HW 1

HW 1 EMA 545
Mechanical Vibrations

Spring 2013
University of Wisconsin, Madison

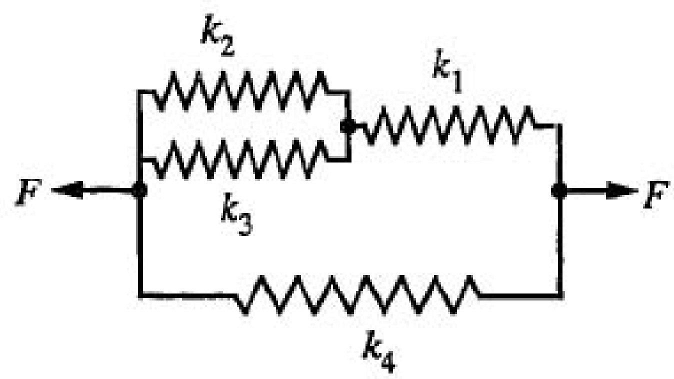
Nasser M. Abbasi

Spring 2013 Compiled on April 19, 2021 at 2:54am [public]

Contents

1 Problem 1 (1.1 book)

1.1 Determine the spring stiffness that is equivalent to the action of the four springs in the sketch.



k_3 and k_2 are in parallel, hence the effective stiffness is

$$k_{23} = k_2 + k_3$$

k_{23} and k_1 are now in series, hence the effective stiffness is

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_{23}} = \frac{k_{23} + k_1}{k_1 k_{23}} = \frac{k_2 + k_3 + k_1}{k_1(k_2 + k_3)} = \frac{k_2 + k_3 + k_1}{k_1 k_2 + k_1 k_3}$$

Therefore

$$k_{123} = \frac{k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1}$$

k_{123} and k_4 are now in parallel, hence the effective stiffness is

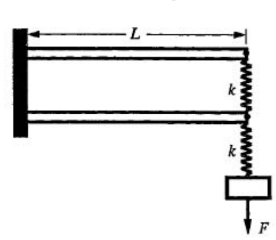
$$\begin{aligned} k_{1234} &= k_4 + k_{123} \\ &= k_4 + \frac{k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1} \end{aligned}$$

Hence the final effective stiffness is

$$k_{eq} = \frac{k_4(k_2 + k_3 + k_1) + k_1 k_2 + k_1 k_3}{k_2 + k_3 + k_1}$$

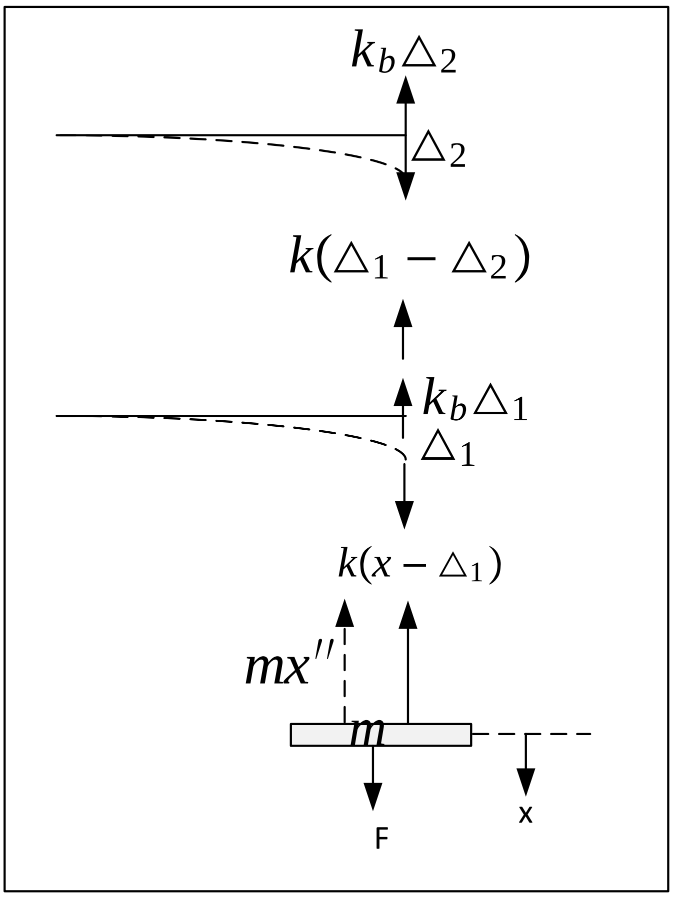
2 Problem 2

Problem 2: Find the equation of motion of the system pictured below. The mass of the block is m and the mass of the beams and springs is negligible. Assume that all of the displacements are very small. (Recall that the displacement of the tip of a cantilever beam, Δ_{tip}, is related to the force at the tip by: $F_{tip}=(3EI/L^3)\Delta_{tip}$)



We start by drawing a free body diagram and taking displacement of mass from the static equilibrium position. Let the displacement of the mass be x and positive pointing upwards.

