns =
       1.2247
       1.2845
       1.3964
PHI =
           -1       -1.2247       0.70711
           -1   -7.4506e-09       -1.4142
           -1        1.2247       0.70711
```

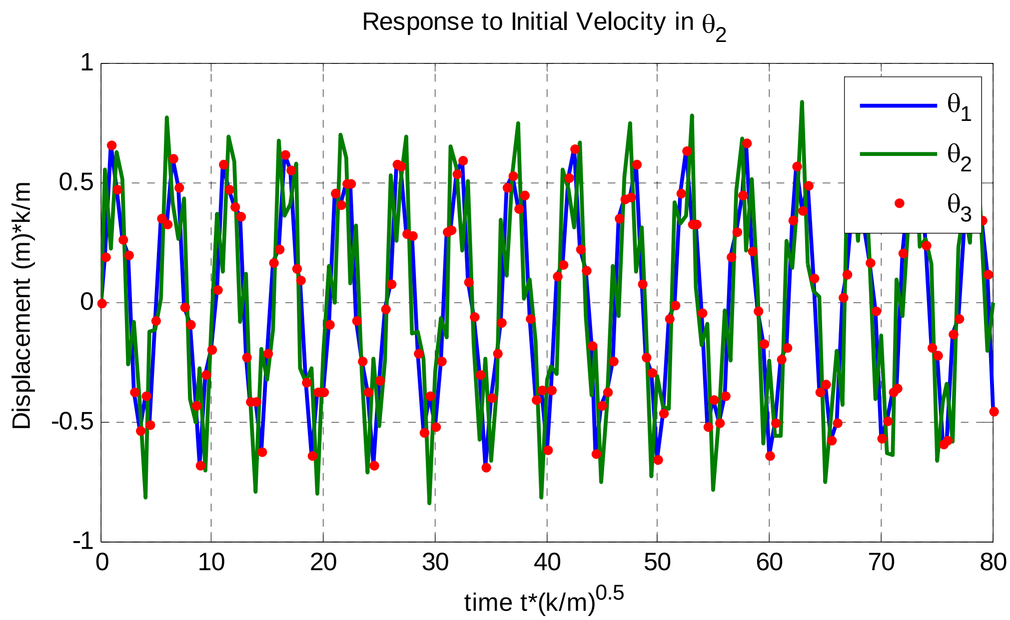
The response is given below for k=0.05 mg/L



The response shows a beating phenomenon, since each bar is influenced by modes 1 and 3 and the modes' frequencies are close (1.22 and 1.39 rad/s).

On the other hand, for k=2*mg/L, the natural frequencies differ by a factor of more than three and the response does not look as simple:

```
wns =
       1.2247
       2.7386
       4.4159
PHI =
           -1       -1.2247       0.70711
           -1   -1.2905e-08       -1.4142
           -1        1.2247       0.70711
```


Response to Initial Velocity in θ_2
Displacement (m)*k/m
time t*(k/m)^0.5
θ_1
θ_2
θ_3
1
0.5
0
-0.5
-1
0
10
20
30
40
50
60
70
80

Exercise 4.43

$$M := \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \qquad K := 1000 \cdot \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \qquad C := \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix}$$

Eigensolution

$\lambda := \text{genvals}(K, M)$ $\phi := \text{genvecs}(K, M)$

$$\phi := \text{submatrix}\left(\text{rsort}\left(\text{stack}\left(\lambda^T, \phi\right), 1\right), 2, \text{rows}(\lambda) + 1, 1, \text{rows}(\lambda)\right)$$

$\lambda := \text{sort}(\lambda)$ $\omega := \sqrt{\lambda}$

$$\lambda^T = \begin{bmatrix} 226.558845 & 308.438284 & 2.075003 \cdot 10^3 \end{bmatrix}$$

$$\omega^T = (15.051872 \quad 17.562411 \quad 45.552199)$$

$$\phi = \begin{bmatrix} -0.586168 & -0.806945 & 0.717 \\ 0.549759 & -0.585395 & -0.386503 \\ -0.595123 & -0.078432 & -0.580109 \end{bmatrix}$$

$j := 1 .. \text{rows}(\lambda)$

