Solving slide 390 example, ME 440 Intermediate Vibration, Fall 2017

Nasser M. Abbasi

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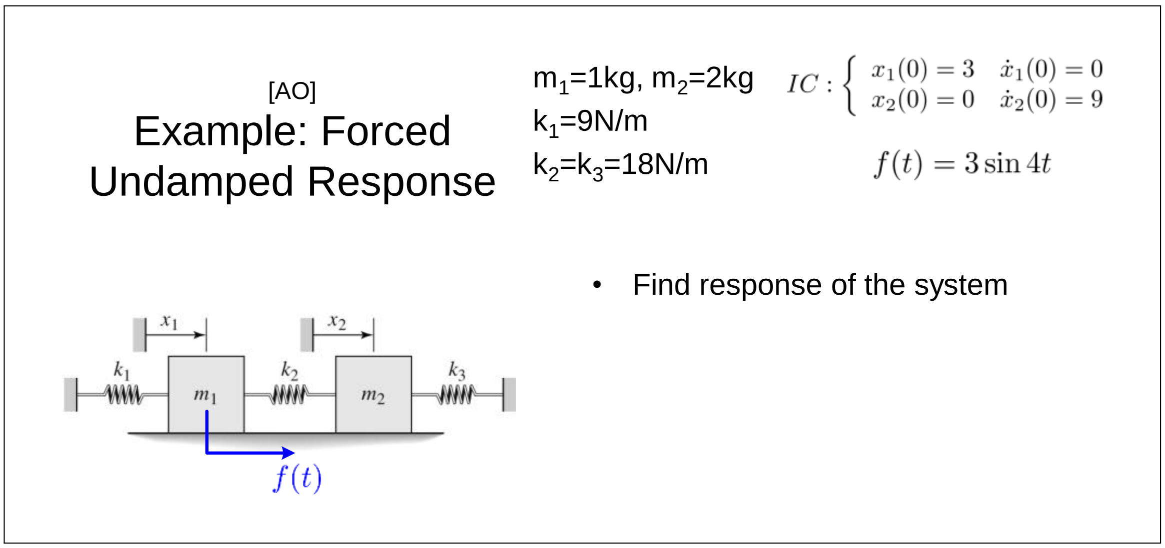

By inspection

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix}$$

And

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The system is

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{1}$$

The above is solved using modal analysis in order to decouple the system. The first step is to determine the eigenvalues.

$$
\begin{aligned}
[A] &= [m]^{-1}[k] \\
&= \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}27 & -18\\ -18 & 36\end{bmatrix} \\
&= \begin{bmatrix}1 & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}27 & -18\\ -18 & 36\end{bmatrix} \\
&= \begin{bmatrix}27 & -18\\ -9 & 18\end{bmatrix}
\end{aligned}
$$

To find the eigenvalues of $[A]$ we solve $|A-\lambda I| = 0$ or

$$
\begin{vmatrix}27-\lambda & -18\\ -9 & 18-\lambda\end{vmatrix} = 0
$$

$$
\lambda^2 - 45\lambda + 324 = 0
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 9 \\
\lambda_2 &= 36
\end{aligned}
$$

Which implies

$$
\begin{aligned}
\omega_{n(1)} &= 3 \text{ rad/s} \\
\omega_{n(2)} &= 9 \text{ rad/s}
\end{aligned}
$$

Now we find the eigenvectors u_i or the shape vectors. For $\underline{\lambda_1 = 9}$

$$
\begin{aligned}
[A]\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} &= \lambda_1\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} \\
\begin{bmatrix}27 & -18\\ -9 & 18\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} &= 9\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} \\
\begin{bmatrix}27u_1 - 18u_2\\ -9u_1 + 18u_2\end{bmatrix} &= \begin{Bmatrix}9u_1\\ 9u_2\end{Bmatrix}
\end{aligned}
$$

