HW8, Math 322, Fall 2016

Nasser M. Abbasi

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1.1 Problem 5.10.1

5.10.1. Consider the Fourier sine series for $f(x) = 1$ on the interval $0 \le x \le L$. How many terms in the series should be kept so that the mean-square error is 1% of $\int_0^L f^2\sigma\,dx$?

The Fourier sin series of $f(x) = 1$ on $0 \le x \le L$ is given by

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Where

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{1}{L}\left(\int_{-L}^{0}(-1)\sin\left(\frac{n\pi}{L}x\right)dx + \int_{0}^{L}(+1)\sin\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\left(-\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{-L}^{0} + \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)\right]_{0}^{L}\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}x\right)\right]_{-L}^{0} - \frac{L}{n\pi}\left[\cos\left(\frac{n\pi}{L}x\right)\right]_{0}^{L}\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[\cos(0) - \cos(n\pi)\right] - \frac{L}{n\pi}\left[\cos(n\pi) - \cos(0)\right]\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}\left[1 - \cos(n\pi)\right] - \frac{L}{n\pi}\left[\cos(n\pi) - 1\right]\right)
\end{aligned}$$

We see that $b_n = 0$ for $n = 2,4,6,\cdots$, and b_n odd for $n = 1,3,5,\cdots$ so we can simplify the above to be

$$\begin{aligned}
b_n &= \frac{1}{L}\left(\frac{L}{n\pi}\left[1-(-1)\right] - \frac{L}{n\pi}\left[-1-1\right]\right) \\
&= \frac{1}{L}\left(\frac{L}{n\pi}[2] - \frac{L}{n\pi}[-2]\right) \\
&= \frac{1}{L}\left(\frac{4L}{n\pi}\right) \\
&= \frac{4}{n\pi}
\end{aligned}$$

Equation (1) becomes

$$f(x) \sim \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi}{L}x\right) \tag{2}$$

mean-square error is, from textbook, page 213, is given by equation 5.10.11

$$E = \int_0^L f^2(x)\,\sigma(x)\,dx - \sum_{n=1,3,5,\cdots}^{\infty} \alpha_n^2 \int_0^L \phi_n^2\sigma(x)\,dx \tag{5.10.11}$$

In this problem, $\phi_n = \sin\left(\frac{n\pi}{L}x\right)$ and $\alpha_n = a_n = \frac{4}{n\pi}$. The above equation becomes

$$\begin{aligned}
E &= \int_0^L f^2(x)\,\sigma(x)\,dx - \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{4}{n\pi}\right)^2\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma(x)\,dx \\
&= \int_0^L f^2(x)\,\sigma dx - \sum_{n=1,3,5,\cdots}^{\infty}\frac{16}{n^2\pi^2}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma dx
\end{aligned}$$

For $\sigma = 1$ we know that

$$\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)\sigma dx = \frac{L}{2}$$

Hence E becomes

$$E = \int_0^L f^2(x)\,\sigma dx - \sum_{n=1,3,5,\cdots}^{\infty}\frac{16}{n^2\pi^2}\frac{L}{2}$$

But $\int_0^L f^2(x)\,\sigma dx$ for $\sigma = 1$ is just $\int_0^L 1^2 dx = L$, and the above becomes

$$\begin{aligned} E &= L - \frac{L}{2}\frac{16}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^2} \\ &= L - \frac{8L}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^2} \end{aligned}$$

We need to find N so that $E = 0.01L$. The above becomes

$$0.01L = L - \frac{8L}{\pi^2}\sum_{n=1,3,5,\cdots}^{N}\frac{1}{n^2}$$

We need now to solve for N in the above

$$\begin{aligned} 0.01L - L &= -\frac{8L}{\pi^2}\sum_{n=1,3,5,\cdots}^{N}\frac{1}{n^2} \\ 0.99L\left(\frac{\pi^2}{8L}\right) &= \sum_{n=1,3,5,\cdots}^{N}\frac{1}{n^2} \\ 1.2214 &= \sum_{n=1,3,5,\cdots}^{N}\frac{1}{n^2} \end{aligned}$$

