

# Solving slide 390 example, ME 440 Intermediate Vibration, Fall 2017

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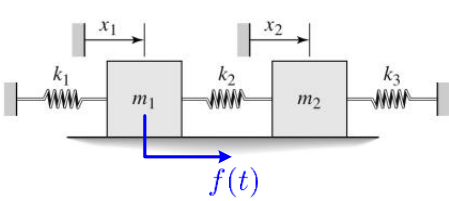
[AO]

## Example: Forced Undamped Response

$m_1=1\text{kg}, m_2=2\text{kg}$   
 $k_1=9\text{N/m}$   
 $k_2=k_3=18\text{N/m}$

IC :  $\begin{cases} x_1(0) = 3 & \dot{x}_1(0) = 0 \\ x_2(0) = 0 & \dot{x}_2(0) = 9 \end{cases}$   
 $f(t) = 3 \sin 4t$

- Find response of the system



By inspection

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix}$$

And

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The system is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix} \quad (1)$$

The above is solved using modal analysis in order to decouple the system. The first step is to determine the eigenvalues.

$$\begin{aligned} [A] &= [m]^{-1} [k] \\ &= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \\ &= \begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix} \end{aligned}$$

To find the eigenvalues of  $[A]$  we solve  $|A - \lambda I| = 0$  or

$$\begin{vmatrix} 27 - \lambda & -18 \\ -9 & 18 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 - 45\lambda + 324 = 0$$

Hence

$$\lambda_1 = 9$$

$$\lambda_2 = 36$$

Which implies

$$\omega_{n(1)} = 3 \text{ rad/s}$$

$$\omega_{n(2)} = 9 \text{ rad/s}$$

Now we find the eigenvectors  $u_i$  or the shape vectors. For  $\lambda_1 = 9$

$$\begin{aligned} [A] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &= \lambda_1 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ \begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &= 9 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ \begin{bmatrix} 27u_1 - 18u_2 \\ -9u_1 + 18u_2 \end{bmatrix} &= \begin{Bmatrix} 9u_1 \\ 9u_2 \end{Bmatrix} \end{aligned}$$

Using first equation only gives

$$27u_1 - 18u_2 = 9u_1$$

We always normalized to  $u_1 = 1$ , hence the above gives

$$27 - 18u_2 = 9$$

$$u_2 = 1$$

Therefore the first eigenvector is

$$\vec{u}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

To find the second eigenvector. For  $\lambda_2 = 36$

$$\begin{aligned} [A] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &= \lambda_2 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ \begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} &= 36 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ \begin{bmatrix} 27u_1 - 18u_2 \\ -9u_1 + 18u_2 \end{bmatrix} &= \begin{Bmatrix} 36u_1 \\ 35u_2 \end{Bmatrix} \end{aligned}$$

Using first equation only gives

$$27u_1 - 18u_2 = 36u_1$$

We always normalized to  $u_1 = 1$ , hence the above gives

$$27 - 18u_2 = 36$$

$$u_2 = -\frac{1}{2}$$

Therefore the second eigenvector is

$$\vec{u}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

Hence the modal matrix is

$$[u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Using the modal matrix, we can now decouple the original system given above in (1) which is

$$[m] \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + [k] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix} \quad (2)$$

Let  $\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [u] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$ , then the above becomes

$$[m][u] \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [k][u] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix}$$

Premultiplying both sides by  $[u]^T$  gives

$$[u]^T [m] [u] \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [u]^T [k] [u] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [u]^T \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix} \quad (4)$$

But

$$[u]^T [m] [u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

And

$$[u]^T [k] [u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix}$$

Then (4) becomes

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3 \sin 4t \\ 3 \sin 4t \end{Bmatrix}$$

Hence we obtain 2 ODEs

$$\begin{aligned} 3\ddot{q}_1(t) + 27q_1(t) &= 3 \sin 4t \\ \frac{3}{2}\ddot{q}_2(t) + 54q_2(t) &= 3 \sin 4t \end{aligned}$$

Or

$$\ddot{q}_1(t) + 9q_1(t) = \sin 4t \quad (5)$$

$$\ddot{q}_2(t) + 36q_2(t) = 2 \sin 4t \quad (6)$$

Note There is a short cut to obtain the above (5,6) equations directly as follows. Starting with (2), we just write

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} \omega_{n(1)}^2 & 0 \\ 0 & \omega_{n(2)}^2 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} &= [u]^{-1} [m]^{-1} \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} \ddot{q}_1(t) + 9q_1(t) \\ \ddot{q}_2(t) + 36q_2(t) \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 3 \sin 4t \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} \sin 4t \\ 2 \sin 4t \end{Bmatrix} \end{aligned}$$

Which is the same as (5,6). This short cut just needs finding  $[u]^{-1} [m]^{-1}$ . Use this short cut for the exam.