Using first equation only gives

$$
27u_1 - 18u_2 = 9u_1
$$

We always normalized to $u_1 = 1$, hence the above gives

$$
\begin{aligned}
27 - 18u_2 &= 9 \\
u_2 &= 1
\end{aligned}
$$

Therefore the first eigenvector is

$$
\vec{u}_1 = \begin{Bmatrix}1\\ 1\end{Bmatrix}
$$

To find the second eigenvector. For $\underline{\lambda_2 = 36}$

$$[A]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \lambda_2 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27 & -18 \\ -9 & 18 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 36 \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{bmatrix} 27u_1 - 18u_2 \\ -9u_1 + 18u_2 \end{bmatrix} = \begin{Bmatrix} 36u_1 \\ 35u_2 \end{Bmatrix}$$

Using first equation only gives

$$27u_1 - 18u_2 = 36u_1$$

We always normalized to $u_1 = 1$, hence the above gives

$$\begin{aligned} 27 - 18u_2 &= 36 \\ u_2 &= -\frac{1}{2} \end{aligned}$$

Therefore the second eigenvector is

$$\vec{u}_2 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \end{Bmatrix}$$

Hence the $\underline{\text{modal matrix}}$ is

$$[u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}$$

Using the modal matrix, we can now decouple the original system given above in (1) which is

$$[m]\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + [k]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{2}$$

Let $\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = [u]\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$, then the above becomes

$$[m][u]\begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [k][u]\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix}$$

Premultiplying both sides by $[u]^T$ gives

$$[u]^T[m][u]\begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + [u]^T[k][u]\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [u]^T\begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix} \tag{4}$$

But

$$[u]^T[m][u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

And

$$[u]^T [k] [u] = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^T \begin{bmatrix} 27 & -18 \\ -18 & 36 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix}$$

Then (4) becomes

$$\begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 54 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} 3\sin 4t \\ 3\sin 4t \end{Bmatrix}$$

Hence we obtain 2 ODEs

$$3\ddot{q}_1(t) + 27q_1(t) = 3\sin 4t$$
$$\frac{3}{2}\ddot{q}_2(t) + 54q_2(t) = 3\sin 4t$$

Or

$$\ddot{q}_1(t) + 9q_1(t) = \sin 4t \tag{5}$$
$$\ddot{q}_2(t) + 36q_2(t) = 2\sin 4t \tag{6}$$

Note There is a short cut to obtain the above (5,6) equations directly as follows. Starting with (2), we just write

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} \omega_{n(1)}^2 & 0 \\ 0 & \omega_{n(2)}^2 \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = [u]^{-1} [m]^{-1} \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \ddot{q}_1(t) + 9q_1(t) \\ \ddot{q}_2(t) + +36q_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{Bmatrix} 3\sin 4t \\ 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} \sin 4t \\ 2\sin 4t \end{Bmatrix}$$

Which is the same as (5,6). This short cut just needs finding $[u]^{-1} [m]^{-1}$. Use this short cut for the exam.

Solving (5)

The homogeneous solution is

$$q_{1,h}(t) = A_1 \cos 3t + B_1 \sin 3t$$

And to find the particular solution, we guess $q_{1,p} = C\sin 4t$, hence $\dot{q}_{1,p} = 4C\cos 4t$ and $\ddot{q}_{1,p} = -16C\sin 4t$. Plug-in in (5) gives

$$-16C\sin 4t + 9(C\sin 4t) = \sin 4t$$
$$-7C_1 \sin 4t = \sin 4t$$
$$C_1 = -\frac{1}{7}$$

Hence $q_{1,p} = -\frac{1}{7}\sin 4t$ and the complete solution is

$$q_1(t) = A_1\cos 3t + B_1 \sin 3t - \frac{1}{7}\sin 4t$$

Now we do the same to solve (6)

The homogeneous solution is

$$q_{2,h}(t) = A_2 \cos 6t + B_2 \sin 6t$$

And to find the particular solution, we guess $q_{2,p} = C \sin 4t$, hence $\dot{q}_{2,p} = 4C \cos 4t$ and $\ddot{q}_{2,p} = -16C \sin 4t$. Plug-in in (6) gives

$$\begin{aligned} -16C \sin 4t + 36 (C \sin 4t) &= 2 \sin 4t \\ 20C_1 \sin 4t &= 2 \sin 4t \\ C_1 &= \frac{1}{10} \end{aligned}$$