A small Mathematica program written which prints the RHS sum for each n, and was visually checked when it reached 1.2214, here is the result

```
In[53]:= data = Table[{i, Sum[1 / n^2, {n, 1, i, 2}]}, {i, 1, 50, 2}] // N;
         Grid[Join[{{"n", "sum"}}, data], Frame → All]
```

Out[54]=

n	sum
1.	1.
3.	1.11111
5.	1.15111
7.	1.17152
9.	1.18386
11.	1.19213
13.	1.19805
15.	1.20249
17.	1.20595
19.	1.20872
21.	1.21099
23.	1.21288
25.	1.21448
27.	1.21585
29.	1.21704
31.	1.21808
33.	1.219
35.	1.21982
37.	1.22055
39.	1.2212
41.	1.2218
43.	1.22234
45.	1.22283
47.	1.22329
49.	1.2237

Counting the number of terms needed to reach 1.2214, we see there are 21 terms (21 rows in the table, since only odd entries are counted, the table above skips the even n values in the sum since these are all zero).

1.2 Problem 5.10.2 (b)

5.10.2. Obtain a formula for an infinite series using Parseval's equality applied to the

(a) Fourier sine series of $f(x) = 1$ on the interval $0 \le x \le L$

*(b) Fourier cosine series of $f(x) = x$ on the interval $0 \le x \le L$

(c) Fourier sine series of $f(x) = x$ on the interval $0 \le x \le L$

Parseval's equality is given by equation 5.10.14, page 214 in textbook

$$\int_a^b f^2 \sigma dx = \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2 \sigma dx \tag{5.10.14}$$

The books uses α_n instead of a_n, but it is the same, these are the coefficients in the Fourier series for $f(x)$. We now need to find the cosine Fourier series for $f(x) = x$. This is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

Where

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ &= \frac{1}{L}\left[\int_{-L}^{0}(-x)\cos\left(\frac{n\pi}{L}x\right)dx + \int_{0}^{L}(+x)\cos\left(\frac{n\pi}{L}x\right)dx\right] \\ &= \frac{1}{L}\left[-\int_{-L}^{0} x\cos\left(\frac{n\pi}{L}x\right)dx + \int_{0}^{L} x\cos\left(\frac{n\pi}{L}x\right)dx\right] \end{aligned}$$

But

$$\begin{aligned} \int_{-L}^{0} x\cos\left(\frac{n\pi}{L}x\right)dx &= -\frac{-1+(-1)^n}{n^2\pi^2}L^2 \\ \int_{0}^{L} x\cos\left(\frac{n\pi}{L}x\right)dx &= \frac{-1+(-1)^n}{n^2\pi^2}L^2 \end{aligned}$$

Hence

$$\begin{aligned} a_n &= \frac{1}{L}\left[2\frac{-1+(-1)^n}{n^2\pi^2}L^2\right] \\ &= \frac{2L}{\pi^2}\frac{\left(-1+(-1)^n\right)}{n^2} \end{aligned}$$

Looking at few terms to see the pattern

$$a_n = \frac{2L}{\pi^2}\left\{\frac{-2}{1}, 0, \frac{-2}{3^2}, 0, \frac{-2}{5^2}, \cdots\right\}$$

Therefore, we can write a_n as

$$a_n = \frac{-4L}{\pi^2 n^2} \qquad n = 1, 3, 5, \cdots$$

And

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &= \frac{1}{2L}\left[\int_{-L}^{0}(-x)\,dx + \int_{0}^{L}(+x)\,dx\right] \\ &= \frac{1}{2L}\left[-\left[\frac{x^2}{2}\right]_{-L}^{0} + \left[\frac{x^2}{2}\right]_{0}^{L}\right] \\ &= \frac{1}{2L}\left[-\left[0-\frac{L^2}{2}\right] + \left[\frac{L^2}{2}-0\right]\right] \\ &= \frac{1}{2L}\left[\frac{L^2}{2}+\frac{L^2}{2}\right] \\ &= \frac{L}{2} \end{aligned}$$

Hence the Fourier series is

$$f(x) = \frac{L}{2} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{-4L}{\pi^2 n^2}\cos\left(\frac{n\pi}{L}x\right)$$

