# HW 10

# EMA 545
# Mechanical Vibrations

# Spring 2013
# University of Wisconsin, Madison

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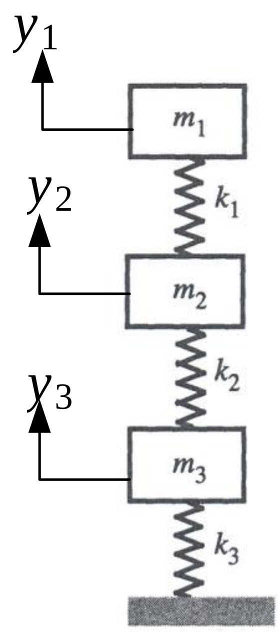
# Contents

# 1 problem 1

**4.8** The system shown in the sketch represents a scale model used to study the vibration of a three-story building. The masses for the model are $m_1 = 100$, $m_2 = 200$, and $m_3 = 300$ kg. The springs are selected such that $k_j = \omega_j^2 m_j$, where $\omega_j = 40, 50$, and 60 rad/s. Determine the characteristic equation, natural frequencies, and mode shapes of this system.




Generalized coordinates are $y_3, y_2, y_1$. Kinetic energy is $T = \frac{1}{2}m_3\left(y_3'\right)^2 + \frac{1}{2}m_2\left(y_2'\right)^2 + \frac{1}{2}m_1\left(y_1'\right)^2$. Potential energy due to springs is $V_{spring} = \frac{1}{2}k_3 y_3^2 + \frac{1}{2}k_2\left(y_2 - y_3\right)^2 + \frac{1}{2}k_1\left(y_1 - y_2\right)^2$. Therefore

$$\begin{aligned} V_{spring} &= \frac{1}{2}k_3 y_3^2 + \frac{1}{2}k_2\left(y_2^2 + y_3^2 - 2y_2 y_3\right) + \frac{1}{2}k_1\left(y_1^2 + y_2^2 - 2y_1 y_2\right) \\ &= y_3^2\left(\frac{1}{2}k_3 + \frac{1}{2}k_2\right) + y_2^2\left(\frac{1}{2}k_2 + \frac{1}{2}k_1\right) + y_1^2\left(\frac{1}{2}k_1\right) + y_1 y_2(-k_1) + y_1 y_3(0) + y_2 y_3(-k_2) \end{aligned}$$

The EOM is

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_2 + k_1 & -k_2 \\ 0 & -k_2 & k_3 + k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Following values are for mass (units in kg) $m_1 = 100, m_2 = 200, m_3 = 300$. Following values are for spring constants (units in N/m) $k_1 = 40^2(100), k_2 = 50^2(200), k_3 = 60^2(300)$. EOM becomes

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} \begin{bmatrix} y_1'' \\ y_2'' \\ y_3'' \end{bmatrix} + \begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Characteristic equation is

$$\begin{aligned} \det\left([K] - \omega^2[M]\right) &= 0 \\ \det\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - \omega^2 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right) &= 0 \\ \det\begin{bmatrix} 160000 - 100\omega^2 & -160000 & 0 \\ -160000 & 660000 - 200\omega^2 & -500000 \\ 0 & -500000 & 1580000 - 300\omega^2 \end{bmatrix} &= 0 \\ -6\times 10^6\omega^6 + 6.1\times 10^{10}\omega^4 - 1.54\times 10^{14}\omega^2 + 8.64\times 10^{16} &= 0 \end{aligned}$$

Positive roots of the above polynomial are the natural frequencies (units in rad/sec).

$$\omega_1 = 28.1$$
$$\omega_2 = 52.6$$
$$\omega_3 = 81.3$$

To obtain mode shapes, the eigenvector associated with each eigenvalue is found. Starting with $\omega_1 = 28.1$

$$\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - 28.1^2 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right)\begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$\begin{bmatrix} 8.1\times10^4 & -160000 & 0 \\ -160000 & 5.02\times10^5 & -500000 \\ 0 & -500000 & 1.34\times10^6 \end{bmatrix}\begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 8.1\times10^4 - 1.6\times10^5\varphi_{21} \\ 5.02\times10^5\varphi_{21} - 5.0\times10^5\varphi_{31} - 1.6\times10^5 \\ 1.34\times10^6\varphi_{31} - 5\times10^5\varphi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{21} = 0.506$ and $\varphi_{31} = 0.188$. First eigenvector is

$$\varphi_1 = \begin{bmatrix} 1 \\ 0.506 \\ 0.188 \end{bmatrix}$$

For $\omega_2 = 52.6$,

$$\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - 52.6^2 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right)\begin{bmatrix} 1 \\ \varphi_{22} \\ \varphi_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$\begin{bmatrix} -1.17\times10^5 & -1.6\times10^5 & 0 \\ -1.6\times10^5 & 1.07\times10^5 & -5.0\times10^5 \\ 0 & -5.0\times10^5 & 7.50\times10^5 \end{bmatrix}\begin{bmatrix} 1 \\ \varphi_{22} \\ \varphi_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1.6\times10^5\varphi_{22} - 1.17\times10^5 \\ 1.07\times10^5\varphi_{22} - 5.0\times10^5\varphi_{32} - 1.6\times10^5 \\ 7.5\times10^5\varphi_{32} - 5.0\times10^5\varphi_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{22} = -0.731$ and $\varphi_{32} = -0.476$ .Second eigenvector is

$$\varphi_2 = \begin{bmatrix} 1 \\ -0.731 \\ -0.476 \end{bmatrix}$$

For $\omega_3 = 81.3$

$$\left(\begin{bmatrix} 160000 & -160000 & 0 \\ -160000 & 660000 & -500000 \\ 0 & -500000 & 1580000 \end{bmatrix} - 81.3^2 \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix}\right) \begin{bmatrix} 1 \\ \varphi_{23} \\ \varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence

$$\begin{bmatrix} -5.01\times10^5 & -1.6\times10^5 & 0 \\ -1.6\times10^5 & -6.62\times10^5 & -5.0\times10^5 \\ 0 & -5.0\times10^5 & -4.03\times10^5 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{23} \\ \varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1.6\times10^5\varphi_{23} - 5.01\times10^5 \\ -6.62\times10^5\varphi_{23} - 5.0\times10^5\varphi_{33} - 1.6\times10^5 \\ -5.0\times10^5\varphi_{23} - 4.03\times10^5\varphi_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{23} = -3.13$ and $\varphi_{32} = \varphi_{33} = 3.82$. Third eigenvector is

$$\varphi_3 = \begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix}$$

Eigenvectors are mass normalized. Mass normalization factors $\mu_i$ are found for each eigenvector

$$\mu_1 = \varphi_1^T[M]\varphi_1 = \begin{bmatrix} 1 \\ 0.506 \\ 0.188 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} \begin{bmatrix} 1 \\ 0.506 \\ 0.188 \end{bmatrix} = 162.$$

and

$$\mu_2 = \varphi_2^T[M]\varphi_2 = \begin{bmatrix} 1 \\ -0.731 \\ -0.476 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} \begin{bmatrix} 1 \\ -0.731 \\ -0.476 \end{bmatrix} = 275.$$

and

$$\mu_3 = \varphi_3^T[M]\varphi_3 = \begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix}^T \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 300 \end{bmatrix} \begin{bmatrix} 1 \\ -3.13 \\ 3.82 \end{bmatrix} = 6.44\times10^3$$

Normalized eigenvectors are

$$\Phi_1 = \frac{\varphi_1}{\sqrt{\mu_1}} = \frac{1}{\sqrt{162}}\begin{bmatrix}1\\0.506\\0.188\end{bmatrix} = \begin{bmatrix}7.86\times10^{-2}\\3.98\times10^{-2}\\1.48\times10^{-2}\end{bmatrix}$$