Let $\triangle_1$ be the downward deflection at right end of the bottom beam. Let $\triangle_2$ be the downward deflection at right end of top beam. The free body diagram is



Applying equilibrium of vertical forces $\sum F_v = 0$ for mass m and noting that inertial forces opposes motion, results in the equation of motion

$$mx'' + k(x - \triangle_1) = F \tag{1}$$

To find an expression for $\triangle_1$ in terms of x, we apply equilibrium of vertical forces at the right end of the lower beam[1]

$$k(x - \triangle_1) = k_b\triangle_1 + k(\triangle_1 - \triangle_2) \tag{2}$$

Similarly, applying equilibrium of vertical forces at the right end of the top beam

$$k(\triangle_1 - \triangle_2) = k_b\triangle_2 \tag{3}$$

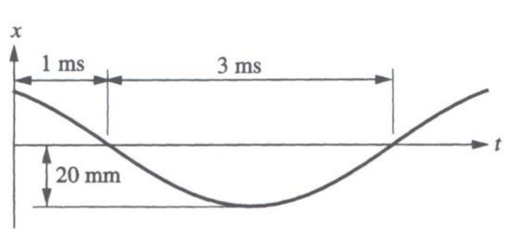
Solving for $\triangle_1, \triangle_2$ from Eqs 2,3 (2 equations, 2 unknowns) gives

$$\triangle_1 = \frac{k(k + k_b)}{k^2 + 3kk_b + k_b^2}x$$

Substituting the above value into Eq 1 results in the equation of motion

$$mx'' + kx\left(1 - \frac{k(k + k_b)}{k^2 + 3kk_b + k_b^2}\right) = F$$

3 Problem 3



EXERCISE 2.3

Use the properties in this trace to determine

(a) The initial values of x and $\dot{x}$.

(b) The value of t at which the minimum value of x first occurs.

(c) The maximum positive value of $\dot{x}$ and the earliest value of t at which this maximum occurs.

(d) The maximum positive value of $\ddot{x}$ and the earliest value of t at which this maximum occurs.

[1] k_b is beam stiffness against vertical displacement at the end and is given as $k_b = \frac{3EI}{L^3}$

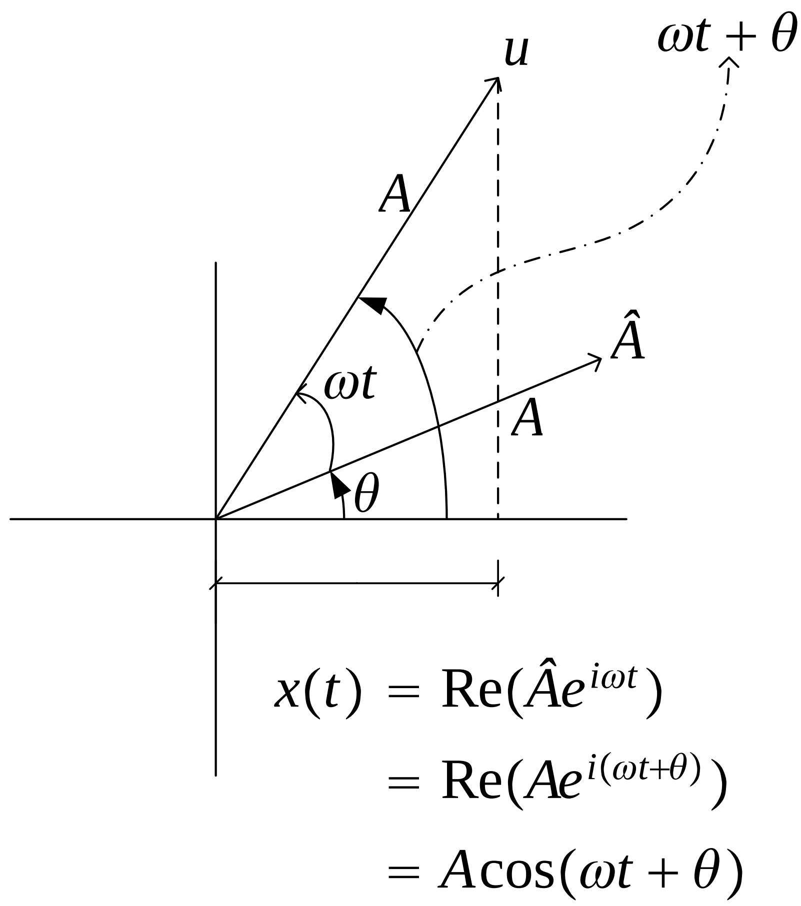
Assuming periodic motion, the period is $T = 6$ ms, or 6×10^{-3} sec. Hence $\omega = \frac{\pi}{3}$ rad/ms Representing this as a cosine signal with phase gives

$$x(t) = A\cos(\omega t + \theta)$$

Then

$$\begin{aligned} x(t) &= \mathrm{Re}[A + \cos(\omega t + \theta)] \\ &= \mathrm{Re}[A e^{i\theta} e^{i\omega t}] \\ &= \mathrm{Re}[\bar{A} e^{i\omega t}] \end{aligned} \tag{4}$$

Where now $\hat{A} = Ae^{i\theta}$.Using phasor diagram



Hence from the diagram we see that for $x(t_0)$ to be zero when $t_0 = 1$ ms, we need to have

$$\omega t_0 + \theta = \frac{\pi}{2}$$

But $\omega = \frac{\pi}{3}$ rad/ms, hence

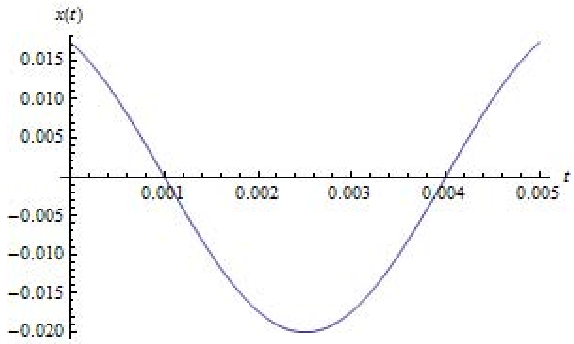
$$\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