We now go back to equation 5.10.14 (but need to add a_0 to it, since there is this extra term with cosine Fourier series)

$$\int_a^b f^2\sigma dx = a_0^2\int_a^b 1^2 dx + \sum_{n=1}^{\infty} a_n^2 \int_a^b \phi_n^2\sigma dx$$

$$\int_0^L x^2 dx = \left(\frac{L}{2}\right)^2\int_0^L dx + \sum_{n=1,3,5,\cdots}^{\infty}\left(\frac{-4L}{\pi^2n^2}\right)^2\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx$$

$$\left[\frac{x^3}{3}\right]_0^L = \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{16L^2}{\pi^4n^4}\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx$$

Since $\int_0^L \cos^2\left(\frac{n\pi}{L}x\right)dx = \frac{L}{2}$ the above becomes

$$\frac{L^3}{3} = \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{16L^2}{\pi^4n^4}\frac{L}{2}$$

$$\frac{L^3}{3} = \left(\frac{L^2}{4}\right)L + \sum_{n=1,3,5,\cdots}^{\infty}\frac{8L^3}{\pi^4n^4}$$

$$\frac{L^3}{3} = \frac{L^3}{4} + \frac{8L^3}{\pi^4}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4}$$

Simplifying

$$\frac{1}{3} = \frac{1}{4} + \frac{8}{\pi^4}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4} = \left(\frac{1}{3}-\frac{1}{4}\right)\frac{\pi^4}{8}$$

$$\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$$

Hence

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots$$

Which agrees with the book solution given in back of book.

1.3 Problem 5.10.6

5.10.6. Assuming that the operations of summation and integration can be interchanged, show that if

$$f = \sum \alpha_n\phi_n \quad \text{and} \quad g = \sum \beta_n\phi_n,$$

then for normalized eigenfunctions

$$\int_a^b fg\sigma\, dx = \sum_{n=1}^{\infty}\alpha_n\beta_n,$$

a generalization of Parseval's equality.

$$\begin{aligned}\int_a^b fg\sigma dx &= \int_a^b\left(\sum_{n=1}^{\infty}\alpha_n\phi_n\right)\left(\sum_{n=1}^{\infty}\beta_n\phi_n\right)\sigma dx\\ &= \int_a^b\left(\alpha_1\phi_1+\alpha_2\phi_2+\cdots\right)\left(\beta_1\phi_1+\beta_2\phi_2+\cdots\right)\sigma dx \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\left(\alpha_1\phi_1+\alpha_2\phi_2+\cdots\right)\left(\beta_1\phi_1+\beta_2\phi_2+\cdots\right) &= \alpha_1\beta_1\phi_1^2+\alpha_1\beta_2\phi_1\phi_2+\alpha_1\beta_3\phi_1\phi_3+\cdots\\ &+\alpha_2\beta_1\phi_2\phi_1+\alpha_2\beta_2\phi_2^2+\alpha_2\beta_3\phi_1\phi_3+\cdots\\ &+\alpha_3\beta_1\phi_3\phi_1+\alpha_3\beta_2\phi_3\phi_2+\alpha_3\beta_3\phi_3^2+\cdots\\ &\vdots\end{aligned}$$

Which means when expanding the product of the two series, only the terms on the diagonal (the terms with $\alpha_i\beta_j\phi_i\phi_j$ with $i=j$) will survive. This due to orthogonality. To show this more clearly, we put the above expansion back into the integral (1) and break up the integral into sum of integrals

$$
\begin{aligned}
\int_a^b fg\sigma dx = &\int_a^b \alpha_1\beta_1\phi_1^2\sigma dx + \int_a^b \alpha_1\beta_2\phi_1\phi_2\sigma dx + \int_a^b \alpha_1\beta_3\phi_1\phi_3\sigma dx + \cdots\\
&+ \int_a^b \alpha_2\beta_1\phi_2\phi_1\sigma dx + \int_a^b \alpha_2\beta_2\phi_2^2\sigma dx + \int_a^b \alpha_2\beta_3\phi_1\phi_3\sigma dx + \cdots\\
&+ \int_a^b \alpha_3\beta_1\phi_3\phi_1\sigma dx + \int_a^b \alpha_3\beta_2\phi_3\phi_2\sigma dx + \int_a^b \alpha_3\beta_3\phi_3^2\sigma dx + \cdots\\
&\vdots
\end{aligned}
$$