Hence $q_{2,p} = \frac{1}{10} \sin 4t$ and the complete solution is

$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t$$

Therefore the solution in principle coordinates is

$$q_1(t) = A_1 \cos 3t + B_1 \sin 3t - \frac{1}{7} \sin 4t \tag{5A}$$

$$q_2(t) = A_2 \cos 6t + B_2 \sin 6t + \frac{1}{10} \sin 4t \tag{6A}$$

Since $\{x\} = [u]\{q\}$, then $\{q\} = [u]^{-1}\{x\}$. Therefore

$$\begin{aligned} \{q(0)\} &= [u]^{-1}\{x(0)\} \\ \begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{Bmatrix} 3 \\ 0 \end{Bmatrix} \\ &= \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \end{aligned}$$

And

$$\begin{aligned} \{\dot{q}(0)\} &= [u]^{-1}\{\dot{x}(0)\} \\ \begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{Bmatrix} &= \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix} \begin{Bmatrix} 0 \\ 9 \end{Bmatrix} \\ &= \begin{Bmatrix} 6 \\ -6 \end{Bmatrix} \end{aligned}$$

Applying first initial conditions to (5A,6A) gives

$$\begin{aligned} 1 &= A_1 \\ 2 &= A_2 \end{aligned}$$

Hence (5A,6A) becomes

$$q_1(t) = \cos 3t + B_1 \sin 3t - \frac{1}{7}\sin 4t \tag{5B}$$

$$q_2(t) = 2\cos 6t + B_2 \sin 6t + \frac{1}{10}\sin 4t \tag{6B}$$

Taking derivatives

$$\dot{q}_1(t) = -3\sin 3t + 3B_1\cos 3t - \frac{4}{7}\cos 4t$$

$$\dot{q}_2(t) = -12\sin 6t + 6B_2\cos 6t + \frac{4}{10}\cos 4t$$

Applying the second initial conditions to the above gives

$$6 = 3B_1 - \frac{4}{7}$$

$$-6 = 6B_2 + \frac{4}{10}$$

Solving gives $B_1 = \frac{46}{21}, B_2 = -\frac{16}{15}$. Hence (5B,6B) become

$$q_1(t) = \cos 3t + \frac{46}{21}\sin 3t - \frac{1}{7}\sin 4t \tag{5C}$$

$$q_2(t) = 2\cos 6t - \frac{16}{15}\sin 6t + \frac{1}{10}\sin 4t \tag{6C}$$

The above is the solution in principle coordinates. Now we transform it back to normal coordinates. Since $\{x\} = [u]\{q\}$, then

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \cos(3t) + \left(\frac{46}{21}\right)\sin(3t) - \left(\frac{1}{7}\right)\sin(4t) \\ 2\cos(6t) - \left(\frac{16}{15}\right)\sin(6t) + \frac{1}{10}\sin 4t \end{Bmatrix}$$

$$= \begin{Bmatrix} \cos 3t + 2\cos 6t + \frac{46}{21}\sin 3t - \frac{3}{70}\sin 4t - \frac{16}{15}\sin 6t \\ \cos 3t - \cos 6t + \frac{46}{21}\sin 3t - \frac{27}{140}\sin 4t + \frac{8}{15}\sin 6t \end{Bmatrix}$$

The above is the final solution. Here is a plot of $x_1(t), x_2(t)$

```
mySol = {x1[t] → Cos[3 t] + 2 Cos[6 t] + 46 / 21 Sin[3 t] - 3 / 70 Sin[4 t] - 16 / 15 Sin[6 t],
   x2[t] → Cos[3 t] - Cos[6 t] + 46 / 21 Sin[3 t] - 27 / 140 Sin[4 t] + 8 / 15 Sin[6 t]};

Plot[{ x1[t] /. mySol, x2[t] /. mySol}, {t, 0, 2}, Frame → True,
 GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"x1(t),x2(t)", None}, {"time (sec)", "Solution to slide 390"}},
 PlotLegends → {"x1(t)", "x2(t)"}, ImageSize → 400, BaseStyle → 14]
```

Out[30]=