To find A we see that the maximum absolute value of $x(t)$ is 20 mm hence $A = 20$ mm or 20×10^{-3} meter. The equation of $x(t)$ when substituting all numerical values becomes

$$x(t) = 20\cos\left(\frac{\pi}{3}t + \frac{\pi}{6}\right) \tag{5}$$

Where units used are radians, milliseconds and mm. This is a plot of the above function

```
parms = f -> 1/(6 10^-3);
Plot[0.02 Cos[2 Pi f t + (Pi/6)] /. parms, {t,0,0.005},
 AxesLabel -> {t,x[t]}, ImageSize -> 300]
```



3.1 part(a)

At $t = 0$, from 4 $x(0) = \text{Re}[\hat{A}] = A\cos(\theta) = 20\cos(\frac{\pi}{6})$ hence

$$x(0) = 17.321 \text{ mm}$$

From 4 $x'(t) = \text{Re}[\omega \hat{A} e^{i\omega t}]$ hence $x'(0) = \text{Re}[\omega \hat{A}] = \omega A\cos(\theta) = 20\frac{\pi}{3}\cos(\frac{\pi}{6})$ giving

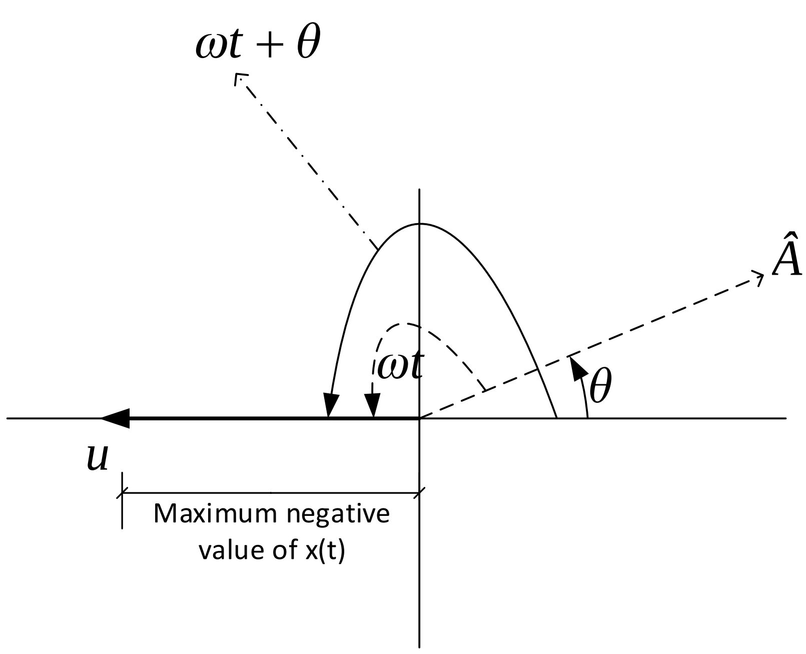
$$x'(0) = 18.138 \text{ m/sec}$$

3.2 part(b)

This can be solved using calculus[2]

$$\begin{aligned} x'(t) &= -2\pi f A \sin\left(2\pi f t + \theta\right) \\ 0 &= -2\pi f A \sin\left(2\pi f t + \frac{\pi}{6}\right) \\ &= -\left(\frac{2\pi}{6 \times 10^{-3}}\right)\left(20 \times 10^{-3}\right) \sin\left(\frac{2\pi}{6 \times 10^{-3}} t + \frac{\pi}{6}\right) \\ 0 &= \sin\left(\frac{2\pi}{6 \times 10^{-3}} t + \frac{\pi}{6}\right) \end{aligned}$$

We solve for t and find t=2.5 ms. But this can be solved more easily by looking at the phasor diagram



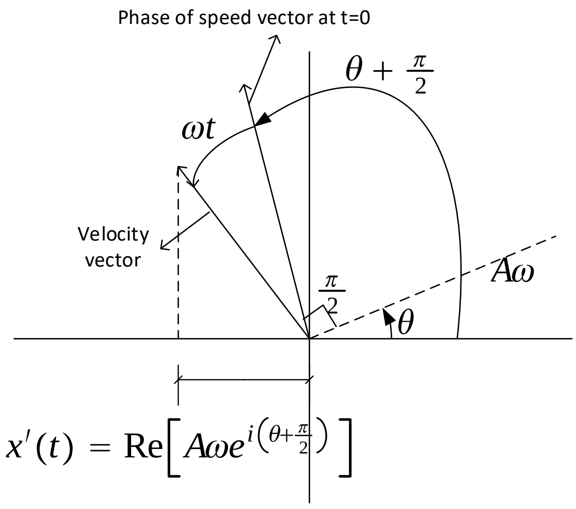
The minimum $x(t)$ (in negative sense and not in absolute value sense) occurs when $\omega t_{min} + \theta = \pi$, hence $t_{min} = \frac{\pi - \theta}{\omega}$, therefore

$$t_{min} = 2.5$$

[2]Taking derivative of $x(t)$ and setting the result to zero and solving for t

3.3 part(c)