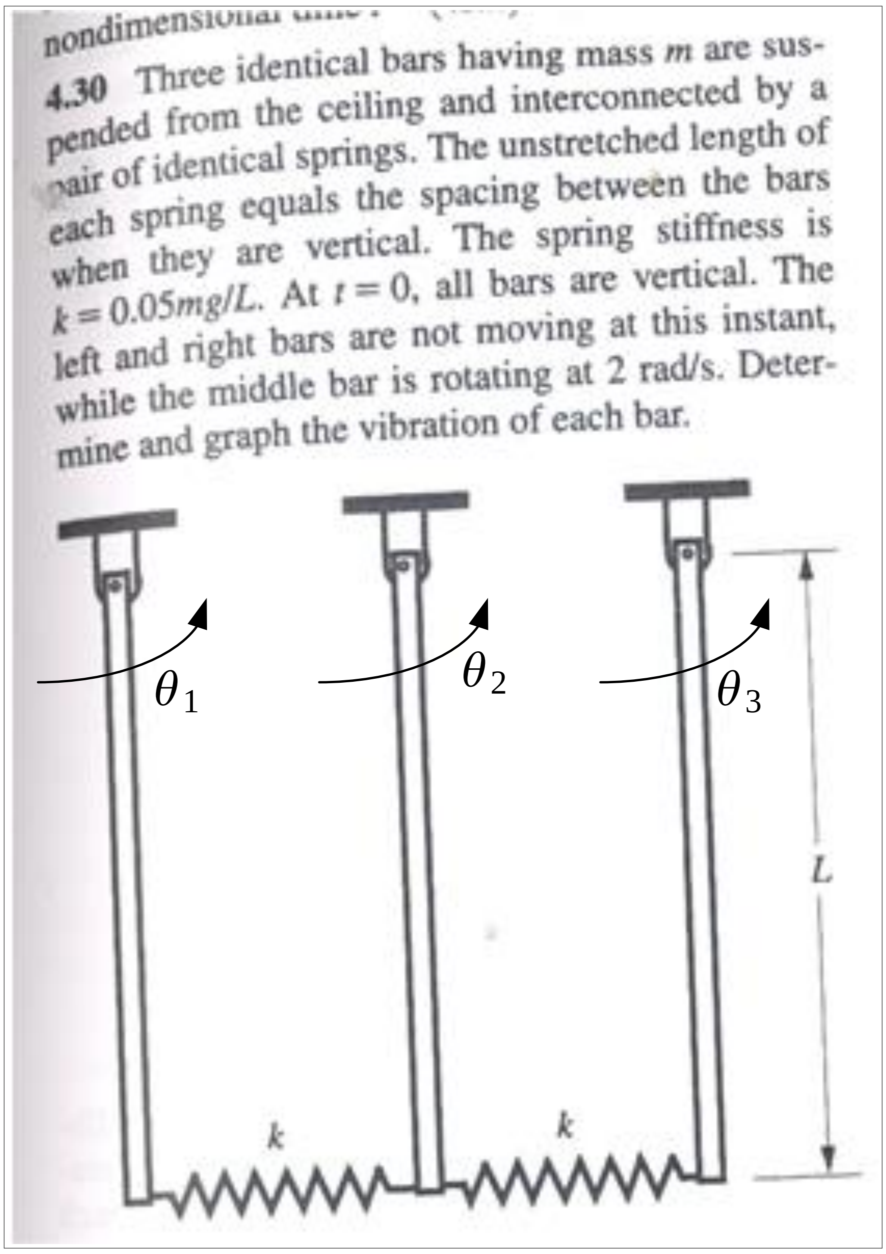
$$\Phi_2 = \frac{\varphi_2}{\sqrt{\mu_2}} = \frac{1}{\sqrt{275.}}\begin{bmatrix}1\\-0.731\\-0.476\end{bmatrix} = \begin{bmatrix}6.03\times10^{-2}\\-4.41\times10^{-2}\\-2.87\times10^{-2}\end{bmatrix}$$

$$\Phi_3 = \frac{\varphi_3}{\sqrt{\mu_3}} = \frac{1}{\sqrt{6.44\times10^{3}}}\begin{bmatrix}1\\-3.13\\3.82\end{bmatrix} = \begin{bmatrix}1.25\times10^{-2}\\-0.039\\4.76\times10^{-2}\end{bmatrix}$$

Verification of the above result follows

```
EDU>> k=[160000 -160000 0;-160000 660000 -500000;0 -500000 1580000];
EDU>> M=[100 0 0;0 200 0;0 0 300];
EDU>> [eigV,lam]=eig(k,M)

eigV =
    0.0786    0.0606    0.0124
    0.0398   -0.0437   -0.0389
    0.0148   -0.0289    0.0477

lam =

   1.0e+03 *
    0.7897         0         0
         0    2.7528         0
         0         0    6.6242
EDU>> sqrt(diag(lam))

ans =

   28.1013
   52.4674
   81.3889
```

## 2 Problem 2


nondimensional time.

**4.30** Three identical bars having mass $m$ are suspended from the ceiling and interconnected by a pair of identical springs. The unstretched length of each spring equals the spacing between the bars when they are vertical. The spring stiffness is $k = 0.05mg/L$. At $t = 0$, all bars are vertical. The left and right bars are not moving at this instant, while the middle bar is rotating at 2 rad/s. Determine and graph the vibration of each bar.


Initial conditions are $\theta_i(0) = 0$ for $i = 1, 2, 3$ and $\theta_1'(0) = \theta_3'(0) = 0$ but $\theta_2'(0) = 2$ rad/sec. The generalized coordinates are shown above. kinetic energy is

$$T = \frac{1}{2}I\left(\theta_1'\right)^2 + \frac{1}{2}I\left(\theta_2'\right)^2 + \frac{1}{2}I\left(\theta_3'\right)^2$$

where $I = \frac{1}{3}mL^2$. Mas matrix becomes

$$[M] = \frac{1}{3}mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix}$$

Spring potential energy is

$$\begin{aligned} V_{spring} &= \frac{1}{2}k(L\theta_2 - L\theta_1)^2 + \frac{1}{2}k(L\theta_3 - L\theta_2)^2 \\ &= \frac{1}{2}kL^2\left(\theta_2^2 + \theta_1^2 - 2\theta_1\theta_2\right) + \frac{1}{2}kL^2\left(\theta_3^2 + \theta_2^2 - 2\theta_2\theta_3\right) \\ &= \theta_1^2\left(\frac{1}{2}kL^2\right) + \theta_2^2\left(\frac{1}{2}kL^2 + \frac{1}{2}kL^2\right) + \theta_3^2\left(\frac{1}{2}kL^2\right) + \theta_1\theta_2\left(-kL^2\right) + \theta_1\theta_3(0) + \theta_2\theta_3\left(-kL^2\right) \end{aligned}$$

Hence stiffness matrix due to spring is

$$[K]_{spring} = kL^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Assume the zero PE for gravity is taken as the top of the bar. Stiffness due to gravity is

$$V_g = -mg\frac{L}{2}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3)$$

$V_{11} = \frac{\partial^2 V_g}{\partial \theta_1^2} = mg\frac{L}{2}(\cos\theta_1)$. Evaluate this at static position $\theta_1 = 0$,hence $V_{11} = m\frac{L}{2}$. Similarly, $V_{22} = V_{33} = m\frac{L}{2}$. All other terms are zero.

Hence stiffness matrix due to gravity is

$$[K]_g = mg\frac{L}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, complete stiffness matrix is

$$kL^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + mg\frac{L}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are no generalized forces. Hence EOM is

$$\frac{1}{3}mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + \left( kL^2 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + mg\frac{L}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## 2.1 Part (a) $k = 0.05\frac{mg}{L}$

For case $k = 0.05\frac{mg}{L}$, Hence for $\sigma = 0.05$ then $k = \sigma\frac{mg}{L}$. EOM becomes

$$\frac{1}{3}mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + \left( \sigma mgL \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + mg\frac{L}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{1}{3}mL^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + mgL \begin{bmatrix} \frac{1}{2} + \sigma & -\sigma & 0 \\ -\sigma & \frac{1}{2} + 2\sigma & -\sigma \\ 0 & -\sigma & \frac{1}{2} + \sigma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + \frac{3g}{L} \begin{bmatrix} \frac{1}{2} + \sigma & -\sigma & 0 \\ -\sigma & \frac{1}{2} + 2\sigma & -\sigma \\ 0 & -\sigma & \frac{1}{2} + \sigma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $L = 1, g = 10$. The above becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1'' \\ \theta_2'' \\ \theta_3'' \end{bmatrix} + \begin{bmatrix} 15+30\sigma & -30\sigma & 0 \\ -30\sigma & 15+60\sigma & -30\sigma \\ 0 & -30\sigma & 15+30\sigma \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Natural frequencies of the system are found by solving the eigenvalue problem.

$$\det\left( \begin{bmatrix} 15+30\sigma & -30\sigma & 0 \\ -30\sigma & 15+60\sigma & -30\sigma \\ 0 & -30\sigma & 15+30\sigma \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Substituting $\sigma = 0.05$ gives

$$\det\left( \begin{bmatrix} 16.5 & -1.5 & 0 \\ -1.5 & 18.0 & -1.5 \\ 0 & -1.5 & 16.5 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det \begin{bmatrix} 16.5-\omega^2 & -1.5 & 0 \\ -1.5 & 18-\omega^2 & -1.5 \\ 0 & -1.5 & 16.5-\omega^2 \end{bmatrix} = 0$$

$$-\omega^6 + 51\omega^4 - 861.75\omega^2 + 4826.3 = 0$$

Positive roots of this polynomial are $\omega = 3.87, \omega = 4.062, \omega = 4.416$.

Associated eigenvectors are found by solving for $\varphi_i$ in $\left([K] - \omega^2[M]\right)\varphi_i = 0$ for each eigenvalue $\omega_i$.

For $\omega_1 = 3.87$

$$\begin{bmatrix} 16.5-\omega_1^2 & -1.5 & 0 \\ -1.5 & 18-\omega_1^2 & -1.5 \\ 0 & -1.5 & 16.5-\omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 16.5-3.87^2 & -1.5 & 0 \\ -1.5 & 18-3.87^2 & -1.5 \\ 0 & -1.5 & 16.5-3.87^2 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.523 & -1.5 & 0 \\ -1.5 & 3.023 & -1.5 \\ 0 & -1.5 & 1.523 \end{bmatrix} \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.523 - 1.5\varphi_{21} \\ 3.023\,\varphi_{21} - 1.5\varphi_{31} - 1.5 \\ 1.523\,\varphi_{31} - 1.5\varphi_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving gives $\varphi_{21} = 1.015\,3$ and $\varphi_{31} = 1.046\,2$. First eigenvector is

$$\varphi_1 = \begin{bmatrix} 1 \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.0153 \\ 1.0462 \end{bmatrix}$$

Similarly, second and the third eigenvectors are found.