The above simplifies to

$$
\int_a^b fg\sigma dx = \int_a^b \alpha_1\beta_1\phi_1^2\sigma dx + \int_a^b \alpha_2\beta_2\phi_2^2\sigma dx + \int_a^b \alpha_3\beta_3\phi_3^2\sigma dx + \cdots \int_a^b \alpha_n\beta_n\phi_n^2\sigma dx + \cdots
$$

Since all other terms vanish due to orthogonality of eigenfunctions. The above simplifies to

$$
\begin{aligned}
\int_a^b fg\sigma dx &= \sum_{n=1}^{\infty}\int_a^b \alpha_n\beta_n\phi_n^2\sigma dx\\
&= \sum_{n=1}^{\infty}\left(\alpha_n\beta_n \int_a^b \phi_n^2\sigma dx\right)
\end{aligned}
$$

Because the eigenfunctions are normalized, then $\int_a^b \phi_n^2\sigma dx = 1$ and the above reduces to the result needed

$$
\int_a^b fg\sigma\, dx = \sum_{n=1}^{\infty}\alpha_n\beta_n
$$

1.4 Problem 7.3.4

7.3.4. Consider the wave equation for a vibrating rectangular membrane ($0 < x < L,\ 0 < y < H$)

$$
\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)
$$

subject to the initial conditions

$$
u(x,y,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = f(x,y).
$$

Solve the initial value problem if

(a) $u(0,y,t) = 0,\quad u(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x,0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$

* (b) $\frac{\partial u}{\partial x}(0,y,t) = 0,\quad \frac{\partial u}{\partial x}(L,y,t) = 0,\quad \frac{\partial u}{\partial y}(x.0,t) = 0,\quad \frac{\partial u}{\partial y}(x,H,t) = 0$

1.4.1 part(a)

Let $u = X(x)Y(y)T(t)$. Substituting this back into the PDE gives

$$
T''XY = c^2(X''YT + Y''XT)
$$

Dividing by $XYT \neq 0$ gives

$$
\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}
$$

Since left side depends on t only and right side depends on (x,y) only, then both must be equal to some constant, say $-\lambda$

$$
\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda
$$

We obtain the following

$$
\begin{aligned}
T'' + c^2\lambda T &= 0\\
\frac{X''}{X} &= -\lambda - \frac{Y''}{Y}
\end{aligned}
$$

Again, looking at the second ODE above, we see that the left side depends on x only, and the right side on y only. Then they must be equal to some constant, say $-\mu$ and we obtain

$$
\frac{X''}{X} = \left(-\lambda - \frac{Y''}{Y}\right) = -\mu
$$

Which results in two ODE's. The first is

$$\begin{aligned} X'' + \mu X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

And the second is

$$\begin{aligned} \lambda + \frac{Y''}{Y} &= \mu \\ Y'' &= Y\mu - \lambda Y \\ Y'' + Y(\lambda - \mu) &= 0 \end{aligned}$$

With B.C.

$$\begin{aligned} Y'(0) &= 0 \\ Y'(H) &= 0 \end{aligned}$$

Starting with the X ODE since it is simpler, the solution is

$$X = c_1 \cos\left(\sqrt{\mu}x\right) + c_2 \sin\left(\sqrt{\mu}x\right)$$

Applying $X(0) = 0$ gives

$$0 = c_1$$

Hence solution is

$$X = c_2 \sin\left(\sqrt{\mu}x\right)$$

Applying $X(L) = 0$ gives

$$0 = c_2 \sin\left(\sqrt{\mu}L\right)$$

For non-trivial solution

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, 3, \cdots$$

And the eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

We now solve the Y ODE.

