my cheat sheet, ME 440 Intermediate Vibration, Fall 2017

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December 30, 2019

0.1 Solution to undamped forced harmonic

0.1.1 Input is $F_0 \cos\omega t$

$$m\ddot{x} + kx = F_0 \cos\omega t$$

This model is single degree of freedom system, undamped, with forced harmonice input. Its solution is given by

$$x(t) = x_h(t) + x_p(t)$$

Where $x_p(t)$ is particular solution and $x_h(t)$ is homogenous solution. We know that

$$x_h(t) = c_1 \cos\omega_n t + c_2 \sin\omega_n t$$

And assuming $x_p(t) = X\cos\omega t$ for the case $\underline{\omega \neq \omega_n}$ Pluggin this into the ODE, we find that

$$X = \frac{x_{st}}{1-r^2}$$

Where $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k_{eq}}$ the static deflection. Hence the solution becomes

$$x(t) = \overbrace{c_1 \cos\omega_n t + c_2 \sin\omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1-r^2}\cos\omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, then (1) at $t = 0$ becomes

$$x_0 = c_1 + \frac{x_{st}}{1-r^2}$$
$$c_1 = x_0 - \frac{x_{st}}{1-r^2}$$

Hence solution (1) now becomes

$$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right)\cos\omega_n t + c_2 \sin\omega_n t + \frac{x_{st}}{1-r^2}\cos\omega t$$

Taking derivative

$$\dot{x}(t) = -\omega_n\left(x_0 - \frac{x_{st}}{1-r^2}\right)\sin\omega_n t + c_2\omega_n \cos\omega_n t - \omega\frac{x_{st}}{1-r^2}\sin\omega t$$

At $t = 0$ the above becomes

$$\dot{x}_0 = c_2\omega_n$$
$$c_2 = \frac{\dot{x}_0}{\omega_n}$$

Therefore the solution now becomes (again, this is for $\omega \neq \omega_n$)

$$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right)\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{x_{st}}{1-r^2}\cos\omega t \tag{2}$$

0.1.2 Input is $F_0 \sin\omega t$

$$m\ddot{x} + kx = F_0 \sin\omega t$$

This model is single degree of freedom system, undamped, with forced harmonice input.

Its solution is given by

$$x(t) = x_h(t) + x_p(t)$$

Where $x_p(t)$ is particular solution and $x_h(t)$ is homogenous solution. We know that

$$x_h(t) = c_1 \cos \omega_n t + c_2 \sin \omega_n t$$

And assuming $x_p(t) = X \sin \omega t$ for the case $\underline{\omega \neq \omega_n}$ Pluggin this into the ODE, we find that

$$X = \frac{x_{st}}{1 - r^2}$$

Where $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k_{eq}}$ the static deflection. Hence the solution becomes

$$x(t) = \overbrace{c_1 \cos \omega_n t + c_2 \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1 - r^2} \sin \omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, then (1) at $t = 0$ becomes

$$x_0 = c_1$$

Hence solution (1) now becomes

$$x(t) = x_0 \cos \omega_n t + c_2 \sin \omega_n t + \frac{x_{st}}{1 - r^2} \sin \omega t$$

Taking derivative

$$\dot{x}(t) = -x_0 \sin \omega_n t + c_2 \omega_n \cos \omega_n t + \omega \frac{x_{st}}{1 - r^2} \cos \omega t$$

At $t = 0$ the above becomes

$$\begin{aligned} \dot{x}_0 &= c_2 \omega_n + \omega \frac{x_{st}}{1 - r^2} \\ c_2 &= \frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{x_{st}}{1 - r^2} \\ &= \frac{\dot{x}_0}{\omega_n} - \frac{r}{1 - r^2} x_{st} \end{aligned}$$

Therefore the solution now becomes (again, this is for $\omega \neq \omega_n$)

$$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1 - r^2} x_{st} \right) \sin \omega_n t + \frac{x_{st}}{1 - r^2} \sin \omega t \tag{2}$$

Notice the difference in the solution. Here is summary

ODE	solution
$m\ddot{x} + kx = F_0 \cos \omega t$	$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \overbrace{\frac{x_{st}}{1-r^2} \cos \omega t}^{x_p}$
$m\ddot{x} + kx = F_0 \sin \omega t$	$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st} \right) \sin \omega_n t + \overbrace{\frac{x_{st}}{1-r^2} \sin \omega t}^{x_p}$

0.2 Solution to underdamped forced harmonic

ODE	particular solution only
$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2}$	$x_p(t) = \frac{a_0}{2} \frac{1}{k}$
$m\ddot{x} + c\dot{x} + kx = a_n \cos(n\omega t)$	$x_p(t) = \frac{a_n}{k} \frac{1}{\sqrt{\left(1-(nr)^2\right)^2 + (2\zeta nr)^2}} \cos\left(n\omega t - \phi_n\right)$
$m\ddot{x} + c\dot{x} + kx = b_n \sin(n\omega t)$	$x_p(t) = \frac{b_n}{k} \frac{1}{\sqrt{\left(1-(nr)^2\right)^2 + (2\zeta nr)^2}} \sin\left(n\omega t - \phi_n\right)$

Where

$$\begin{aligned} r &= \frac{\omega}{\omega_n} \\ \phi_n &= \tan^{-1} \left(\frac{2\zeta nr}{1 - (nr)^2} \right) \end{aligned}$$

0.3 unit Impulse respones

For undamped system $m\ddot{x}+kx=\delta(t)$ the response (solution) is (notes calls these $g(t)$)

$$g(t)=\frac{1}{m\omega_n}\sin(\omega_n t)$$

And for an underdamped $m\ddot{x}+c\dot{x}+kx=\delta(t)$ the response is

$$g(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$$

0.4 Duhamel Integral

For arbitray forcing function $F(t)$ which can be of any forum, the response of the system to $F(t)$, assuming the system was at rest is

$$x_{conv}(t)=\int_0^t F(\tau)\,g(t-\tau)\,d\tau$$

0.4.1 Some definitions

DLF Dynamic locad factor. $DLF=\frac{x(t)}{x_{st}}$. But we really only care for the maximum DLF. When the input is constant (step input), the $DLF_{\max}=2$.

Response spectrum Plots the $\text{DLF}_{\max}$ on the y axis vs $\frac{t}{T}$ where T is the period of the system on the x axis. This is done for typical inputs such as unit step, triangle, half sine, etc...