Eigenvectors are mass normalized. First the mass normalization factors $\mu_i$ are found for each eigenvector

$$\mu_1 = \varphi_1^T[M]\varphi_1$$
$$= \begin{bmatrix} 1 \\ 1.0153 \\ 1.0462 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.0153 \\ 1.0462 \end{bmatrix} = 3.1254$$

Normalized eigenvector is

$$\Phi_1 = \frac{\varphi_1}{\sqrt{3.1254}} = \frac{1}{\sqrt{3.792}} \begin{bmatrix} 1 \\ 1.0153 \\ 1.0462 \end{bmatrix} = \begin{bmatrix} 0.51353 \\ 0.52139 \\ 0.53726 \end{bmatrix}$$

Verification of the above result (Matlab result is more accurate due to more accurate method used)

```
EDU>>  k=[0.55 -0.05 0;-0.05 0.6 -0.05;0 -0.05 0.55];
EDU>> M=eye(3);
EDU>> [eigV,lam]=eig(k,M)

eigV =

   -0.5774   -0.7071    0.4082
   -0.5774   -0.0000   -0.8165
   -0.5774    0.7071    0.4082

EDU>> sqrt(diag(lam))

ans =

    0.7071
    0.7416
    0.8062
```

Transformation matrix (based on Matlab more accurate result) is

$$\Phi = [\Phi_1\Phi_2\Phi_3] = \begin{bmatrix} -0.577 & -0.7073 & 0.4082 \\ -0.577 & 0 & -0.8165 \\ -0.577 & 0.7069 & 0.4082 \end{bmatrix}$$

Mapping from physical coordinates $\theta$ to modal coordinates $\eta$ is

$$= [\Phi]$$

Bold face is used to indicate a column vector. EOM's are written in modal coordinates resulting in

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} 0.7071^2 & 0 & 0 \\ 0 & 0.7416^2 & 0 \\ 0 & 0 & 0.8062^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Initial conditions are transformed to modal coordinates using (0) = $[\Phi]^T[M]$(0) and '(0) = $[\Phi]^T[M]$'(0), since (0) = 0 then (0) = 0, however '(0) is not all zero, hence

$$\begin{bmatrix}\eta_1'(0)\\ \eta_2'(0)\\ \eta_3'(0)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7073 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.7069 & 0.4082\end{bmatrix}^T \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 2\\ 0\end{bmatrix}$$
$$= \begin{bmatrix}-1.154\\ 0\\ -1.633\end{bmatrix}$$

Initial conditions in modal coordinates  are found. The solution can be found. The solution to $\eta'' + \omega^2\eta = 0$ with initial conditions $\eta(0)$ and $\eta'(0)$ is $\eta(t) = \eta(0)\cos\omega t + \frac{\eta'(0)}{\omega}\sin\omega t$. Therefore modal solutions are

$$\begin{aligned}\eta_1(t) &= \frac{-1.154}{0.7071}\sin(0.7071t) = -1.632\sin(0.707\,t)\\ \eta_2(t) &= 0\\ \eta_3(t) &= \frac{-1.633}{0.8062}\sin(0.8062t) = -2.026\sin(0.8062t)\end{aligned}$$

Solution in the normal coordinates is

$$\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\\ \theta_3(t)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7073 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.7069 & 0.4082\end{bmatrix}\begin{bmatrix}-1.632\sin(0.707\,t)\\ 0\\ -2.026\sin(0.8062t)\end{bmatrix}$$
$$= \begin{bmatrix}0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\\ 0.94166\sin(0.707\,t) + 1.6542\sin(0.8062t)\\ 0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\end{bmatrix}$$

## 2.2 Part (b) $k = 2\frac{mg}{L}$

Using part (a), but with $\sigma = 2$ results in

$$\det\left(\begin{bmatrix}15+30\sigma & -30\sigma & 0\\ -30\sigma & 15+60\sigma & -30\sigma\\ 0 & -30\sigma & 15+30\sigma\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}15+30(2) & -30(2) & 0\\ -30(2) & 15+60(2) & -30(2)\\ 0 & -30(2) & 15+30(2)\end{bmatrix} - \omega^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right) = 0$$
$$\det\begin{bmatrix}75.0-\omega^2 & -60.0 & 0\\ -60.0 & 135.0-\omega^2 & -60.0\\ 0 & -60.0 & 75.0-\omega^2\end{bmatrix} = 0$$

Similar steps as repeated as part (a) above. The final result are shown below using Matlab

```
EDU>> k=[75 -60 0;-60 135 -60;0 -60 75]
EDU>> M=eye(3);
[eigV,lam]=eig(k,M)

eigV =

   -0.5774   -0.7071    0.4082
   -0.5774    0.0000   -0.8165
   -0.5774    0.7071    0.4082
EDU>> sqrt(diag(lam))

    3.8730
    8.6603
   13.9642
```

Transformation matrix is $\Phi = [\Phi_1 \Phi_2 \Phi_3] = \begin{bmatrix} -0.577 & -0.7071 & 0.4082 \\ -0.577 & 0 & -0.8165 \\ -0.577 & 0.7071 & 0.4082 \end{bmatrix}$. Mapping from $\theta$ to modal coordinates $\eta$ is

$$= [\Phi]$$

Bold face is used to indicate a column vector. EOM's are written in modal coordinates resulting in

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_2^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{bmatrix} + \begin{bmatrix} 3.8730^2 & 0 & 0 \\ 0 & 8.6603^2 & 0 \\ 0 & 0 & 13.9642^2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Initial conditions are transformed to modal coordinates using $(0) = [\Phi]^T [M] \mathbf{x}(0)$ and $'(0) = [\Phi]^T [M] \theta'(0)$, since $\theta(0) = 0$ then $(0) = 0$, however $\theta'(0)$ is not all zero. Similar to part (a), initial conditions are found

$$\begin{bmatrix} \eta_1'(0) \\ \eta_2'(0) \\ \eta_3'(0) \end{bmatrix} = \begin{bmatrix} -1.154 \\ 0 \\ -1.633 \end{bmatrix}$$

The solution to $\eta'' + \lambda^2 \eta = 0$ with initial conditions $\eta(0)$ and $\eta'(0)$ is given by $\eta(t) = \eta(0)\cos\lambda t + \frac{\eta'(0)}{\lambda}\sin\lambda t$. The solutions are

$$\begin{aligned} \eta_1(t) &= \frac{-1.154}{3.8730}\sin(3.873t) = -0.29796\sin(3.873\,t) \\ \eta_2(t) &= 0 \\ \eta_3(t) &= \frac{-1.633}{13.9642}\sin(13.9642) = -0.11694\sin(13.9642) \end{aligned}$$

Solution in the physical coordinates is

$$\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\\ \theta_3(t)\end{bmatrix} = \begin{bmatrix}-0.577 & -0.7071 & 0.4082\\ -0.577 & 0 & -0.8165\\ -0.577 & 0.7071 & 0.4082\end{bmatrix}\begin{bmatrix}-0.29796\sin(3.8730\,t)\\ 0\\ -0.11694\sin(13.9642t)\end{bmatrix}$$

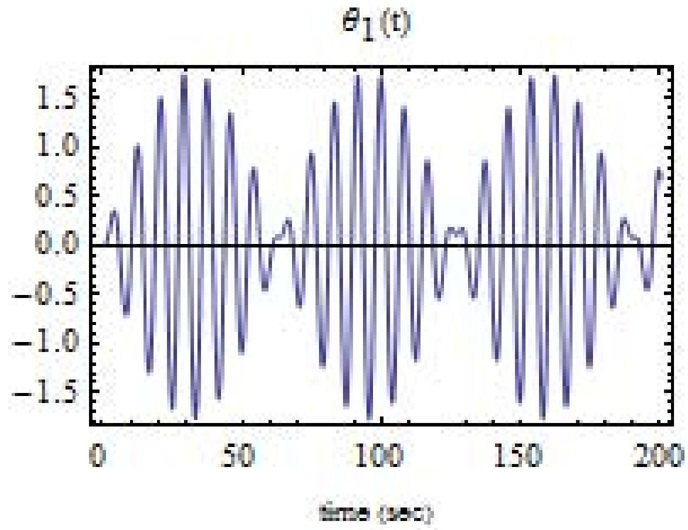
$$= \begin{bmatrix}0.17192\sin(3.873t) - 4.7735\times10^{-2}\sin(13.964t)\\ 9.5482\times10^{-2}\sin(13.964t) + 0.17192\sin(3.873t)\\ 0.17192\sin(3.873t) - 4.7735\times10^{-2}\sin(13.964t)\end{bmatrix}$$