This is solved in a similar way by treating the speed as the rotating vector in complex plan. Since $x'(t) = \text{Re}\left(A\omega e^{i\left(\omega t+\theta+\frac{\pi}{2}\right)}\right)$ then in complex plan as follows



The difference is that the velocity vector has phase of $\theta + \frac{\pi}{2}$ instead of θ as was the case with the position vector, and the amplitude is $A\omega$ instead of A. Hence the first time the speed vector will have the maximum value is when

$$\theta + \frac{\pi}{2} + \omega t = 2\pi$$

Hence

$$\begin{aligned} t &= \frac{2\pi - \frac{\pi}{2} - \theta}{\omega} \\ &= \frac{2\pi - \frac{\pi}{2} - \frac{\pi}{6}}{\frac{\pi}{3}} \end{aligned}$$

Hence $t = 4$ ms and the amplitude is given by $A\omega = 20\frac{\pi}{3}$ hence $A\omega = 20.944$ meter/sec

3.4 part(d)

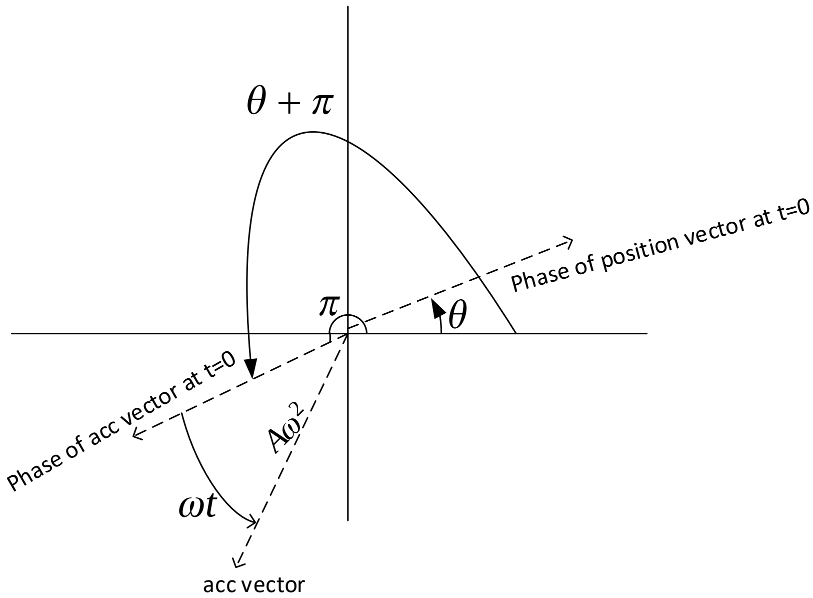
Now treating the acceleration as the rotating vector in complex plan

$$\begin{aligned} x(t) &= \text{Re}\left(Ae^{i(\theta+\omega t)}\right) \\ x'(t) &= \text{Re}\left(iA\omega e^{i(\theta+\omega t)}\right) \\ x''(t) &= \text{Re}\left(-A\omega^2 e^{i(\theta+\omega t)}\right) \end{aligned}$$

But $-1 = e^{i\pi}$ This adds a π to the phase resulting in

$$x''(t) = \text{Re}\left(A\omega^2 e^{i(\theta+\omega t+\pi)}\right)$$

Representing $x''(t)$ in complex plan gives



The first time the $x''(t)$ vector will have the maximum value is when

$$\theta + \pi + \omega t = 2\pi$$

Hence

$$\begin{aligned} t &= \frac{2\pi - \pi - \theta}{\omega} \\ &= \frac{\pi - \frac{\pi}{6}}{\frac{\pi}{6}} \end{aligned}$$

Hence $t = 2.5$ ms and the amplitude is

$$\begin{aligned} A\omega^2 &= 20 \text{ mm}\left(\frac{\pi}{3} \text{ rad/msec}\right)^2 \\ &= 21.933 \times 10^3 \text{ meter/sec}^2 \end{aligned}$$

4 Problem 4 (2.5 book)

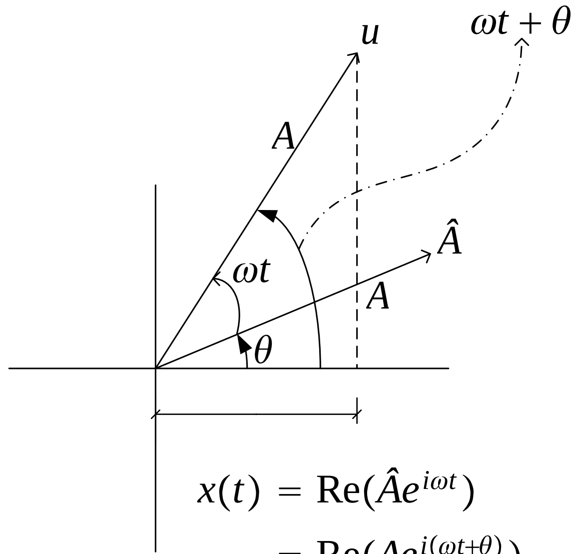
2.5 An oscilloscope trace indicates that the voltage output v from a sensor varies harmonically, with zeroes occurring every 8 ms. The first zero of v occurs at $t = 5.5$ ms, the amplitude of the signal is 1.2 V, and $v > 0$ at $t = 0$.