Solving (5)

The homogeneous solution is

$$q_{1,h}(t) = A_1 \cos 3t + B_1 \sin 3t$$

And to find the particular solution, we guess  $q_{1,p} = C \sin 4t$ , hence  $\dot{q}_{1,p} = 4C \cos 4t$  and  $\ddot{q}_{1,p} = -16C \sin 4t$ . Plug-in in (5) gives

$$\begin{aligned} -16C \sin 4t + 9(C \sin 4t) &= \sin 4t \\ -7C_1 \sin 4t &= \sin 4t \\ C_1 &= -\frac{1}{7} \end{aligned}$$

Hence  $q_{1,p} = -\frac{1}{7} \sin 4t$  and the complete solution is

$$q_1(t) = A_1 \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t$$

Now we do the same to solve (6)

The homogeneous solution is

$$q_{2,h}(t) = A_2 \cos 6t + B_2 \sin 6t$$

And to find the particular solution, we guess  $q_{2,p} = C \sin 4t$ , hence  $\dot{q}_{2,p} = 4C \cos 4t$  and  $\ddot{q}_{2,p} =$

$-16C \sin 4t$ . Plug-in in (6) gives

$$\begin{aligned} -16C \sin 4t + 36(C \sin 4t) &= 2 \sin 4t \\ 20C_1 \sin 4t &= 2 \sin 4t \\ C_1 &= \frac{1}{10} \end{aligned}$$

Hence  $q_{2,p} = \frac{1}{10} \sin 4t$  and the complete solution is

$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t$$

Therefore the solution in principle coordinates is

$$q_1(t) = A_1 \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t \quad (5A)$$

$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t \quad (6A)$$

Since  $\{x\} = [u] \{q\}$ , then  $\{q\} = [u]^{-1} \{x\}$ . Therefore

$$\begin{aligned} \{q(0)\} &= [u]^{-1} \{x(0)\} \\ \begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} \{\dot{q}(0)\} &= [u]^{-1} \{\dot{x}(0)\} \\ \begin{pmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{pmatrix} &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{pmatrix} 0 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ -6 \end{pmatrix} \end{aligned}$$

Applying first initial conditions to (5A,6A) gives

$$1 = A_1$$

$$2 = A_2$$

Hence (5A,6A) becomes

$$q_1(t) = \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t \quad (5B)$$

$$q_2(t) = 2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t \quad (6B)$$

Taking derivatives

$$\dot{q}_1(t) = -3 \sin 3t + 3B_1 \cos 3t - \frac{4}{7} \cos 4t$$

$$\dot{q}_2(t) = -12 \sin 6t + 6B_2 \cos 6t + \frac{4}{10} \cos 4t$$

Applying the second initial conditions to the above gives

$$6 = 3B_1 - \frac{4}{7}$$

$$-6 = 6B_2 + \frac{4}{10}$$

Solving gives  $B_1 = \frac{46}{21}$ ,  $B_2 = -\frac{16}{15}$ . Hence (5B,6B) become

$$q_1(t) = \cos 3t + \frac{46}{21} \sin 3t - \frac{1}{7} \sin 4t \quad (5C)$$

$$q_2(t) = 2 \cos 6t - \frac{16}{15} \sin 6t + \frac{1}{10} \sin 4t \quad (6C)$$

The above is the solution in principle coordinates. Now we transform it back to normal coordinates.

Since  $\{x\} = [u] \{q\}$ , then

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \cos(3t) + \left(\frac{46}{21}\right) \sin(3t) - \left(\frac{1}{7}\right) \sin(4t) \\ 2 \cos(6t) - \left(\frac{16}{15}\right) \sin(6t) + \frac{1}{10} \sin(4t) \end{Bmatrix} \\ &= \begin{Bmatrix} \cos 3t + 2 \cos 6t + \frac{46}{21} \sin 3t - \frac{3}{70} \sin 4t - \frac{16}{15} \sin 6t \\ \cos 3t - \cos 6t + \frac{46}{21} \sin 3t - \frac{27}{140} \sin 4t + \frac{8}{15} \sin 6t \end{Bmatrix} \end{aligned}$$

The above is the final solution. Here is a plot of  $x_1(t), x_2(t)$