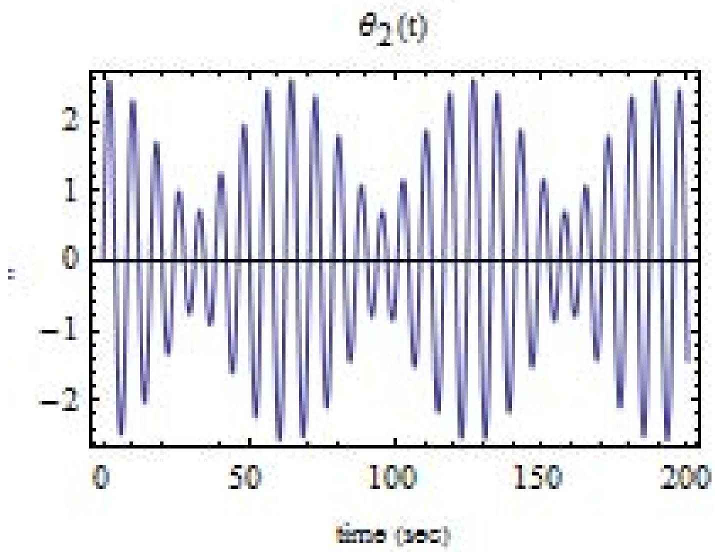
Summary table

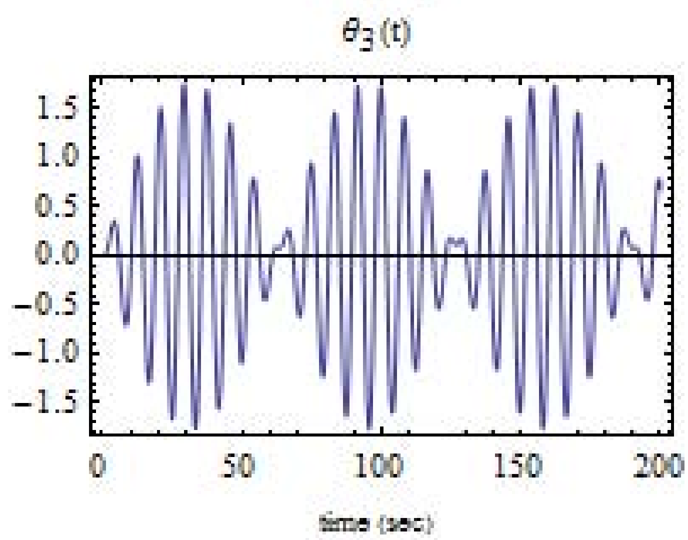
| $k$ | frequencies | [Φ] | solutions in $\theta$ |
|---|---|---|---|
| $0.05\frac{mg}{L}$ | $\begin{Bmatrix}0.7071\\ 0.7416\\ 0.8062\end{Bmatrix}$ | $\begin{bmatrix}-0.5774 & -0.7071 & 0.4082\\ -0.5774 & 0 & -0.8165\\ -0.5774 & 0.7071 & 0.4082\end{bmatrix}$ | $\begin{bmatrix}0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\\ 0.94166\sin(0.707\,t) + 1.6542\sin(0.8062t)\\ 0.94166\sin(0.707\,t) - 0.82701\sin(0.8062t)\end{bmatrix}$ |
| $2\frac{mg}{L}$ | $\begin{Bmatrix}3.8730\\ 8.6603\\ 13.9642\end{Bmatrix}$ | $\begin{bmatrix}-0.5774 & -0.7071 & 0.4082\\ -0.5774 & 0 & -0.8165\\ -0.5774 & 0.7071 & 0.4082\end{bmatrix}$ | $\begin{bmatrix}0.17192\sin(3.873t) - 4.7735\times10^{-2}\sin(13.964t\\ 9.5482\times10^{-2}\sin(13.964t) + 0.17192\sin(3.873t\\ 0.17192\sin(3.873t) - 4.7735\times10^{-2}\sin(13.964t\end{bmatrix}$ |

Even though the normalized natural frequencies are different, the shape functions are the same.

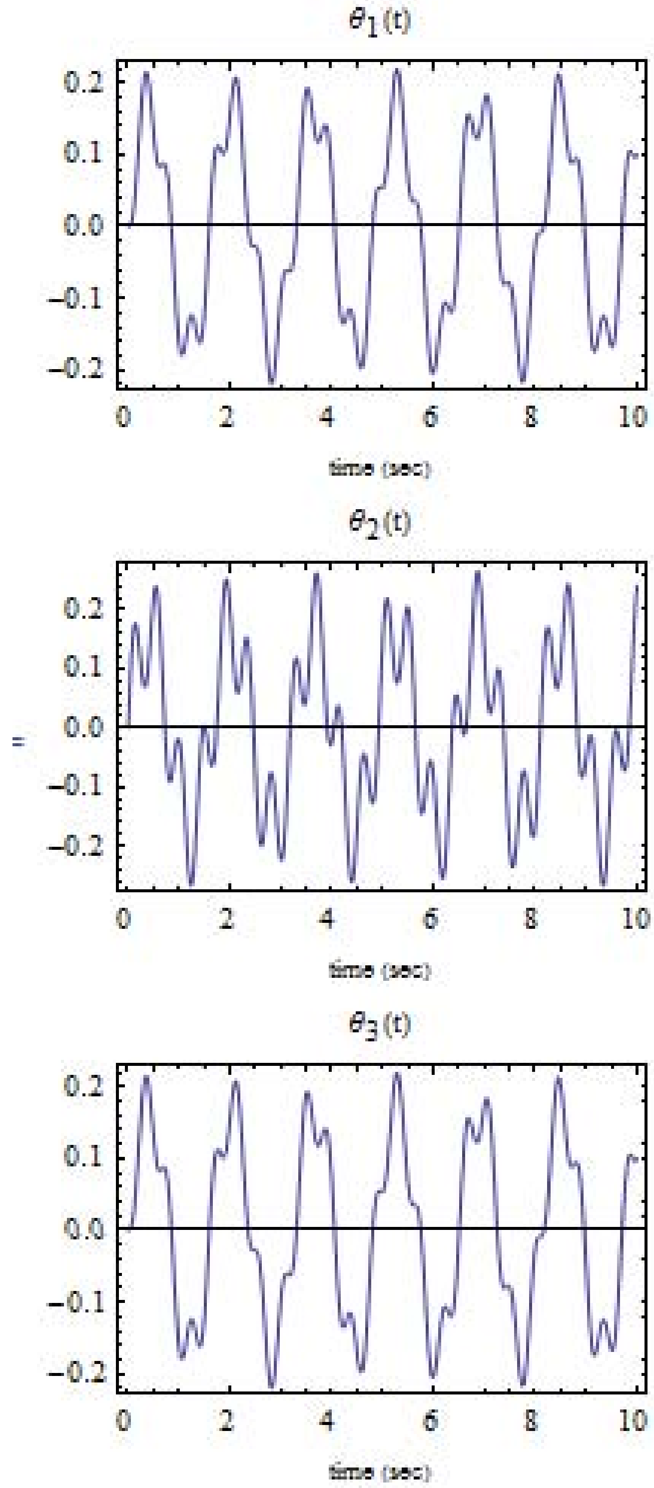
Plots of the solutions of $\theta_i(t)$ for both cases are made. For the case of $k = 0.05\frac{mg}{L}$




For $k = 2\frac{mg}{L}$




In addition, a small program is written to animate both the full solution and the modal solutions. The program to animate the full solution is at `http://12000.org/my_courses/univ_wisconson_madison/spring_2013/EMA_545_Mechanical_Vibrations/HWs/HW10/HW10p2.m.txt` while the program that animate the modal solution is number 112 at bottom of this page `http://12000.org/my_notes/my_matlab_functions/index.htm`

# 3 Problem 3

**4.43** The following properties are known for a certain three-degree-of-freedom system:

$$[M] = \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \text{kg},$$

$$[K] = \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \text{kN/m},$$

$$[C] = \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} \text{N-s/m},$$

$$\{Q\} = \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix} \text{N}$$

The system was initially at rest at its static equilibrium position. Use the light damping approximation to determine the response. Graph each generalized coordinate as a function of time. From that result, estimate the time required to attain the steady-state condition.

EOM is

$$\begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \begin{Bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{Bmatrix} + \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} \begin{Bmatrix} x_1' \\ x_2' \\ x_3' \end{Bmatrix} + 10^3 \begin{bmatrix} 300 & 0 & -200 \\ 0 & 500 & 300 \\ -200 & 300 & 700 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 200\cos(16t) \\ 0 \\ 0 \end{Bmatrix}$$

Initial conditions are $\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} x_1'(0) \\ x_2'(0) \\ x_3'(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$.

Solve the eigenvalue problem to determine the natural frequencies of the system

$$\det\left([K] - \omega^2 [M]\right) = 0$$

$$\det\left( \begin{bmatrix} 300\times 10^3 & 0 & -200\times 10^3 \\ 0 & 500\times 10^3 & 300\times 10^3 \\ -200\times 10^3 & 300\times 10^3 & 700\times 10^3 \end{bmatrix} - \omega^2 \begin{bmatrix} 600 & 400 & 200 \\ 400 & 1200 & 0 \\ 200 & 0 & 800 \end{bmatrix} \right) = 0$$

$$-4.0\times 10^8 \omega^6 + 1.044\times 10^{12} \omega^4 - 4.72\times 10^{14} \omega^2 + 5.8\times 10^{16} = 0$$

Positive roots are $\{\omega = 15.052, \omega = 17.562, \omega = 45.552\}$. For each natural frequency the corresponding eigenvector is found. A program is now used to compute these values.

```
EDU>> k = [300 0 -200;0 500 300;-200 300 700]*10^3;
M = [600 400 200;400 1200 0;200 0 800];
C = [500 300 -400;300 900 600;-400 600 1300];
[PHI,lam] = eig(k,M);
PHI
lam   = sqrt(diag(lam))
CC    = PHI'*C*PHI;
zeta1 = CC(1,1)/(2*lam(1))
zeta2 = CC(2,2)/(2*lam(2))
zeta3 = CC(3,3)/(2*lam(3))

PHI =
   -0.0216    0.0232   -0.0373
    0.0203    0.0168    0.0201
   -0.0220    0.0023    0.0302

lam =
   15.0519
   17.5624
   45.5522


zeta1 =
    0.0018

zeta2 =
    0.0219

zeta3 =
    0.0376
```