(a) Express this signal as a complex exponential. Write the complex amplitude in polar and rectangular forms.

(b) Express the time rate of change of the voltage as a complex exponential. Write this quantity in polar and rectangular forms.

4.1 part(a)

The function of the signal is converted to complex exponential. A sin or cos can be used to represent the signal as long as we are consistent. Assuming the signal is $x(t) = A\cos(\omega t + \theta)$, plotting the general representation of the position vector in complex plan gives



The complex representation of the position vector is

$$x(t) = \text{Re}\left[Ae^{i(\omega t+\theta)}\right]$$

We are given that $\omega = \frac{2\pi}{T} = \frac{2\pi}{16}$, and since $x(t_0)$ has first zero at $t_0 = 5.5$ ms this means from looking at the above diagram that

$$\theta + \omega t_0 = \frac{\pi}{2}$$

Hence $\theta = \frac{\pi}{2} - (\omega t_0) = \frac{\pi}{2} - (\frac{\pi}{8}\frac{55}{10})$ which gives

$$\theta = \frac{-3\pi}{16}\text{radians}$$

Hence the signal is

$$\begin{aligned} x(t) &= \text{Re}\left[Ae^{i(\omega t+\theta)}\right] \\ &= \text{Re}\left[1.2e^{i\left(\frac{\pi}{8}t-\frac{3\pi}{16}\right)}\right] \\ &= \text{Re}\left[1.2e^{-i\frac{3\pi}{16}}e^{i\frac{\pi}{8}t}\right] \\ &= \text{Re}\left[\hat{A}e^{i\frac{\pi}{8}t}\right] \end{aligned}$$

Where $\hat{A} = 1.2e^{-i\frac{3\pi}{16}}$ is the complex amplitude in polar coordinates. In rectangular coordinates it becomes

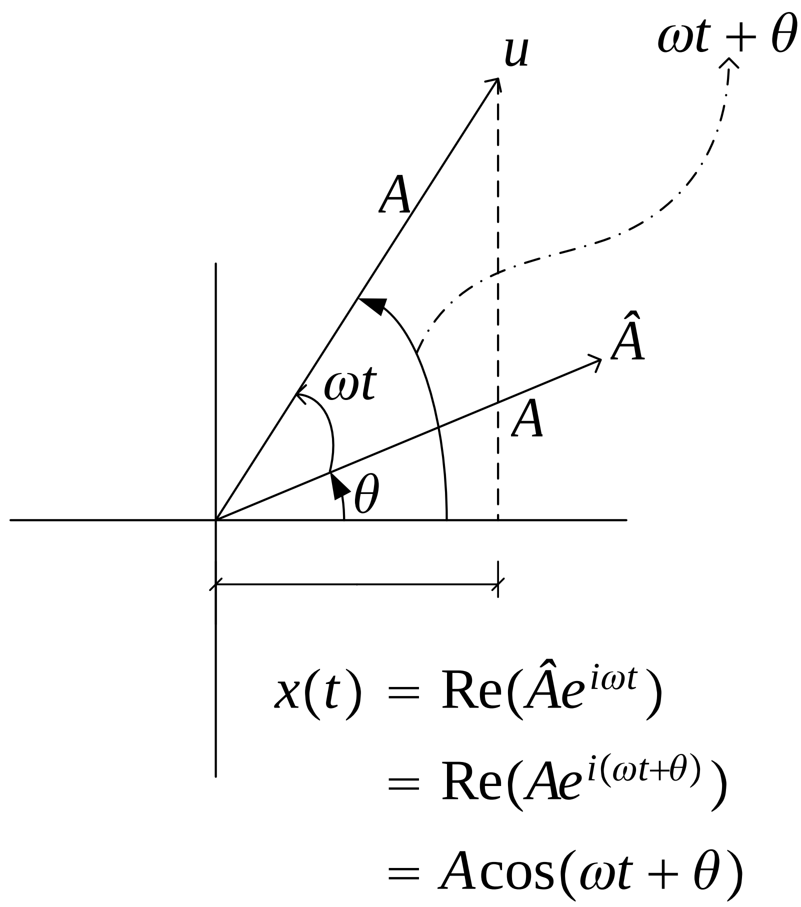
$$\begin{aligned} \hat{A} &= 1.2e^{-i\frac{3\pi}{16}} \\ &= 1.2\left(\cos\left(\frac{3\pi}{16}\right) - i\sin\left(\frac{3\pi}{16}\right)\right) \\ &= 1.2(0.831 - i0.5556) \\ &= \boxed{0.9977 - i0.6667} \end{aligned}$$

Hence

$$\begin{aligned} x(t) &= \text{Re}\left[(0.998 - i0.668)\left(\cos\frac{\pi}{8}t + i\sin\frac{\pi}{8}t\right)\right] \\ &= \text{Re}\left[\left(0.998\cos\frac{\pi}{8}t + 0.668\sin\frac{\pi}{8}t\right)+ \right. \\ &\left. i\left(0.998\sin\frac{\pi}{8}t - 0.668\cos\frac{\pi}{8}t\right)\right] \end{aligned}$$

Here is a plot of the signal for 20 ms

```
w = Pi/8;
f = 1.2 Cos[w t - 3 Pi/16];
Plot[f, {t, 0, 20}, AxesLabel -> {t, x[t]},
                    ImageSize -> 300,
                    GridLines -> Automatic,
     GridLinesStyle->{{Dashed,Gray},{Dashed,Gray}},
     PlotStyle -> Red]
```