$$\Phi^{\langle j \rangle} := \text{if}\left[\phi_{1,j} \geq 0, \frac{\phi^{\langle j \rangle}}{\sqrt{\left|\left(\left(\phi^{\langle j \rangle}\right)^T \cdot M \cdot \phi^{\langle j \rangle}\right)_{1,1}\right|}}, -\frac{\phi^{\langle j \rangle}}{\sqrt{\left|\left(\left(\phi^{\langle j \rangle}\right)^T \cdot M \cdot \phi^{\langle j \rangle}\right)_{1,1}\right|}}\right]$$

$$\Phi = \begin{bmatrix} 0.021637 & 0.023197 & 0.037332 \\ -0.020293 & 0.016828 & -0.020124 \\ 0.021968 & 2.254691 \cdot 10^{-3} & -0.030205 \end{bmatrix} \qquad \Phi^T \cdot M \cdot \Phi - \text{identity}(\text{rows}(\lambda)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Light damping approximation

$$C' := \Phi^T \cdot C \cdot \Phi$$

$$C' = \begin{bmatrix} 0.053409 & -0.052984 & -0.413171 \\ -0.052984 & 0.768463 & 2.49957 \cdot 10^{-3} \\ -0.413171 & 2.49957 \cdot 10^{-3} & 3.428128 \end{bmatrix}$$

$$\zeta_j := \frac{C'_{j,j}}{2 \cdot \omega_j} \qquad \zeta^T = \begin{bmatrix} 1.774153 \cdot 10^{-3} & 0.021878 & 0.037629 \end{bmatrix}$$

Unit cosine response from Appendix B

$$c_{trans}(t,\omega,\omega_{nat},\zeta) := \frac{\left[\begin{array}{l}(\omega_{nat}^2-\omega^2)\cdot\cos(\omega\cdot t)+2\cdot\zeta\cdot\omega_{nat}\cdot\omega\cdot\sin(\omega\cdot t)\ \dots \\ + -\exp(-\zeta\cdot\omega_{nat}\cdot t)\cdot\left[\begin{array}{l}(\omega_{nat}^2-\omega^2)\cdot\left(\cos\left(\omega_{nat}\cdot\sqrt{1-\zeta^2}\cdot t\right)\right)\ \dots \\ +\frac{\zeta}{\sqrt{1-\zeta^2}}\cdot(\omega_{nat}^2+\omega^2)\cdot\sin\left(\omega_{nat}\cdot\sqrt{1-\zeta^2}\cdot t\right)\end{array}\right]\end{array}\right]}{\left[(\omega_{nat}^2-\omega^2)^2+4\cdot\zeta^2\cdot\omega_{nat}^2\cdot\omega^2\right]}$$

Generalized force coefficents

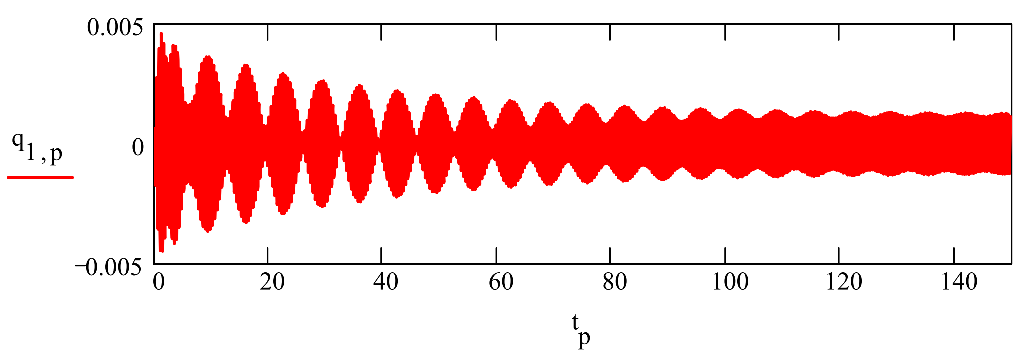