$[\Phi] = \begin{bmatrix} -0.0216 & 0.0232 & -0.0373 \\ 0.0203 & 0.0168 & 0.0201 \\ -0.0220 & 0.0023 & 0.0302 \end{bmatrix}$. In modal coordinates EOM is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + [\Phi]^T \begin{bmatrix} 500 & 300 & -400 \\ 300 & 900 & 600 \\ -400 & 600 & 1300 \end{bmatrix} [\Phi] \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} 5.419 \times 10^{-2} & 5.331 \times 10^{-2} & -0.416 \\ 5.331 \times 10^{-2} & 0.768 & -3.52 \times 10^{-4} \\ -0.4156 & -3.52 \times 10^{-4} & 3.428 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \eta_1'' \\ \eta_2'' \\ \eta_3'' \end{Bmatrix} + \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} \begin{Bmatrix} \eta_1' \\ \eta_2' \\ \eta_3' \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 200\cos(1 \\ 0 \\ 0 \end{Bmatrix}$$

In the above $2\zeta_1\omega_1 = 0.0542, 2\zeta_2\omega_2 = 0.7676$ and $2\zeta_3\omega_3 = 3.4247$. Hence $\zeta_1 = \frac{5.4193 \times 10^{-2}}{2(15.0519)} = 0.0018$ and $\zeta_2 = \frac{0.76755}{2(17.5624)} = 0.0219$ and $\zeta_3 = \frac{3.4247}{2(45.5522)} = 0.0376$

Final EOM in modal coordinates is

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{Bmatrix}\eta_1''\\\eta_2''\\\eta_3''\end{Bmatrix}+\begin{bmatrix}0.0542&0&0\\0&0.768&0\\0&0&3.425\end{bmatrix}\begin{Bmatrix}\eta_1'\\\eta_2'\\\eta_3'\end{Bmatrix}+\begin{bmatrix}226.56&0&0\\0&308.44&0\\0&0&2075\end{bmatrix}\begin{Bmatrix}\eta_1\\\eta_2\\\eta_3\end{Bmatrix}=\begin{Bmatrix}-4.32\cos(16.0t)\\4.64\cos(16.0t)\\-7.46\cos(16.0t)\end{Bmatrix}$$

EOM's to solve are

$$\begin{aligned}\eta_1''+2\zeta_1\omega_1\eta_1'+\omega_1^2\eta_1&=-4.32\cos(16.0t)\\\eta_2''+2\zeta_2\omega_2\eta_2'+\omega_2^2\eta_2&=4.64\cos(16.0t)\\\eta_3''+2\zeta_3\omega_3\eta_3'+\omega_3^2\eta_3&=-7.46\cos(16.0t)\end{aligned}$$

Initial conditions are zero. The solution in modal coordinates is given in appendix B for underdamped case. Complete solution for the case of underdamped is given in appendix B as

$$\eta(t)=\frac{F_0}{\beta^2+4\zeta^2\omega^2\varpi^2}\left\{\beta\cos(\varpi t)+2\zeta\omega\varpi\sin(\varpi t)-e^{-\zeta\omega t}\left[\beta\cos(\omega_d t)+\frac{\zeta\omega\beta}{\omega_d}\sin(\omega_d t)\right]\right\}h(t)$$

$\beta=\left(\omega^2-\varpi^2\right), \omega_d=\omega\sqrt{1-\zeta^2}$.

The solutions in modal coordinates are now found. Recall that $\omega_1=15.0519, \omega_2=17.5624, \omega_3=45.5522$ and $\zeta_1=0.0018, \zeta_2=0.0219$ and $\zeta_3=0.0376$

Next step is to transform the solution to the physical coordinates using $q_j=\sum_{m=1}^{3}\Phi\left(j,m\right)\eta(m)$, or

$$\mathbf{q}=[\Phi]$$

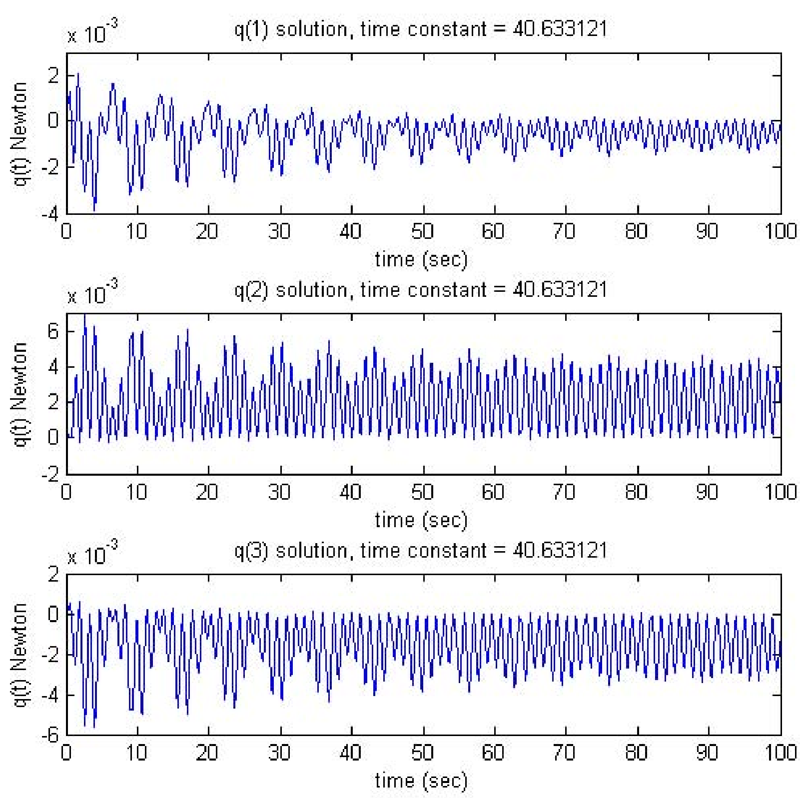
In component form

$$\begin{aligned}q_1(t)&=\Phi(1,1)\eta_1(t)+\Phi(1,2)\eta_2(t)+\Phi(1,3)\eta_3(t)\\q_2(t)&=\Phi(2,1)\eta_1(t)+\Phi(2,2)\eta_2(t)+\Phi(2,3)\eta_3(t)\\q_3(t)&=\Phi(3,1)\eta_1(t)+\Phi(3,2)\eta_2(t)+\Phi(3,3)\eta_3(t)\end{aligned}$$

Program was written to complete the computation and make plots. Here is the result showing plots of each of the above $q_i(t)$ vs. time

```
function nma_HW10_problem_3_EMA_545()
%solve for q(t) using modal analysis, by Nasser M. Abbasi
close all;

syms t;
N = 3;
k = [300 0 -200;0 500 300;-200 300 700]*10^3;
M = [600 400 200;400 1200 0;200 0 800];
C = [500 300 -400;300 900 600;-400 600 1300];
wF = 16;
F = [200*cos(wF*t); 0; 0];

[PHI,lam] = eig(k,M);
lam   = sqrt(diag(lam));
CC    = PHI'*C*PHI

F = PHI.'*F;
eta = sym(zeros(N, 1));
time_constant = zeros(3,1);

for i=1:N
```

```
    w    = lam(i);
    b    = w^2-wF^2;
    zeta = CC(i,i)/(2*w);
    wd   = w*sqrt(1-zeta^2);
    eta(i) = F(i)/(b^2+4*zeta^2*w^2*wF^2) * ...
        ( b*cos(wF*t)+2*zeta*w*wF*sin(wF*t)- ...
        exp(-zeta*w*t)* (  b*cos(wd*t)+ zeta*w*b/wd * sin(wd*t)  ) ...
        );
    time_constant(i) = 1/(zeta*w);
end

q=PHI*eta;
time_constant
time_constant = sum(time_constant);

% plot the generalized solutions
lims= [-0.004 0.003;
      -0.002  0.007;
      -0.006  0.002
      ];

for i=1:N
    subplot(3,1,i);
    ezplot(q(i),[0,100]);
    ylim(lims(i,:));
    title(sprintf('q(%d) solution, time constant = %f',i,time_constant));
    xlabel('time (sec)');
    ylabel('q(t) Newton');
end

end
```




From above, the time to reach steady state is about 90 seconds based on looking at $q_1(t)$ since that takes the longest time to each steady state out of the three coordinates.

The time constant for each $\eta_i(t)$ solution was calculated giving $\tau_1 = \frac{1}{\zeta_1 \omega_1} = 37.4471$ and

$\tau_2 = 2.602$ and $\tau_3 = 0.58$. The first time constant $\tau_1 = 37.4$ seconds dominated the result in the response in the physical coordinates.

This means the dominant time constant found in modal analysis is one to use to estimate how long it will take for the response in physical coordinates to reach steady state. Each modal solution contributes to each physical solution. The one with the longest time constant affects more than any other mode how long the physical solution takes to reach steady state.

