HW 4, ME 440 Intermediate Vibration, Fall 2017

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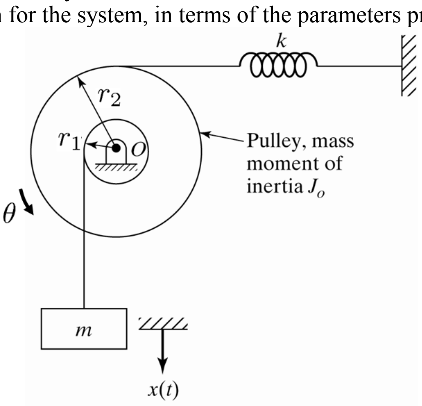
December 30, 2019

0.1 Problem 1

Problem 1

The pulley is in fixed axis rotation about Point O. Using energy concepts and θ as the generalized coordinate, determine
a) the natural frequency of the system shown below, and
b) the equation of motion for the system, in terms of the parameters provided.



0.1.1 Part a

Using Rayleigh method, we need to find $T_{\max}$ and $U_{\max}$ where T is the kinetic energy of the system and U is the potential energy and then solve for ω_n by setting $T_{\max} = U_{\max}$.

Kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_o\dot{\theta}^2$$

But $x = r_1\theta$, therefore $\dot{x} = r_1\dot{\theta}$ and the above becomes

$$T = \frac{1}{2}m\left(r_1\dot{\theta}\right)^2 + \frac{1}{2}J_o\dot{\theta}^2 \tag{1}$$

And potential energy only comes from the spring, since we assume x is measured from static equilibrium. Hence

$$\begin{aligned} U &= \frac{1}{2}kx^2 \\ &= \frac{1}{2}k\left(r_2\theta\right)^2 \end{aligned} \tag{2}$$

To get ω_n into (1) and (2), we now assume that motion is harmonic, hence $\theta = \theta_{\max}\sin\left(\omega_n t\right)$, Therefore $\dot{\theta} = \theta_{\max}\omega_n\cos\left(\omega_n t\right)$ and rewriting (1,2) using these expressions results in

$$\begin{aligned} T &= \frac{1}{2}m\left(r_1\theta_{\max}\omega_n\cos\left(\omega_n t\right)\right)^2 + \frac{1}{2}J_o\left(\theta_{\max}\omega_n\cos\left(\omega_n t\right)\right)^2 \\ U &= \frac{1}{2}k\left(r_2\left(\theta_{\max}\sin\left(\omega_n t\right)\right)\right)^2 \end{aligned}$$

Hence, maximum is when $\theta=\theta_{\max}$ and $\dot{\theta}=\theta_{\max}\omega_n$ and the above becomes

$$
\begin{aligned}
T_{\max} &= \frac{1}{2}mr_1^2\theta_{\max}^2\omega_n^2+\frac{1}{2}J_o\theta_{\max}^2\omega_n^2\\
U_{\max} &= \frac{1}{2}kr_2^2\theta_{\max}^2
\end{aligned}
$$

Now

$$
\begin{aligned}
T_{\max} &= U_{\max}\\
\frac{1}{2}mr_1^2\theta_{\max}^2\omega_n^2+\frac{1}{2}J_o\theta_{\max}^2\omega_n^2 &= \frac{1}{2}kr_2^2\theta_{\max}\\
mr_1^2\omega_n^2+J_o\omega_n^2 &= kr_2^2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\omega_n^2 &= \frac{kr_2^2}{mr_1^2+J_o}\\
\omega_n &= \sqrt{\frac{kr_2^2}{mr_1^2+J_o}}
\end{aligned}
$$

0.1.2 Part b

The equation of motion is given by

$$
\frac{d}{dt}(T+U)=0
$$

We found T,U in part (a), therefore the above becomes

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}m\left(r_1\dot{\theta}\right)^2+\frac{1}{2}J_o\dot{\theta}^2+\frac{1}{2}k\left(r_2\theta\right)^2\right) &= 0\\
mr_1^2\dot{\theta}\ddot{\theta}+J_o\dot{\theta}\ddot{\theta}+kr_2^2\theta\dot{\theta} &= 0
\end{aligned}
$$

For non trivial motion $\dot{\theta}\neq 0$ for all time, hence we can divide throughout by $\dot{\theta}$ and obtain

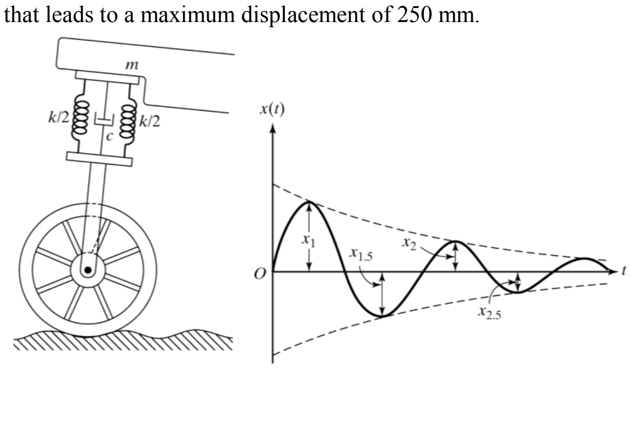
$$
\begin{aligned}
mr_1^2\ddot{\theta}+J_o\ddot{\theta}+kr_2^2\theta &= 0\\
\ddot{\theta}\left(mr_1^2+J_o\right)+kr_2^2\theta &= 0\\
\ddot{\theta}+\frac{kr_2^2}{mr_1^2+J_o}\theta &= 0
\end{aligned}
$$

The above is the equation of motion.

