HW 3, ME 440 Intermediate Vibration, Fall 2017

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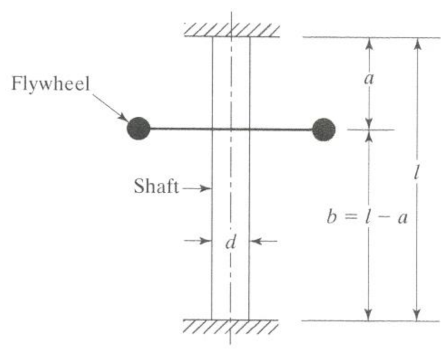
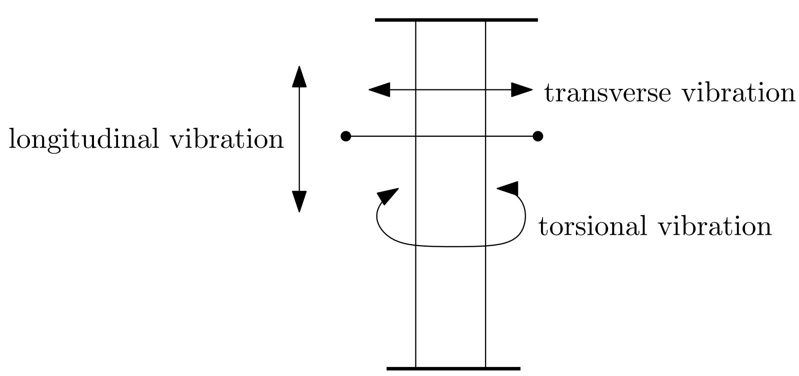
0.1 Problem 1

Problem 1

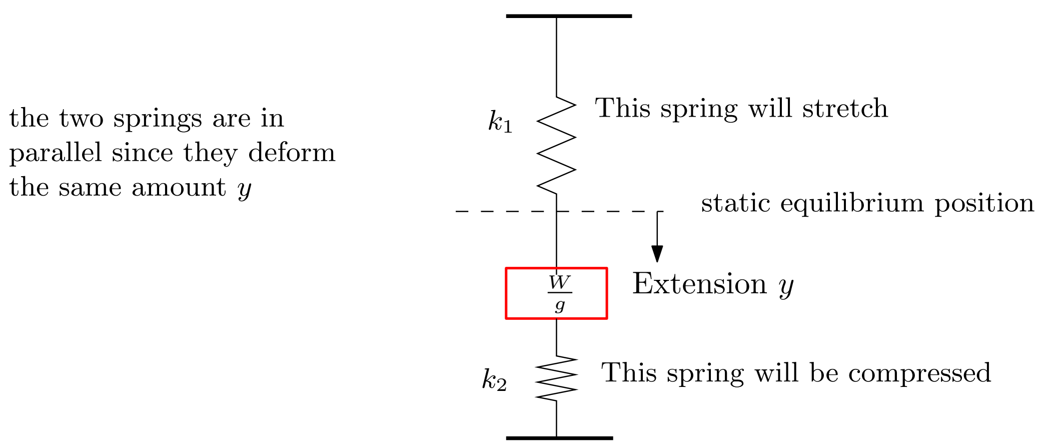
A flywheel is mounted on a vertical shaft, as shown below. The shaft has a diameter d and length l and is fixed at both ends. The flywheel has a weight of W and a radius of gyration of r. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.



We need to find the natural frequency of vibration for the following cases



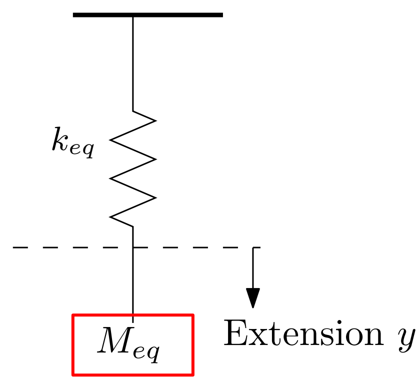
<u>longitudinal</u> In this mode the system can be modeled as the following