# 4 Problem 4

**4.47** The mass matrix, natural frequencies, and unnormalized vibration modes for a two-degree-of-freedom system are

$$[M] = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \text{kg}$$

$$\omega_1 = 15.68 \text{ rad/s}, \qquad \omega_2 = 40.78 \text{ rad/s}$$

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}, \qquad \{\phi_2\} = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

The modal damping ratios are estimated to be $\zeta_1 = \zeta_2 = 0.08$. The system is subjected to a harmonic excitation for which the generalized forces are $Q_1 = 50 \sin(20t)$, $Q_2 = 100 \cos(20t)$ N. Determine the steady-state response of the generalized coordinates. Express the result in the form $q_n = \text{Re}[Y_n \exp(i20t)]$ and give the values of $Y_1$ and $Y_2$.

$$[M] = \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \text{kg}, \omega_1 = 15.68 \text{ rad/sec}, \omega_2 = 40.78 \text{ rad/sec}. \,_1 = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}, \,_2 = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}$$

$$\mu_1 = \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix}^T \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ 1.366 \end{Bmatrix} = 4.2678$$

$$\mu_2 = \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix}^T \begin{bmatrix} 5 & -3 \\ -3 & 4 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = 7.7318$$

Normalized eigenvectors are

$$\,_1 = \frac{1}{\sqrt{\mu_1}}\,_1 = \frac{1}{\sqrt{4.2678}} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = \begin{Bmatrix} 0.48406 \\ -0.17717 \end{Bmatrix}$$

$$\,_2 = \frac{1}{\sqrt{\mu_2}}\,_2 = \frac{1}{\sqrt{7.7318}} \begin{Bmatrix} 1 \\ -0.366 \end{Bmatrix} = \begin{Bmatrix} 0.35963 \\ -0.13163 \end{Bmatrix}$$

Hence

$$[\Phi] = [_{12}] = \begin{bmatrix} 0.484\,06 & 0.359\,63 \\ -0.177\,17 & -0.131\,63 \end{bmatrix}$$

EOM in modal coordinates is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 2(0.08)(15.68) & 0 \\ 0 & 2(0.08)(40.78) \end{bmatrix}\begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 15.68^2 & 0 \\ 0 & 40.78^2 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = [\Phi]^T \begin{Bmatrix} 50\sin(20t) \\ 100\cos(20t) \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \eta_1'' \\ \eta_2'' \end{Bmatrix} + \begin{bmatrix} 2.509 & 0 \\ 0 & 6.524\,8 \end{bmatrix}\begin{Bmatrix} \eta_1' \\ \eta_2' \end{Bmatrix} + \begin{bmatrix} 245.86 & 0 \\ 0 & 1663 \end{bmatrix}\begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix} = \begin{bmatrix} 24.203\sin(20.0t) - 17.717\,\mathrm{c} \\ 17.982\sin(20.0t) - 13.163\,\mathrm{c} \end{bmatrix}$$

The two EOMs to solve are

$$\eta_1''(t) + 2.509\eta_1'(t) + 245.86\eta_1(t) = 24.203\sin(20t) - 17.717\cos(20t) = \mathrm{Re}\left\{\frac{24.203}{i}e^{i20t}\right\} + \mathrm{Re}\left\{-17.717e^{i20t}\right\}$$

$$\eta_2''(t) + 6.525\eta_2'(t) + 1663\eta_2(t) = 17.982\sin(20t) - 13.163\cos(20t) = \mathrm{Re}\left\{\frac{17.982}{i}e^{i20t}\right\} + \mathrm{Re}\left\{-13.163e^{i20t}\right\}$$

Hence

$$\eta_1''(t) + 2.509\eta_1'(t) + 245.86\eta_1(t) = 24.203\sin(20t) - 17.717\cos(20t) = \mathrm{Re}\left\{(-24.203i - 17.717)e^{i20t}\right\}$$

$$\eta_2''(t) + 6.525\eta_2'(t) + 1663\eta_2(t) = 17.982\sin(20t) - 13.163\cos(20t) = \mathrm{Re}\left\{(-17.982i - 13.163)e^{i20t}\right\}$$

In matrix form

$$[I]'' + [C]' + [K] = \mathrm{Re}\left\{\mathbf{F}e^{i\varpi t}\right\}$$

Where $\varpi = 20$ rad/sec. $\mathbf{F}$ is the complex amplitude of the input

$$\mathbf{F} = \begin{Bmatrix} -24.203i - 17.717 \\ -17.982i - 13.163 \end{Bmatrix}$$

Using method of transfer functions (since steady state response is needed), response is

$$= \mathrm{Re}\left\{\mathbf{X}e^{i20t}\right\}$$

Where

$$X_j = \frac{F_j}{-\varpi^2 + 2i\zeta_j\omega_j\varpi + \omega_j^2}$$

Steady state solutions in modal coordinates is

$$\begin{aligned} \eta_1(t) &= \mathrm{Re}\left\{\frac{-24.203i - 17.717}{-\varpi^2 + 2.5088i\varpi + 245.86}e^{i\varpi t}\right\} \\ &= \mathrm{Re}\left\{\frac{-24.203i - 17.717}{-400 + 50.176i + 245.86}e^{i\varpi t}\right\} \\ &= \mathrm{Re}\left\{\left(5.77\times10^{-2} + 0.176i\right)e^{i\varpi t}\right\} \\ \eta_2(t) &= \mathrm{Re}\left\{\frac{-17.982i - 13.163}{-\varpi^2 + 6.525i\varpi + 1663}e^{i\varpi t}\right\} \\ &= \mathrm{Re}\left\{\frac{-17.982i - 13.163}{-400 + 130.5i + 1663}e^{i\varpi t}\right\} \\ &= \mathrm{Re}\left\{\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)e^{i\varpi t}\right\} \end{aligned}$$

Solutions are transformed back to normal coordinates

$$\mathbf{q} = [\Phi]$$

Hence

$$\begin{aligned} q_j(t) &= \sum_n^2 \Phi(j,n)\eta(n) \\ &= \sum_n^2 \Phi(j,n)\,\mathrm{Re}\{X(n)e^{i\omega t}\} \\ &= \mathrm{Re}\sum_n^2 \Phi(j,n)X(n)e^{i\omega t} \end{aligned}$$

Since$[\Phi] = \begin{bmatrix} 0.484\,06 & 0.359\,63 \\ -0.177\,17 & -0.131\,63 \end{bmatrix}$ then

$$q_1(t) = \mathrm{Re}\left(\left\{0.484\,06\left(5.77\times10^{-2} + 0.176i\right) + 0.359\,63\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)\right\}e^{i20t}\right)$$
$$q_2(t) = \mathrm{Re}\left(\left\{-0.177\,17\left(5.77\times10^{-2} + 0.176i\right) - 0.131\,63\left(-1.178\times10^{-2} - 1.302\times10^{-2}i\right)\right\}e^{i20t}\right)$$

or

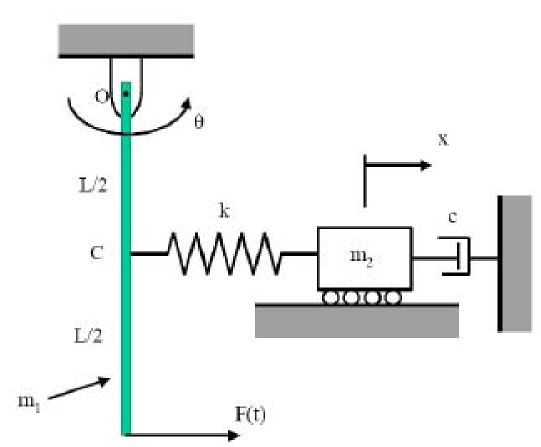
$$q_1(t) = \mathrm{Re}\left(\left\{2.369\times10^{-2} + 8.051\times10^{-2}i\right\}e^{i20t}\right)$$
$$q_2(t) = \mathrm{Re}\left(\left\{-8.672\times10^{-3} - 2.947\times10^{-2}i\right\}e^{i20t}\right)$$

Therefore

$$Y_1 = 2.369\times10^{-2} + 8.051\times10^{-2}i$$
$$Y_2 = -8.672\times10^{-3} - 2.947\times10^{-2}i$$

sectionProblem 5

5.) (Covering material from Chapter 5)
A uniform rod of length L and mass $m_1$ is attached to a cart having mass $m_2$ by means of a spring k. A viscous damper c resists the motion of the cart.




a.) Let F(t)=Re[Fexp(iωt)], x(t) = Re[Xexp(iωt)] and θ(t)=Re[Yexp(iωt)]. Find analytical expressions for the complex transfer functions X/F and Y/F.
b.) Find the magnitude and phase of the response of x and θ when the system is forced at its natural frequencies $\omega=\omega_1$ and $\omega=\omega_2$. Compare these values to the eigenvectors for modes 1 and 2. Use the following numerical values: $m_1=m_2=1$ kg, k=3 N/m, L=1 m, g=9.81 m/s^2, and c=0.1 N-s/m.

c.) Plot the transfer functions Y/F and X/F over a range of frequencies encompassing both modes of vibration. Use the plot to determine at what frequency $m_2$ acts as a vibration absorber for the rod. How does that frequency compare with the natural frequency that the system would have if the rod were held fixed: $\omega_{cart} = (k/m_2)^{1/2}$ ?