0.2 Problem 2

Problem 2

An underdamped shock absorber is to be designed for motorcycle of mass 200 kg. When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as illustrated below. Determine the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 seconds and the amplitude x_1 is to be reduced to ¼ in one half cycle (i.e., $x_{1.5}=x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.



First part

The first step is to determine damping ratio ζ. This is done using logarithmic decrement.

Since $X_{1.5} = \frac{1}{4}X_1$ and $X_2 = \frac{1}{4}X_{1.5}$ then

$$
\begin{aligned}
X_2 &= \frac{1}{4}\left(\frac{1}{4}X_1\right) \\
&= \frac{1}{16}X_1
\end{aligned}
$$

Using

$$\frac{X_1}{X_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+t_2)}}$$

Where $t_2 = t_1 + \tau_d$ and τ_d is damped period. Therefore the above becomes

$$
\begin{aligned}
\frac{X_1}{\frac{1}{16}X_1} &= \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n t_1}e^{-\zeta\omega_n\tau_d}} = e^{\zeta\omega_n\tau_d} \\
\ln(16) &= \zeta\omega_n\tau_d
\end{aligned}
$$

Taking log of both sides gives

$$\ln(16) = \zeta\omega_n\tau_d \tag{1}$$

But

$$
\begin{aligned}
\tau_d &= \frac{2\pi}{\omega_d} \\
&= \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}
\end{aligned}
$$

And (1) simplifies to

$$
\begin{aligned}
\ln(16) &= \zeta\omega_n\frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} \\
2.7726 &= \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}
\end{aligned}
$$

Squaring both sides and solving for ζ gives

$$
\begin{aligned}
(2.7726)^2\left(1-\zeta^2\right) &= 4\pi^2\zeta^2 \\
\zeta^2\left(4\pi^2+7.6873\right) &= 7.6873 \\
\zeta^2 &= \frac{7.6873}{4\pi^2+7.6873}
\end{aligned}
$$

Taking the positive root results in

$$
\begin{aligned}
\zeta &= \sqrt{\frac{7.6873}{4\pi^2+7.6873}} \\
&= 0.40371
\end{aligned}
$$

Now that ζ is know, ω_n can be found, since we are told that $\tau_d = 2$ seconds. Using

$$\tau_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Then solving for ω_n from the above gives

$$
\begin{aligned}
2 &= \frac{2\pi}{\omega_n\sqrt{1-0.40371^2}} \\
\omega_n &= \frac{\pi}{\sqrt{1-0.40371^2}} \\
&= 3.4339 \text{ rad/sec}
\end{aligned}
$$

Now we are ready to find the stiffness coefficient k and damping coefficient c. Using

$$\zeta = \frac{c}{2\omega_n m}$$

Then

$$\begin{aligned} c &= 2\zeta\omega_n m \\ &= 2(0.40371)(3.4339)(200) \\ &= 554.52 \text{ N-s/m} \end{aligned}$$

But since

$$\omega_n^2 = \frac{k}{m}$$

Then k is now found

$$\begin{aligned} k &= \omega_n^2 m \\ &= (3.4339)^2(200) \\ &= 2358.3 \text{ N/m} \end{aligned}$$

Second part

Maximum displacement occurs at time t_1 as given by (from textbook)

$$\sin\omega_d t_1 = \sqrt{1-\zeta^2}$$

Hence

$$\begin{aligned} \omega_d t_1 &= \arcsin\left(\sqrt{1-\zeta^2}\right) \\ t_1 &= \frac{1}{\omega_n\sqrt{1-\zeta^2}}\arcsin\left(\sqrt{1-\zeta^2}\right) \\ &= \frac{1}{3.4339\sqrt{1-0.40371^2}}\arcsin\left(\sqrt{1-0.40371^2}\right) \\ &= 0.36772 \text{ sec} \end{aligned}$$

Since

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t) \tag{2}$$

Then at maximum displacement, where $x = 0.25$ m, the above becomes

$$\begin{aligned} x_{\max}(t_1) &= Xe^{-\zeta\omega_n t_1}\sin(\omega_d t_1) \\ \frac{x_{\max}e^{\zeta\omega_n t_1}}{\sin(\omega_d t_1)} &= X \end{aligned}$$

Plug-in numerical values to solve for maximum displacement X gives

$$\begin{aligned} X &= \frac{0.25\exp(0.40371\times 3.4339\times 0.36772)}{\sin\left(\left(3.4339\sqrt{1-0.40371^2}\right)(0.36772)\right)} \\ &= 0.45495 \text{ m} \end{aligned}$$

From (2), the velocity is found

$$\begin{aligned} \dot{x}(t) &= -\zeta\omega_n Xe^{-\zeta\omega_n t}\sin(\omega_d t) + Xe^{-\zeta\omega_n t}\omega_d\cos(\omega_d t) \\ &= Xe^{-\zeta\omega_n t}(\omega_d\cos(\omega_d t) - \zeta\omega_n\sin(\omega_d t)) \end{aligned}$$

At $t = 0$ the above gives

$$\begin{aligned} \dot{x}(0) &= X\omega_d \\ &= X\left(\omega_n\sqrt{1-\zeta^2}\right) \end{aligned}$$

Plug-in in numerical values

$$\begin{aligned} \dot{x}(0) &= 0.45495\left(3.4339\sqrt{1-0.40371^2}\right) \\ &= 1.4293 \text{ m/s} \end{aligned}$$