4.2 part(b)

From above it was found that

$$x(t) = \text{Re}\left[Ae^{i(\omega t+\theta)}\right]$$

Hence

$$\begin{aligned} x'(t) &= \text{Re}\left[i\omega Ae^{i(\omega t+\theta)}\right] \\ &= \text{Re}\left[e^{i\frac{\pi}{2}}\omega Ae^{i\theta}e^{i\omega t}\right] \\ &= \text{Re}\left[\omega Ae^{i(\frac{\pi}{2}+\theta)}e^{i\omega t}\right] \\ &= \text{Re}\left[\hat{A}e^{i\omega t}\right] \end{aligned}$$

Where $\hat{A} = \omega Ae^{i(\frac{\pi}{2}+\theta)}$ Replacing numerical values gives $\hat{A} = \frac{\pi}{8}(1.2)e^{i(\frac{\pi}{2}-\frac{3}{16}\pi)} = 0.471e^{i0.983}$ and

$$\begin{aligned} x'(t) &= \text{Re}\left[0.471e^{i0.983}e^{i\omega t}\right] \\ &= \text{Re}\left[0.471e^{i0.983}e^{i\frac{\pi}{8}t}\right] \\ &= \text{Re}\left[0.471e^{i0.983}e^{i0.3923t}\right] \end{aligned}$$

In rectangular coordinates, the above becomes

$$
\begin{aligned}
x'(t) &= \mathrm{Re}\Big[0.471(\cos 0.983 + i\sin 0.983) \\
&\quad (\cos 0.3923t + i\sin 0.3923t)\Big] \\
&= \mathrm{Re}[(0.261 + 0.392i)(\cos 0.392t + i\sin 0.392t)] \\
&= \mathrm{Re}\Big[(0.261\cos 0.392t - 0.392\sin 0.392t) \\
&\quad + i(0.261\sin 0.392t + 0.392\cos 0.392t)\Big]
\end{aligned}
$$

4.3 part(c)

To find the maximum rate of the signal

$$x'(t) = \mathrm{Re}\left[\hat{A}e^{i\omega t}\right]$$

Then the maximum $x'(t)$ is $|\hat{A}|$ which is

$$
\begin{aligned}
|\hat{A}| &= |0.261 + 0.392i| \\
&= \sqrt{0.261^2 + 0.392^2} \\
&= 0.471
\end{aligned}
$$

Hence maximum $x'(t)$ is 0.471 v/ms or 471 volt/sec.

Maximum velocity in simple harmonic motion occurs when $x(t) = 0$. This occurs at $t = 5.5$ ms and at 8 ms henceforth. Hence maximum speed occurs at

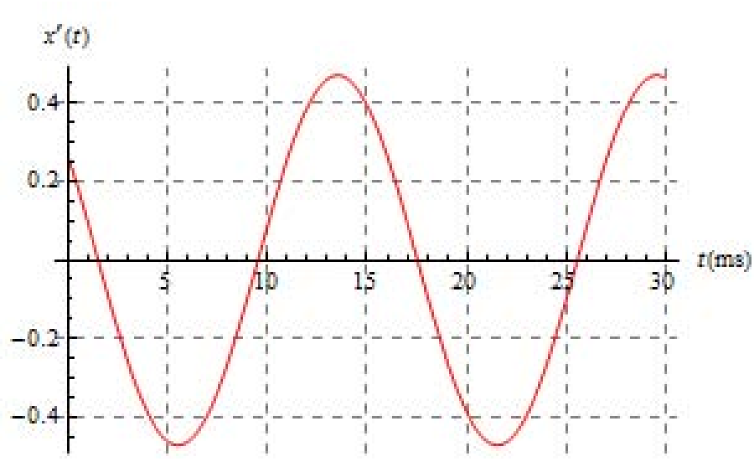
$$t = 5.5 + n(8)$$

for $n = 0, 1, 2, \cdots$ this results in

$$t = 5.5, 13, 5, 21.5, \cdots \text{ ms}$$

Here is a plot of $x'(t)$ in units of volt/ms

```
f = 0.261 Cos[0.392 t] - 0.392 Sin[0.392 t];
Plot[f, {t, 0, 30},
 AxesLabel -> {Row[{t, "(ms)"}], x'[t]},
 ImageSize -> 300, GridLines -> Automatic,
 GridLinesStyle -> {{Dashed, Gray}, {Dashed, Gray}},
 PlotStyle -> Red]
```



5 Problem 5 (2.8 book)

2.8 Suppose $q = 0.01\sin(50t) - 0.02\cos(50t - 0.3\pi)$.

(a) Write q in complex exponential form. What is the complex amplitude?

(b) What is the time interval separating instants at which $q = 0$?

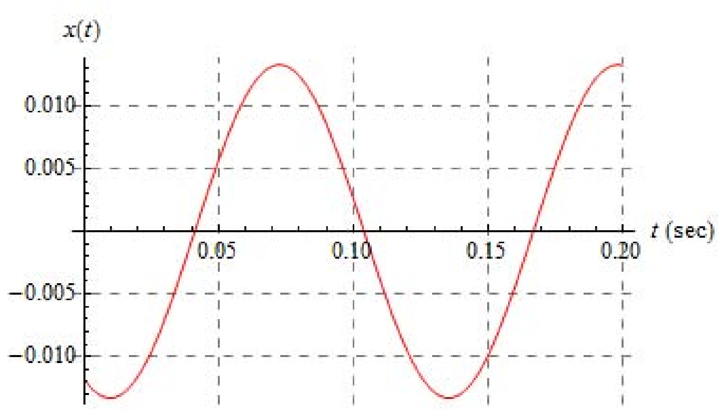
(c) What is the earliest positive t at which $q = 0$?

