

Solving slide 148 example, ME 440 Intermediate Vibration, Fall 2017

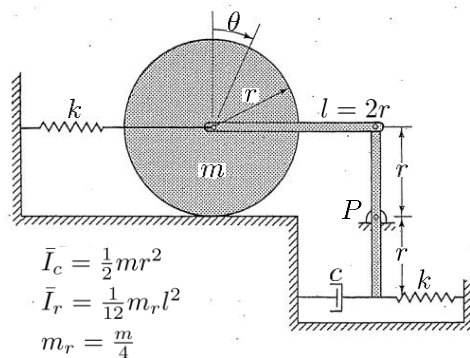
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Example, Deriving EOM

- Cylinder of radius r rolls without slip. Mass of each rod is $m_r = m/4$
- Assume small oscillation and ignore the very small rotational effect of the horizontal bar

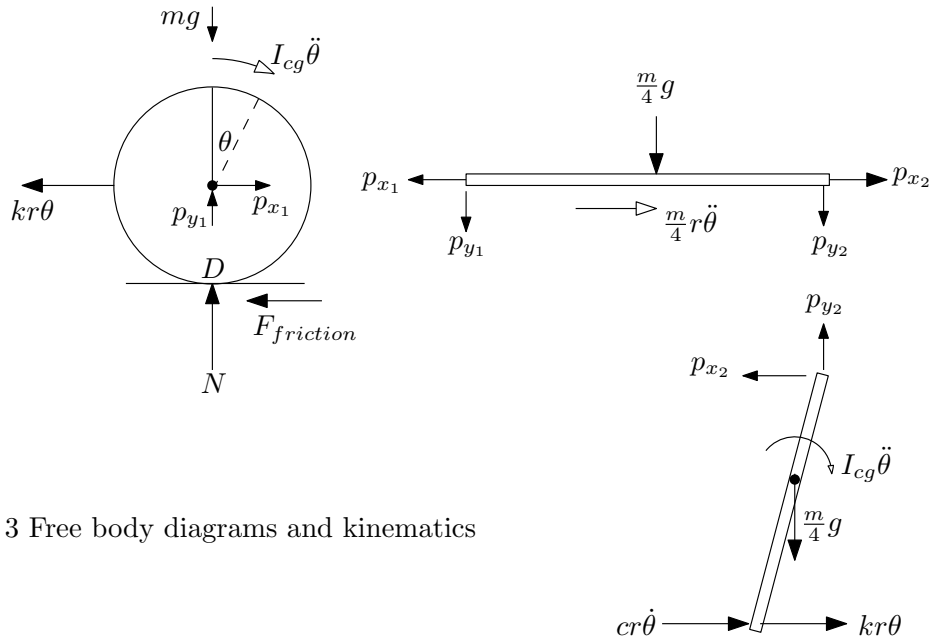


- For this system:
 - Derive EOM
 - Show that the model's natural frequency and damping ratio are

$$\omega_n = \sqrt{\frac{12k}{11m}} \quad \zeta = \frac{3c}{\sqrt{132km}}$$

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We will solve this using 3 separate bodies. So there are three free body diagrams as shown below



3 Free body diagrams and kinematics

In this diagram, it is assumed the horizontal bar only moves in the x direction and this is all for small angle θ . Now we apply Newton laws to each body.

For disk, we apply $\tau = I_o\ddot{\theta}$ but using the point D on the figure to take moments around in order to get rid of the friction F and N terms. This gives (using counter clock wise as positive)

$$\begin{aligned}
 (kr\theta)r - p_{x1}r &= -I_o\ddot{\theta} \\
 kr^2\theta - p_{x1}r &= -(I_{cg} + mr^2)\ddot{\theta} \\
 &= -\left(\frac{1}{2}mr^2 + mr^2\right)\ddot{\theta} \\
 &= -\frac{3}{2}mr^2\ddot{\theta}
 \end{aligned} \tag{1}$$

We now move to the second body, which is the horizontal bar.

$$\begin{aligned}
 \sum F_x &= m_{bar}\ddot{x} \\
 -p_{x1} + p_{x2} &= \frac{m}{4}r\ddot{\theta}
 \end{aligned} \tag{2}$$