## 4.1 Part(a)

First step is to determine EOM. The kinetic energy $T$ is

$$T = \frac{1}{2}I(\theta')^2 + \frac{1}{2}m_2(x')^2$$

$I = \frac{1}{3}m_1L^2$.Assuming small angle, stiff spring approximation and zero gravity datum at the level where pendulum is hinged, spring potential energy $V$ is

$$\begin{aligned} V &= \frac{1}{2}k\left(x - \frac{L}{2}\theta\right)^2 \\ &= \frac{1}{2}k\left(x^2 + \frac{L^2}{4}\theta^2 - xL\theta\right) \\ &= \theta^2\left(\frac{L^2}{8}k\right) + x^2\left(\frac{1}{2}k\right) + x\theta\left(-\frac{kL}{2}\right) \end{aligned}$$

Stiffness matrix due to spring is

$$K_{spring} = \begin{bmatrix} \frac{L^2}{4}k & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}$$

Potential energy due to gravity is $V_g = -mg\frac{L}{2}\cos\theta$. Hence $V_{g_{11}} = \frac{\partial^2 V_g}{\partial^2\theta} = \left(mg\frac{L}{2}\cos\theta\right)_{\theta=0} = mg\frac{L}{2}$. All other terms are zero. The stiffness matrix due to gravity is

$$K_{spring} = \begin{bmatrix} mg\frac{L}{2} & 0 \\ 0 & 0 \end{bmatrix}$$

Combined stiffness matrix is

$$K = \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}$$

EOM is

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} Q_\theta \\ Q_x \end{Bmatrix}$$

Generalized forces are now found. $Q_\theta = FL$ since $F$ is only external forces acting on the first d.o.f. $\theta$ and the work done by this force is $FL\delta\theta$ for small virtual angle. For $Q_x$ work is done only by damper and acts to remove energy, hence negative in sign. $Q_x = -cx'$. The above becomes

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + mg\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} FL \\ -cx' \end{Bmatrix}$$

Rearranging

$$\begin{bmatrix} I & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \theta'' \\ x'' \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{Bmatrix} \theta' \\ x' \end{Bmatrix} + \begin{bmatrix} \frac{L^2}{4}k + m_2g\frac{L}{2} & -\frac{kL}{2} \\ -\frac{kL}{2} & k \end{bmatrix}\begin{Bmatrix} \theta \\ x \end{Bmatrix} = \begin{Bmatrix} FL \\ 0 \end{Bmatrix}$$

Each EOM is

$$I\theta'' + \left(\frac{L^2}{4}k + m_1 g\frac{L}{2}\right)\theta - \frac{kL}{2}x = FL$$
$$m_2 x'' + cx' - \frac{kL}{2}\theta + kx = 0$$

Units checking: First EOM. each term must have units of torque. $\frac{L^2}{4}k\theta$ have units of torque OK. $mg\frac{L}{2}\theta$ have units of torque OK. $kLx$ have units of torque OK.

second EOM Each term must have units of force. $cx'$ have units of force OK. $kL\theta$ have units of force, OK. $kx$ have units of force, OK.

Transfer function is now found Let $x = \operatorname{Re}\left\{Xe^{i\varpi t}\right\}, \theta = \operatorname{Re}\left\{Ye^{i\varpi t}\right\}, F = \operatorname{Re}\left\{\hat{F}e^{i\varpi t}\right\}$.Substitute in the above EOM

$$\operatorname{Re}\left\{\left[\left(-I\varpi^2 Y\right) + \left(\frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}X\right]e^{i\varpi t}\right\} = \operatorname{Re}\left\{\hat{F}Le^{i\varpi t}\right\}$$
$$\operatorname{Re}\left\{\left[-m_2\varpi^2 X + ic\varpi X - \frac{kL}{2}Y + kX\right]e^{i\varpi t}\right\} = 0$$

Simplify

$$\left(-I\varpi^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}X = \hat{F}L \tag{1}$$
$$\left(-m_2\varpi^2 + ic\varpi + k\right)X = \frac{kL}{2}Y \tag{2}$$

The above two equations are solved to obtain the required transfer functions $X/F$ and $Y/F$ . To obtain $Y/F$, the second equation solved for $X$ in terms of $Y$

$$X = \frac{\frac{kL}{2}}{-m_2\varpi^2 + ic\varpi + k}Y$$

$X$ in first equation is replaced by the giving

$$\left(-I\varpi^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)Y - \frac{kL}{2}\frac{\frac{kL}{2}}{-m_2\varpi^2 + ic\varpi + k}Y = \hat{F}L$$
$$\left(-I\varpi^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2} - \frac{\frac{k^2L^2}{4}}{-m_2\varpi^2 + ic\varpi + k}\right)Y = \hat{F}L$$

Hence

$$Y = \frac{1}{\left(-\frac{1}{3}m_1 L\varpi^2 + \frac{L}{4}k + \frac{m_2 g}{2} - \frac{k^2L/4}{-m_2\varpi^2 + ic\varpi + k}\right)}\hat{F}$$

To obtain the transfer function $X/F$, the second equation is solved for $Y$ in terms of $X$

$$Y = \frac{\left(-m_2\varpi^2 + ic\varpi + k\right)}{kL/2}X$$

This is substituted in the first equation giving

$$\left(-I\varpi^2 + \frac{L^2}{4}k + m_2 g\frac{L}{2}\right)\frac{\left(-m_2\varpi^2 + ic\varpi + k\right)}{kL/2}X - \frac{kL}{2}X = \hat{F}L$$
$$\left[\frac{\left(-\frac{1}{3}m_1 L\varpi^2 + \frac{L}{4}k + \frac{m_2 g}{2}\right)\left(-m_2\varpi^2 + ic\varpi + k\right)}{k/2} - \frac{kL}{2}\right]X = \hat{F}L$$

Hence

$$X = \frac{kL}{\left(-\frac{1}{3}m_1 L\varpi^2 + \frac{L}{4}k + \frac{m_2 g}{2}\right)\left(-m_2\varpi^2 + ic\varpi + k\right) - k^2 L}\hat{F}$$

This complete part(a). These are the analytical expressions for the transfer functions.

## 4.2 Part(b)

Let $m_1 = m_2 = 1$ kg, $k = 3$ N/m, $L = 1$ m, $g = 9.81$ m/s$^2$, $c = 0.1$ N-s/m.

A program was written to plot the magnitude and phase spectrums of $x(t)$ and $\theta(t)$ using the above numerical values. This was done for a range of forcing frequencies to cover both natural frequencies and beyond. Natural frequencies are found by solving the eigenvalue problem $\det\left([K] - \omega^2[M]\right) = 0$

$$\omega_1 = 1.1308 \text{ rad/sec}$$
$$\omega_2 = 4.3228 \text{ rad/sec}$$

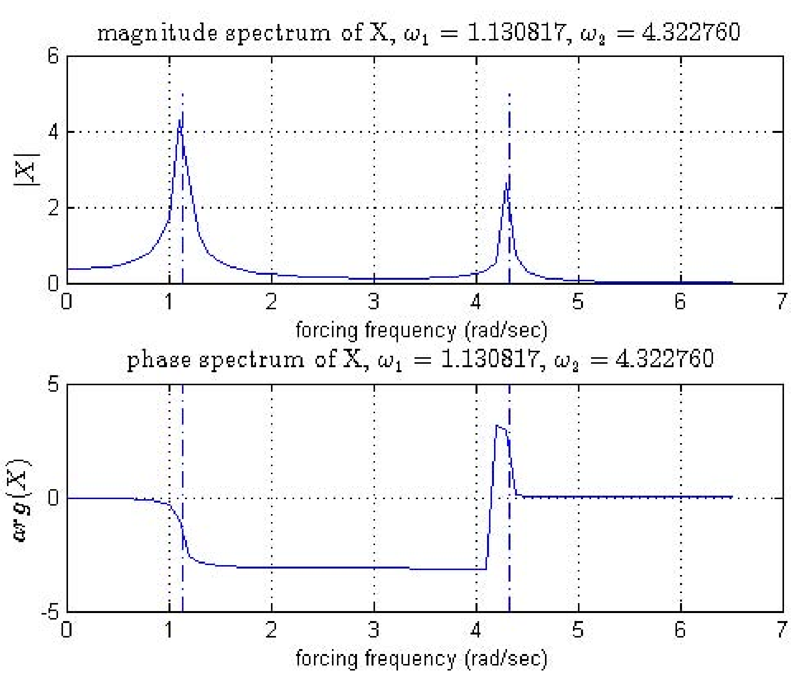
The magnitude and phase of each transfer function are evaluated when $\varpi = \omega_1$ and when $\varpi = \omega_2$.$F = 1$ was assumed since its numerical value was not given. Result is shown below. From these plots, magnitude and phase values are determined at the natural frequencies.