(d) What is the largest value of q that will occur, and what is the earliest positive t at which this maximum occurs?

5.1 part(a)

This is a plot of the signal

```
f = 0.01 Sin[50 t] - 0.02 Cos[50 t - 0.3 Pi];
Plot[f, {t, 0, 0.2},
        AxesLabel -> {Row[{t , " (sec)"}], x[t]},
        ImageSize -> 300,
        GridLines -> Automatic,
        GridLinesStyle->{{Dashed,Gray},{Dashed,Gray}},
        PlotStyle -> Red]
```



$$
\begin{aligned}
q &= 0.01\sin(50t) - 0.02\cos(50t - 0.3\pi) \\
&= \mathrm{Re}\left[\frac{0.01}{i}e^{i50t} - 0.02e^{i(50t-0.3\pi)}\right] \\
&= \mathrm{Re}\left[0.01e^{-i\frac{\pi}{2}}e^{i50t} - 0.02e^{i50t}e^{-i0.3\pi}\right] \\
&= \mathrm{Re}\left[\left(0.01e^{-i\frac{\pi}{2}} - 0.02e^{-i0.3\pi}\right)e^{i50t}\right] \\
&= \mathrm{Re}\left[\hat{A}e^{i50t}\right]
\end{aligned}
$$

Hence the complex amplitude is

$$\hat{A} = 0.01e^{-i\frac{\pi}{2}} - 0.02e^{-i0.3\pi}$$

5.2 part(b)

From above, we see that

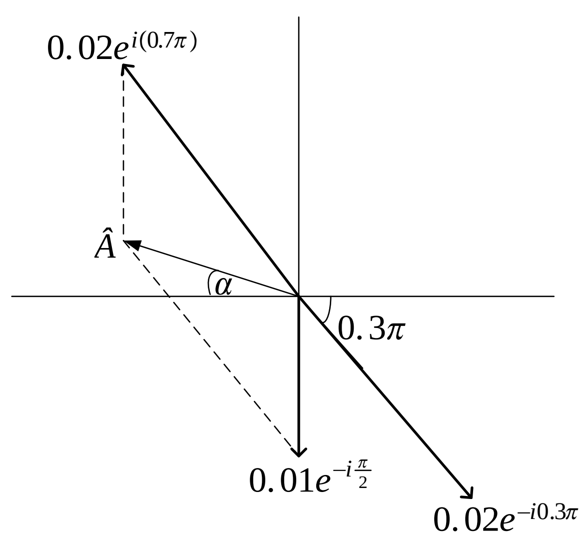
$$\omega = 50 \text{ rad/sec}$$

Hence $f = \frac{50}{2\pi}$Hz, or the period $T = \frac{2\pi}{50} = 0.126$ sec, therefore the time period separating the zeros is $\frac{0.126}{2} = 0.063$ sec or 63 ms

5.3 part(c)

The complex phase $\hat{A}$ can be found by adding the vector $0.01e^{-i\frac{\pi}{2}}$ and $-0.02e^{-i\frac{3\pi}{10}}$ by completing the parallelogram as shown in this diagram. $\hat{A} = -0.02\cos 0.7\pi + i(-0.01 + 0.02\sin 0.7\pi)$, hence the angle α that $\hat{A}$ makes with the horizontal is

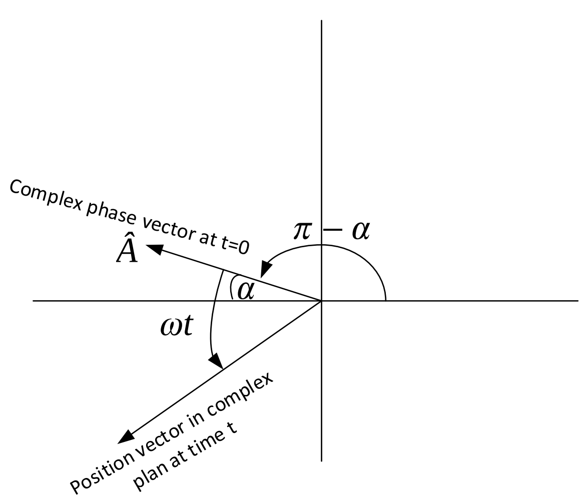
$$\begin{aligned}\tan^{-1}\left(\frac{-0.01 + 0.02\sin 0.7\pi}{-0.02\cos 0.7\pi}\right) &= \arctan(0.526)\\ &= 0.484 \text{ radian}\\ &= 27.73 \text{ degree}\end{aligned}$$



and the amplitude is

$$\sqrt{(-0.01 + 0.02\sin 0.7\pi)^2 + (0.02\cos 0.7\pi)^2} = 0.0133\text{V}$$

To find the earliest time q will be zero, we need to find the time the complex position vector will take to rotate and reach the imaginary axis.