$$Y'' + \left(\lambda - \mu_n\right) Y = 0$$

Assuming that $(\lambda - \mu) > 0$ for all λ, μ, (we know this is the only case, since only positive $\lambda - \mu_n$ will be possible when B.C. are homogeneous Dirichlet). Then, for $(\lambda - \mu) > 0$, the solution is

$$\begin{aligned} Y(y) &= c_1 \cos\left(\sqrt{\lambda - \mu_n} y\right) + c_2 \sin\left(\sqrt{\lambda - \mu_n} y\right) \\ Y' &= -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} y\right) + c_2 \sqrt{\lambda - \mu_n} \cos\left(\sqrt{\lambda - \mu_n} y\right) \end{aligned}$$

Applying B.C. $Y'(0) = 0$ the above becomes

$$0 = c_2 \sqrt{\lambda - \mu_n}$$

Hence $c_2 = 0$ and the solution becomes

$$\begin{aligned} Y &= c_1 \cos\left(\sqrt{\lambda - \mu_n} y\right) \\ Y' &= -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} y\right) \end{aligned}$$

Applying second B.C. $Y'(H) = 0$ gives

$$0 = -c_1 \sqrt{\lambda - \mu_n} \sin\left(\sqrt{\lambda - \mu_n} H\right)$$

For non-trivial solution we want

$$\begin{aligned} \sin\left(\sqrt{\lambda - \mu_n} H\right) &= 0 \\ \sqrt{\lambda_{nm} - \mu_n} &= m\frac{\pi}{H} \\ \lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \\ \lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2 + \mu_n \qquad m = 0, 1, 2, \cdots \end{aligned}$$

Hence the eigenfunctions are

$$Y_{nm} = \cos\left(m\frac{\pi}{H}y\right) \qquad m = 0, 1, 2, \cdots, n = 1, 2, 3, \cdots$$

For each n,m, we find solution of $T'' + c^2\lambda_{nm}T = 0$.The solution is

$$T_{nm}(t) = A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)$$

Putting all these results together gives

$$\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}(t)\,X_{nm}(x)\,Y_{nm}(y)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}\cos\left(c\sqrt{\lambda_{n,m}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{n,m}}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}$$

We now apply initial conditions to find A_{nm}, B_{nm}. At $t = 0$

$$\begin{aligned}
u(x,y,0) &= 0\\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}$$

Hence

$$A_{nm} = 0$$

And the solution becomes

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

Taking derivative of the solution w.r.t. time t gives

$$\frac{\partial}{\partial t}u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

At $t = 0$ the above becomes

$$\alpha(x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

Multiplying both sides by $\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$ and integrating gives

$$\begin{aligned}
\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\,dxdy &= c\sqrt{\lambda_{nm}}B_{nm}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\int_0^L\int_0^H \sin^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)\,dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\int_0^L\int_0^H \sin^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)\,dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right)
\end{aligned}$$

Hence

$$B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy$$

Summary of solution

$$\begin{aligned}
X_n(x) &= \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots\\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots\\
Y_{nm}(y) &= \cos\left(m\frac{\pi}{H}y\right) \qquad m = 0,1,2,\cdots\\
\lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \qquad m = 0,1,2,\cdots, n = 1,2,3,\cdots\\
T_{nm}(t) &= B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\\
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
\end{aligned}$$

1.4.2 Part (b)

In this case we have

$$\begin{aligned} X'' + \mu X &= 0 \\ X'(0) &= 0 \\ X'(L) &= 0 \end{aligned}$$

And the second spatial ODE is

$$\begin{aligned} \lambda + \frac{Y''}{Y} &= \mu \\ Y'' &= Y\mu - \lambda Y \\ Y'' + Y(\lambda - \mu) &= 0 \end{aligned}$$

With B.C.

$$\begin{aligned} Y'(0) &= 0 \\ Y'(H) &= 0 \end{aligned}$$

Starting with the X ODE. The solution is

$$\begin{aligned} X &= c_1 \cos\left(\sqrt{\mu}x\right) + c_2 \sin\left(\sqrt{\mu}x\right) \\ X' &= -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}x\right) + c_2\sqrt{\mu}\cos\left(\sqrt{\mu}x\right) \end{aligned}$$