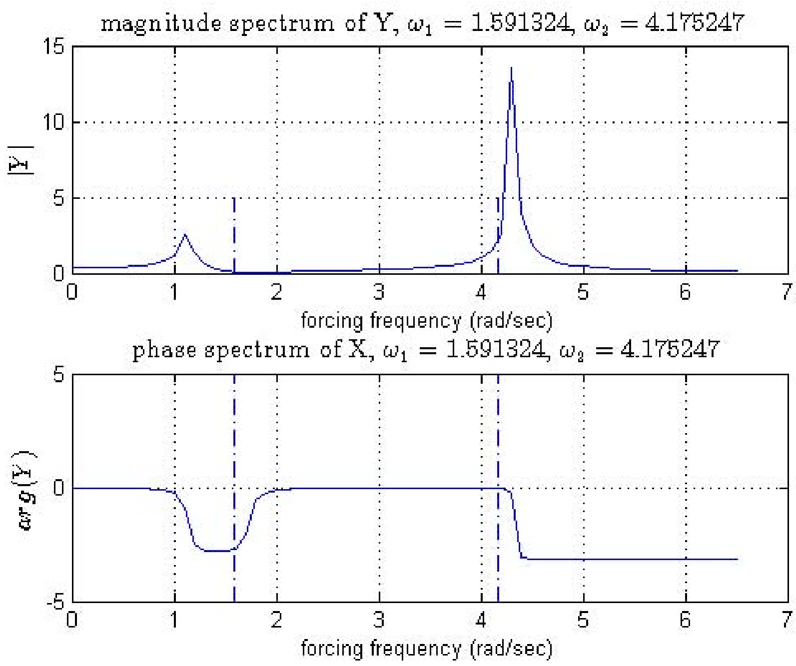
$$x(t) = \text{Re}\left\{Xe^{i\varpi t}\right\}$$
$$\theta(t) = \text{Re}\left\{Ye^{i\varpi t}\right\}$$

Table of results

| response | magnitude at $\omega_1$ | phase at $\omega_1$ | magnitude at $\omega_2$ | phase at $\omega_2$ |
|---|---|---|---|---|
| $x(t)$ | 4.25 | $-83^0$ | 2.62 | $131.7^0$ |
| $\theta(t)$ | 2.55 | $-80^0$ | 11.5 | $-50^0$ |
| ratio | 4.25/2.55 = 1.6667 | | 11.5/2.62 = 4.3893 | |

Plots used to obtain these results






The function used to generate the plots

```
function nma_HW10_problem_5_EMA_545_spectrum()
%plots the spectrums of problem 5, HW10, by Nasser M. Abbasi
close all;

c  = 0.1;
g  = 9.81;
L  = 1;
k  = 3;
m1 = 1;
m2 = 1;
F  = 1;

M  = [1/3*m1*L^2 0;0 m2];
K  = [L^2/4*k+m2*g*L/2  -k*L/2;-k*L/2 k];
C  = [0 0;0 c];

[PHI,w] = eig(K,M);
lam     = sqrt(diag(w))

I = sqrt(-1);
X = @(wf) ((k*L)./((-1/3*m1*L*wf.^2+L/4*k+m2*g/2).*(-m2*wf.^2+I*c*wf+k)- (k^2*L))*F;
Y = @(wf) (1./((-1/3*m1*L*wf.^2+L/4*k+m2*g/2-( k^2*L./(-m2*wf.^2+I*c*wf+k)))))*F;

N = 2;

for i=1:N
    figure(i);
    wf   = 0:0.1:6.5;

    if i==1
       name_='X';
       tf_ = X(wf);
    else
       name_='Y';
       tf_ = Y(wf);
    end

    subplot(2,1,1);
```

```
    plot(wf,abs(tf_));
    hold on;
    line([lam(1) lam(1)],[0 5],'LineStyle','-.');
    line([lam(2) lam(2)],[0 5],'LineStyle','-.');
    title(sprintf('magnitude spectrum of %c, $\\omega_1=%f$, $\\omega_2=%f$',name_,lar
    xlabel('forcing frequency (rad/sec)');
    ylabel(sprintf('$|%c|$',name_),'interpreter','latex','FontSize',12);
    grid;

    subplot(2,1,2);
    plot(wf,angle(tf_));
    line([lam(1) lam(1)],[-5 5],'LineStyle','-.');
    line([lam(2) lam(2)],[-5 5],'LineStyle','-.');
    title(sprintf('phase spectrum of X, $\\omega_1=%f$, $\\omega_2=%f$',lam(1),lam(2))
    xlabel('forcing frequency (rad/sec)');
    ylabel(sprintf('$arg(%c)$',name_),'interpreter','latex','FontSize',12);
    grid;
end

end
```

Eigenvectors $\Phi_1$ and $\Phi_1$ are now found, using modal analysis, which de-couples the EOM. The ratio of one component of the same eigenvector to its other component is found and compared with the result found above. The eigenvectors found are

$$\Phi_1 = \begin{Bmatrix} -0.5446 \\ -0.9493 \end{Bmatrix}$$

$$\Phi_2 = \begin{Bmatrix} -1.6442 \\ 0.3145 \end{Bmatrix}$$

The ratios are 0.9493/0.5446 = 1.7431 and 1.6442/0.3145 = 5.2280. Compare these to the ratios found

| response | magnitude at $\omega_1$ | phase at $\omega_1$ | magnitude at $\omega_2$ | phase at $\omega_2$ |
|---|---|---|---|---|
| $x(t)$ | 4.25 | $-83^0$ | 2.62 | $131.7^0$ |
| $\theta(t)$ | 2.55 | $-80^0$ | 11.5 | $-50^0$ |
| ratio | 4.25/2.55 = 1.6667 | | 11.5/2.62 = 4.3893 | |

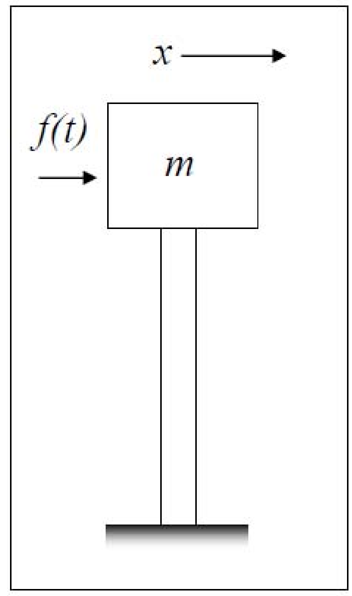
These ratios are close to each others. Ratio $\Phi_{1j}/\Phi_{2j}$ shows how much one dof (1) will change relative to dof (2) in mode $j$

## 4.3 Part(c)

Transfer functions are plotted in part(a). From magnitude spectrum of $Y$ it is seen that $|Y| = 0$ when $\varpi$ between 1.5 and 2.0 rad/sec and also when $\varpi > 6$ rad/sec. $\omega_{cart} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{3}{1}} = 1.7321$ rad/sec. This agrees with range found in plots. When $\sqrt{\frac{k}{m_2}} = 1.73$, top mass acts as vibration absorber, and rod will not oscillate when $F(t)$ is at this specific frequency.

# 5 Problem 6

**6.)** Consider **Exercise 3.45 and 3.46** in the text (you solved this in problem #3 in HW#6). Use the steady-state displacement that you computed using FFT techniques for $\tau = 3\pi/\omega_n$ to compute the maximum stress in the spring. Assume that the spring is a cantilever beam (in bending) modeled after one of the pillars supporting the ERB, which have length L=40m, rectangular cross section with height h, equal to the width b=h=0.6m, and is constructed from a material with modulus E=30 GPa and ultimate tensile strength $\sigma$=40 MPa. (The mass of the beam is assumed to be included in *m*, so its density is not needed.) Let the mass *m* be such that the natural frequency of the mass-spring system is $\omega_n$=0.2Hz. What is the amplitude of the force, P, such that the beam fails due to the dynamic load? Compare that to the static load required to cause the beam to fail (also in bending).




From HW6, problem 3

$$f(t) = \begin{cases} \frac{P}{\tau}t & 0 < t < \tau \\ 0 & \tau < t < 2\tau \end{cases}$$

Let $y_{ss}(t)$ be the solution from problem 3 found using FFT technique. Let the full solution for deflection of the above pillar be

$$\chi(y,t) = y(t)\psi(y)$$

$y(t)$ is the time dependent (dynamic) part of the solution. This solution is $y_{ss}(t)$ found in problem 3. $\psi(y)$ is solution due to static loading. Also called the shape function. For cantilever beam with static force $P$ at its end, deflection curve due to static loading $P$ at end is

$$\psi(x) = \frac{P}{6EI}\left(3Lx^2 - x^3\right)$$

Internal bending moment $M(x,t) = EI\frac{d^2\chi(x,t)}{dx^2}$ and direct stress $\sigma = \frac{M(x,t)c}{I}$ where $c$ is the section modulus. Assume $c = \frac{h}{2}$. For yield, let $\sigma = 40MPa$, then

$$\begin{aligned} M(x,t) &= \frac{\sigma I}{c} \\ EI\frac{d^2\chi(x,t)}{dx^2} &= \frac{\sigma I}{c} \end{aligned}$$

$I = \frac{1}{12}bh^3$.

$$\begin{aligned} \frac{d^2\chi(x,t)}{dx^2} &= y_{ss}(t)\frac{d^2}{dx^2}\frac{P}{6EI}\left(3Lx^2 - x^3\right) \\ &= y_{ss}(t)\frac{PL}{EI} \end{aligned}$$

Solve for $P$ at yield

$$\begin{aligned} y_{ss}(t)\frac{PL}{EI} &= \frac{\sigma_{yield}I}{c} \\ P &= \frac{\sigma_{yield}I}{y_{ss}(t)\frac{h}{2}L}EI \end{aligned}$$

$y_{ss}(t)$ from problem 3 has maximum value of 1.8 at $t = 10$ sec. Given numerical values in the problem and using this maximum value of $y_{ss}(t)$ then $P$ can be found from above.

I am not sure this is the correct approach to solve this.We did not have any practice or examples on solving this type of vibration problem before. Need more time to study this subject.