Hence we need to solve

$$\begin{aligned}
\pi - \alpha + \omega t_0 &= \frac{3}{2}\pi \\
t_0 &= \frac{\frac{3}{2}\pi - \pi + 0.48402}{50} \\
&= 0.0411 \text{ s}
\end{aligned}$$

Therefore

$$t = 41.1 \text{ ms}$$

5.4 part(d)

The largest value of q is the absolute value of its complex amplitude. We found this above as

$$\left|\hat{A}\right| = 0.0133 \text{ Volt}$$

To find when this occur first time, the time the position vector will align with the real axis in the positive direction is found. Hence solving for t_0 from

$$\begin{aligned}
\pi - \alpha + \omega t_0 &= 2\pi \\
t_0 &= \frac{2\pi - \pi + 0.484}{50}
\end{aligned}$$

Gives $t = 72.5$ ms. Another way would be to take derivative of $qt)$ and set that to zero and solve for first t which satisfy the equation.

6 Problem 6 (2.10 book)

> **2.10** Two parts of the harmonic motion of a system are $x_1 = 8\sin(10t - 5\pi/6)$, $x_2 = 12\cos(10t + \phi)$. Find the phase angle ϕ for which $x = x_1 + x_2$ is a pure sine function. What is the amplitude of x in that case?

$$\begin{aligned}
x_1 &= 8\sin\left(10t - \frac{5}{6}\pi\right) \\
x_2 &= 12\cos\left(10t + \phi\right)
\end{aligned}$$

Let $\omega = 10$, hence

$$\begin{aligned}
x &= x_1 + x_2 \\
&= \operatorname{Re}\left[\frac{8}{i}e^{i\left(\omega t - \frac{5}{6}\pi\right)}\right] + \operatorname{Re}\left[12e^{i\left(\omega t + \phi\right)}\right] \\
&= \operatorname{Re}\left[\frac{8}{i}e^{i\left(\omega t - \frac{5}{6}\pi\right)} + 12e^{i\left(\omega t + \phi\right)}\right] \\
&= \operatorname{Re}\left[8e^{-i\frac{\pi}{2}}e^{i\left(\omega t - \frac{5}{6}\pi\right)} + 12e^{i\left(\omega t + \phi\right)}\right] \\
&= \operatorname{Re}\left[8e^{-i\frac{\pi}{2}}e^{i\omega t}e^{-i\frac{5}{6}\pi} + 12e^{i\omega t}e^{i\phi}\right] \\
&= \operatorname{Re}\left[\left(8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i\phi}\right)e^{i\omega t}\right] \\
&= \operatorname{Re}\left[\hat{A}e^{i\omega t}\right] \qquad (6)
\end{aligned}$$

Where

$$\begin{aligned}\hat{A} &= 8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i\phi}\\ &= (-4 + 6.928i) + 12\left(\cos\phi + i\sin\phi\right)\\ &= \left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\end{aligned}$$

Hence Eq 6 becomes

$$x = \operatorname{Re}\left[\left\{\left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\right\}e^{i\omega t}\right]$$

To convert to sin we multiply and divide by i hence

$$\begin{aligned}x &= \operatorname{Re}\left[\left\{\left(-4 + 12\cos\phi\right) + i\left(6.928 + \sin\phi\right)\right\}i\frac{e^{i\omega t}}{i}\right]\\ &= \operatorname{Re}\left[\left\{-\left(6.928 + \sin\phi\right) + i\left(-4 + 12\cos\phi\right)\right\}\frac{e^{i\omega t}}{i}\right] \qquad (7)\end{aligned}$$

The complex number $-\left(6.928 + \sin\phi\right) + i\left(-4 + 12\cos\phi\right)$ can be written in polar form as $ke^{i\beta}$ where $K = \sqrt{\left(6.928 + \sin\phi\right)^2 + \left(-4 + 12\cos\phi\right)^2}$ and $\beta = \tan^{-1}\left(\frac{-4+12\cos\phi}{-\left(6.928+\sin\phi\right)}\right)$, hence Eq 7 becomes

$$\begin{aligned}x &= \operatorname{Re}\left[ke^{i\beta}\frac{e^{i\omega t}}{i}\right]\\ &= \operatorname{Re}\left[k\frac{e^{i\left(\omega t+\beta\right)}}{i}\right]\\ &= k\sin\left(\omega t + \beta\right)\end{aligned}$$

or in full form

$$x = \sqrt{\left(6.928 + \sin\phi\right)^2 + \left(-4 + 12\cos\phi\right)^2}\,\sin\left(\omega t + \tan^{-1}\left(\frac{-4 + 12\cos\phi}{-\left(6.928 + \sin\phi\right)}\right)\right)$$

For pure sine function we need $\frac{-4+12\cos\phi}{-\left(6.928+\sin\phi\right)} = 0$ or $12\cos\phi = 4$ or $\cos\phi = \frac{1}{3}$, hence

$$\begin{aligned}\phi &= 1.23096 \text{ radian}\\ &= 70.529^{\circ}\end{aligned}$$

The amplitude can also be found from the complex amplitude above when $\phi = 1.23096$ as follows

$$\begin{aligned}\left|8e^{-i\left(\frac{4}{3}\pi\right)} + 12e^{i1.23096}\right| &= \left|-6.592\times 10^{-6} + 18.242i\right|\\ &= \sqrt{\left(-6.592\times 10^{-6}\right)^2 + (18.242)^2}\\ &= 18.242\end{aligned}$$