$$F := (200\ \ 0\ \ 0)^T$$

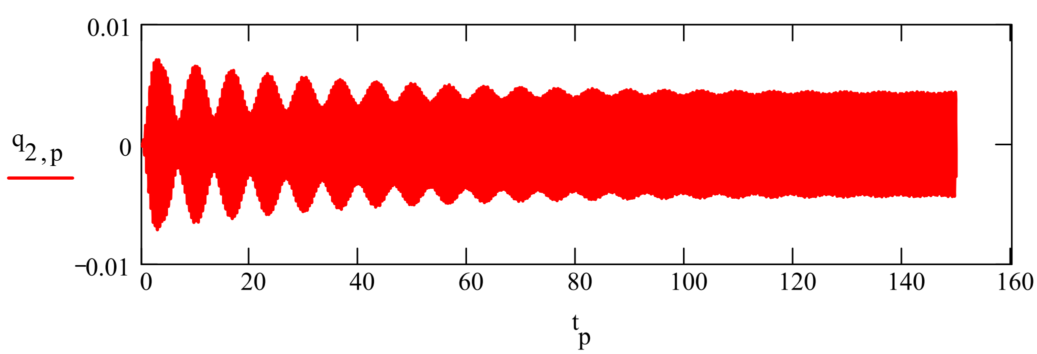
Transient solution for modal coordinates when ω = 16 rad/s:

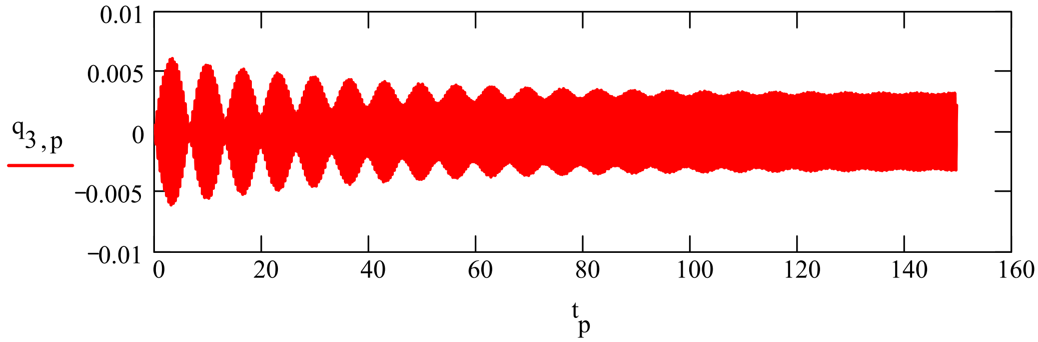
$$T_{max} := \frac{4}{\min\left(\overrightarrow{(\zeta\cdot\omega)}\right)} \qquad \Delta t := \frac{1}{4}\cdot\frac{2\cdot\pi}{\max(\omega)} \qquad P := \text{ceil}\left(\frac{T_{max}}{\Delta t}\right)$$

$$p := 1..P \qquad t_p := (p-1)\cdot\Delta t \qquad P = 4.344\cdot 10^3$$

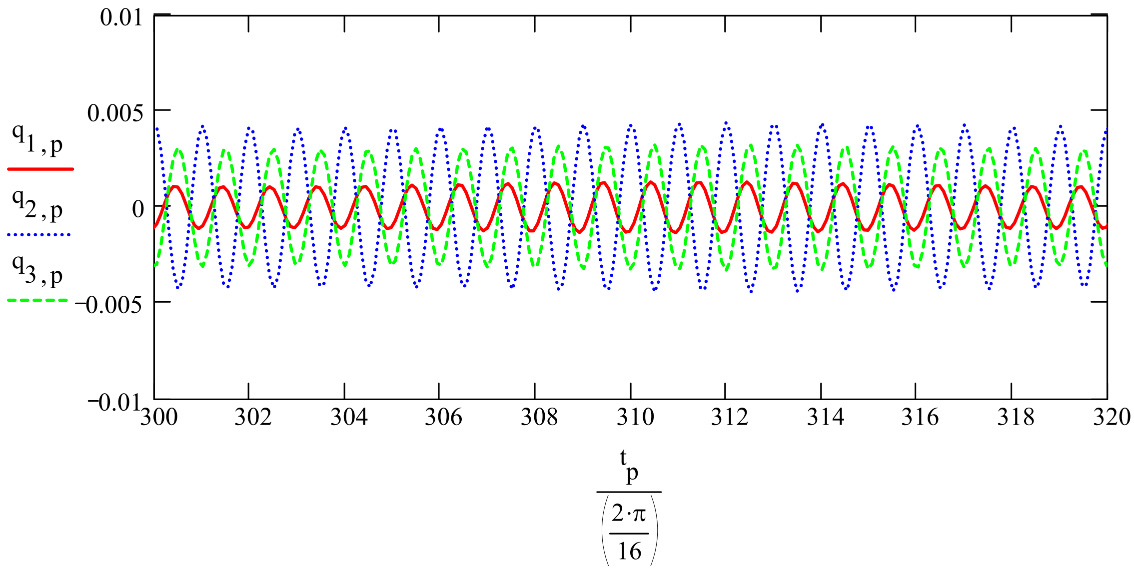
$$\eta_{j,p} := \Phi_{1,j}\cdot F_1\cdot c_{trans}(t_p,16,\omega_j,\zeta_j) \qquad q := \Phi\cdot\eta$$







Plot responses vs. t nondimensionalized by the forced period to check for steady-state



Response seems to periodic at ω=16 rad/s after 300 forced cycles

$$300\cdot\frac{2\cdot\pi}{16} = 117.809725$$

Exercise 4.47

$$M := \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \qquad \omega_1 := 15.38 \qquad \omega_2 := 40.78$$

$$\phi := \begin{bmatrix} 1 & 1 \\ 1.366 & -0.366 \end{bmatrix} \qquad \zeta_1 := 0.08 \qquad \zeta_2 := 0.08$$

$j := 1 .. 2$

$$\Phi^{\langle j \rangle} := \frac{\phi^{\langle j \rangle}}{\sqrt{\left(\phi^{\langle j \rangle^T} \cdot M \cdot \phi^{\langle j \rangle}\right)_{1,1}}} \qquad \Phi = \begin{bmatrix} 0.484057 & 0.359633 \\ 0.661222 & -0.131626 \end{bmatrix}$$

$$F := \begin{bmatrix} \frac{50}{i} \\ 100 \end{bmatrix} \qquad \Omega := 20$$

$$X_j := \frac{\left(\Phi^{\langle j \rangle^T} \cdot F\right)_{1,1}}{\left(\omega_j\right)^2 + 2i \cdot \zeta_j \cdot \omega_j \cdot \Omega - \Omega^2}$$

$$Y := \Phi \cdot X \qquad Y = \begin{bmatrix} -0.203556 + 6.975392i \cdot 10^{-3} \\ -0.270729 + 0.017639i \end{bmatrix}$$

Solution P4e01 → column failure 5/3/2013

Given:

$m\ddot{x} + c\dot{x} + kx = f(t)$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{f(t)}{m}$

non-dimensionalize:

$t' = \omega_n t$, $dt' = \omega_n dt$ } $\frac{dx}{dt} = \frac{dx}{dt'}\frac{dt'}{dt} = \frac{dx}{dt'}\omega_n$

EOM becomes:

$\omega_n^2 x'' + 2\zeta\omega_n^2 x' + \omega_n^2 x = \frac{f(t)}{m}$ $\omega_n^2 \cdot m = K$

$x'' + 2\zeta x' + x = \frac{f(t)}{K}$

- Now the effective nat. freq. is $\omega_n = 1$
- Everywhere we had t before, use $t' = \omega_n t$

$\longrightarrow \tau = \frac{3\pi}{\omega_n} \longrightarrow \tau' = 3\pi$

Solve for response to $f(t)$, with $K = 1$, $P = 1$

$\max(x(t)) = 1.88P$ (Matlab) if $K = 2 \longrightarrow x(t) = (1.88 \cdot P/K)$ meters etc...

$\max(x(t)) = (1.88\, P/K)$ (amplification factor = 1.88) (Small since the 3rd harmonic is on resonance, not the first)

For the column given:

$K = \frac{3EI}{L^3}$ $E = 30\,GPa$ $L = 40\,m$ $I = bh^3/12 = (0.6m)^4/12$

$K = 182\,KN/m$

(could use this to find m w/ $f_n = 0.2\,Hz$)

$m = \frac{K}{\omega_n^2} = 115{,}000\,Kg$

for ref, mass of column $= 2300\,kg/m^3 \cdot 0.6^2 \cdot 40 = 33{,}000\,Kg$

$\omega_n^2 = \frac{K}{m}$

stress: (static)

$\sigma_{max} = \frac{Mc}{I} = \frac{FL(h/2)}{I}$

$\sigma_{max} = \frac{K x_{max} L \cdot h/2}{h^4/12} = \frac{K \cdot 1.88 LP/K}{h^3/6} = \frac{6 \cdot 1.88\, PL}{h^3}$

$P_{max}|_{dyn} = \sigma_{max} \cdot \frac{h^3}{L}\left(\frac{1}{6 \cdot 1.88}\right) =$ **$19\,KN = P_{max}|_{dyn}$**

$P_{max}|_{static} \Rightarrow \sigma_{max} = \frac{PL\,h/2}{h^4/12} \longrightarrow P_{max}|_{static} = \frac{\sigma h^3}{6L} = 19\,kN \cdot 1.88$

$P_{max}|_{static} = 36\,KN$

✓ (If the beam were cantilevered horizontally, it couldn't hold it's own weight. probably reasonable)

→ steel reinforcement would help a lot...