From (2) we solve for p_{x1} and plug it into (1)

$$p_{x1} = p_{x2} - \frac{m}{4}r\ddot{\theta}$$

Hence (1) now becomes

$$\begin{aligned}
 kr^2\theta - \left(p_{x_2} - \frac{m}{4}r\dot{\theta}\right)r &= -\frac{3}{2}mr^2\ddot{\theta} \\
 kr^2\theta - p_{x_2}r &= -\left(\frac{3}{2}mr^2 + \frac{m}{4}r^2\right)\ddot{\theta} \\
 &= -\frac{7}{4}mr^2\ddot{\theta}
 \end{aligned} \tag{3}$$

To find p_{x_2} , we use the third body, the vertical bar. Taking moments about C.G. of bar using counter clock wise as positive gives

$$\begin{aligned}
 \tau &= -I_{CG}\ddot{\theta} \\
 (kr\theta)r \cos \theta + (cr\dot{\theta})r \cos \theta + p_{x_2}r \cos \theta + p_{y_2}r \sin \theta &= -\frac{1}{12}\left(\frac{m}{4}\right)(2r)^2\ddot{\theta} \\
 &= -\frac{1}{12}mr^2\ddot{\theta}
 \end{aligned}$$

For small angle the above becomes

$$kr^2\theta + cr^2\dot{\theta} + p_{x_2}r + p_{y_2}r\theta = -\frac{m}{12}r^2\ddot{\theta} \tag{4}$$

p_{y_2} is now found from vertical balance of horizontal bar. Since it does not move vertically and assumed to only move horizontally, then

$$\begin{aligned}
 \sum F_y &= 0 \\
 -p_{y_1} - p_{y_2} - \frac{m}{4}g &= 0
 \end{aligned}$$

Due to symmetry, $p_{y_1} = p_{y_2}$ and the above becomes

$$\begin{aligned}
 -2p_{y_2} &= \frac{m}{4}g \\
 p_{y_2} &= -\frac{m}{8}g
 \end{aligned}$$

Plugging this value for p_{y_2} into (4) and solving for p_{x_2} gives

$$\begin{aligned}
 kr^2\theta + cr^2\dot{\theta} + p_{x_2}r - \frac{m}{8}gr\theta &= -\frac{m}{12}r^2\ddot{\theta} \\
 p_{x_2} &= \frac{1}{r}\left(-\frac{m}{12}r^2\ddot{\theta} + \frac{m}{8}gr\theta - kr^2\theta - cr^2\dot{\theta}\right)
 \end{aligned}$$

Plugging the above into (3) gives the equation of motion for disk

$$\begin{aligned}
 kr^2\theta - \left(-\frac{m}{12}r^2\ddot{\theta} + \frac{m}{8}gr\theta - kr^2\theta - cr^2\dot{\theta}\right) &= -\frac{7}{4}mr^2\ddot{\theta} \\
 kr^2\theta + \frac{m}{12}r^2\ddot{\theta} - \frac{m}{8}gr\theta + kr^2\theta + cr^2\dot{\theta} &= -\frac{7}{4}mr^2\ddot{\theta} \\
 \theta\left(2kr^2 - \frac{m}{8}gr\right) + cr^2\dot{\theta} &= -\frac{7}{4}mr^2\ddot{\theta} - \frac{m}{12}r^2\ddot{\theta} \\
 \frac{11}{6}mr^2\ddot{\theta} + cr^2\dot{\theta} + \theta\left(2kr^2 - \frac{m}{8}gr\right) &= 0
 \end{aligned}$$

Or

$$\ddot{\theta} + \frac{6c}{11m}\dot{\theta} + \theta\left(\frac{12k}{11m} - \frac{3g}{44r}\right) = 0$$

Writing the above in the standard form $\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$ we see that

$$\omega_n^2 = \sqrt{\frac{12k}{11m} - \frac{3g}{44r}}$$

And

$$\begin{aligned} 2\zeta\omega_n &= \frac{6c}{11m} \\ \zeta &= \frac{3c}{11m\omega_n} \\ &= \frac{3c}{11m\sqrt{\frac{12k}{11m} - \frac{3g}{44r}}} \\ &= \frac{3c}{\sqrt{132km - \frac{363gm^2}{44r}}} \end{aligned}$$