Since both springs are in parallel, then the equivalent spring stiffness is

$$k_{eq} = k_1 + k_2$$

The equivalent mass is just the mass of the flywheel $\frac{W}{g}$. Hence the overall system can now be modeled as follows



Which has the equation of motion

$$m_{eq}\ddot{y} + k_{eq}y = 0$$

$$\ddot{y} + \frac{k_{eq}}{m_{eq}}y = 0$$

Therefore

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

We now just need to determine $k_{eq} = k_1 + k_2$. But from mechanics of materials we know that $k_1 = \frac{AE}{a}$ and $k_2 = \frac{AE}{b}$. Therefore the above becomes

$$\omega_n = \sqrt{\frac{\frac{AE}{a} + \frac{AE}{b}}{\frac{W}{g}}}$$

$$= \sqrt{\frac{gAE}{W}\left(\frac{1}{a} + \frac{1}{b}\right)}$$

Transverse In this mode the system can be modeled as beam with fixed ends with load W at distance a from one end and distance b from the other end. From tables, the stiffness coefficient in this case is given by

$$k_{eq} = 3EI\left(\frac{L}{ab}\right)^3$$

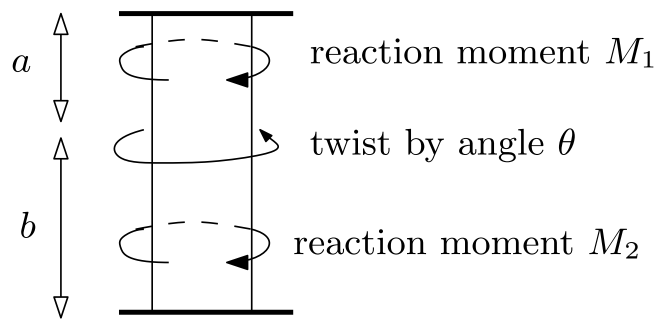
The equivalent mass remains as before which is just the mass of the flywheel $\frac{W}{g}$. Therefore, as above we find the natural frequency as

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

Or

$$\boxed{\omega_n = \sqrt{\frac{3gEI}{W}\left(\frac{L}{ab}\right)^3}}$$

<u>Torsional</u> In this mode, the flywheel is twisted by some degree θ, and therefore the top part of the beam and the bottom part of the beam will resist this twist by applying moment against the twist as shown in this diagram



From mechanics of materials, there is relation between the twisting angle and resisting torque by beam which is given by

$$M = \frac{GJ}{L}\theta$$

Where here θ is the twist angle (radians) and M is the torque (Nm) and L is length of beam and G is modulus of rigidity (N per m^2) and J is the second moment of area of the cross section (m^4) about its center. Therefore total moments is

$$\begin{aligned} M_1 + M_2 &= \frac{GJ}{a}\theta + \frac{GJ}{a}\theta \\ &= GJ\theta\left(\frac{1}{a} + \frac{1}{b}\right) \end{aligned}$$

Comparing the above to definition of stiffness which is $F = K\Delta$ but in this problem $\Delta \equiv \theta$ and $F \equiv (M_1 + M_2)$, then we see that the equivalent stiffness is

$$k_{eq} = GJ\left(\frac{1}{a} + \frac{1}{b}\right)$$

We now need the equivalent mass. In this case it is the <u>mass moment of inertia</u> of flywheel. We are given that radius of gyration is r, hence

$$m_{eq} = \frac{W}{g}r^2$$

We now have all the pieces needed to find ω_n

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_{eq}}{m_{eq}}} \\ &= \sqrt{\frac{GJ\left(\frac{1}{a} + \frac{1}{b}\right)}{\frac{W}{g}r^2}} \\ &= \sqrt{\frac{gGJ}{Wr^2}\left(\frac{1}{a} + \frac{1}{b}\right)} \end{aligned}$$

From tables, for circular bar of radius d, we see that $J = \frac{\pi}{32} d^4$. Hence the above becomes

$$\boxed{\omega_n = \sqrt{\frac{gG\pi d^4}{32Wr^2}\left(\frac{1}{a} + \frac{1}{b}\right)}}$$

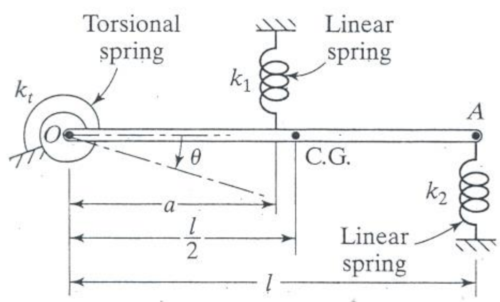
<u>Summary of results</u>

case	ω_n
longitudinal	$\sqrt{\frac{gAE}{W}\left(\frac{1}{a} + \frac{1}{b}\right)}$
Transverse	$\sqrt{\frac{g}{W}(3EI)\left(\frac{L}{ab}\right)^3}$
Torsional	$\sqrt{\frac{g}{W}\frac{G}{r^2}\frac{\pi d^4}{32}\left(\frac{1}{a} + \frac{1}{b}\right)}$

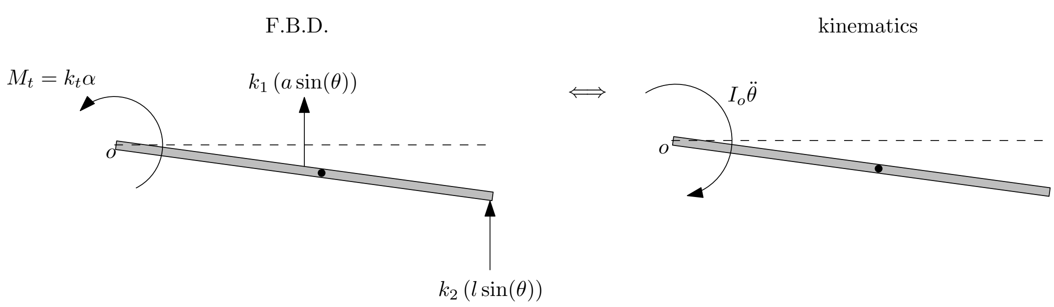
0.2 Problem 2

Problem 2

The uniform rigid bar OA of length L and mass m is pinned about point O. Using Newton's Second Law, find the equation of motion for the system using the generalized coordinate θ and also find the system's natural frequency.



The first step is to draw the free body diagram and the kinematic diagram



Taking moments about the joint O, noting that positive is anti-clockwise gives

$$k_t\theta + k_1 (a\sin\theta)\, a + k_2 (L\sin\theta)\, L = -I_o\ddot{\theta} \tag{1}$$

Using parallel axis theorem,

$$\begin{aligned} I_o &= I_{CG} + m\left(\frac{L}{2}\right)^2 \\ &= \frac{1}{12}mL^2 + m\frac{L^2}{4} \\ &= \frac{1}{3}L^2 m \end{aligned}$$

Hence (1) becomes

$$\frac{1}{3}L^2 m\ddot{\theta} + k_t\theta + k_1 (a\sin\theta)\, a + k_2 (L\sin\theta)\, L = 0$$

For small angle approximation the above becomes (we have to apply small angle approximation in order to obtain the form that allows us to determine ω_n^2, since this only works for linear equations of motion).

$$\begin{aligned}
\frac{1}{3}L^2 m\ddot{\theta} + k_t\theta + k_1 a^2\theta + k_2 L^2\theta &= 0\\
\frac{1}{3}L^2 m\ddot{\theta} + \theta\left(k_t + k_1 a^2 + k_2 L^2\right) &= 0\\
\ddot{\theta} + \theta\frac{3\left(k_t + k_1 a^2 + k_2 L^2\right)}{mL^2} &= 0
\end{aligned}$$

Comparing the above to standard form of linearized $\ddot{\theta} + \omega_n^2\theta = 0$ we see that the natural frequency (radians per second) is

$$\boxed{\omega_n = \sqrt{\frac{3\left(k_t + k_1 a^2 + k_2 L^2\right)}{mL^2}}}$$