First B.C. gives

$$0 = c_2\sqrt{\mu}$$

Hence $c_2 = 0$ and the solution becomes

$$\begin{aligned} X &= c_1 \cos\left(\sqrt{\mu}x\right) \\ X' &= -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}x\right) \end{aligned}$$

Second B.C. gives

$$0 = -c_1\sqrt{\mu}\sin\left(\sqrt{\mu}L\right)$$

Hence

$$\begin{aligned} \sqrt{\mu}L &= n\pi \\ \mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 0,1,2,\cdots \end{aligned}$$

Now for the Y solution. This is the same as part (a).

$$\begin{aligned} Y_{nm}(y) &= \cos\left(m\frac{\pi}{H}y\right) \\ \lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \\ \lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2 + \mu_n \\ &= \left(m\frac{\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad m = 0,1,2,\cdots, n = 0,1,2,\cdots \end{aligned}$$

For each n,m, we find solution of $T'' + c^2\lambda_{nm}T = 0$.When $n = 0, m = 0$, $\lambda_{nm} = 0$ and the ODE becomes

$$T'' = 0$$

With solution

$$T = At + B$$

And total solution is

$$\begin{aligned} u(x,y,t) &= T_{nm}(t)\, X_{nm}(x)\, Y_{nm}(y) \\ &= T_{00}(t)\, X_{00}(x)\, Y_{00}(y) \\ &= (At + B) \end{aligned}$$

Since $X_{00}(x) = 1$ and $Y_{00}(y) = 1$. Applying initial conditions gives

$$u(x,y,0) = 0 = B$$

Therefore the solution is $u(x,y,t) = At$. Applying second initial conditions gives

$$A = \alpha(x,y)$$

Hence the time solution for $n = m = 0$ is

$$T_{00} = t\alpha(x,y)$$

For each $n, m,$ other than $n = m = 0$, the time solution of $T'' + c^2\lambda_{nm}T = 0$ is

$$T_{nm}(t) = A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)$$

Putting all these results together, we obtain

$$\begin{aligned}
u\left(x,y,t\right) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_{nm}(t)\,X_{nm}(x)\,Y_{nm}\left(y\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\right]\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)\\
&+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)
\end{aligned}$$

The difference in part(b) from part(a), is that the space solutions eigenfunctions are now all cosine instead of cosine and sine. When the eigenfunction is cos the sum starts from zero. When eigenfunction is sin the sum starts from 1. Now initial conditions are applied as in part (a).

$$u\left(x,y,0\right) = 0 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

Hence $A_{nm} = 0$. And the solution becomes

$$u\left(x,y,t\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

Taking derivative of the solution w.r.t. time

$$\frac{\partial}{\partial t}u\left(x,y,t\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

At $t = 0$ the above becomes

$$\alpha\left(x,y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c\sqrt{\lambda_{nm}}B_{nm}\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$$

Multiplying both sides by $\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)$ and integrating gives

$$\begin{aligned}
\int_0^L\int_0^H \alpha\left(x,y\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy &= c\sqrt{\lambda_{nm}}B_{nm}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^L\int_0^H \cos^2\left(\frac{n\pi}{L}x\right)\cos^2\left(m\frac{\pi}{H}y\right)dxdy\\
&= c\sqrt{\lambda_{nm}}B_{nm}\left(\frac{L}{2}\right)\left(\frac{H}{2}\right)
\end{aligned}$$

Hence

$$B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy$$

<u>Summary of solution</u>

$$\begin{aligned}
X_n(x) &= \cos\left(\frac{n\pi}{L}x\right) && n = 0,1,2,\cdots\\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 && n = 0,1,2,\cdots\\
Y_{nm}\left(y\right) &= \cos\left(m\frac{\pi}{H}y\right) && m = 0,1,2,\cdots\\
\lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 && m = 0,1,2,\cdots,n = 0,1,2,\cdots\\
T_{nm}(t) &= \begin{cases} t\alpha\left(x,y\right) & n = m = 0\\ B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) & \text{otherwise}\end{cases}\\
u\left(x,y,t\right) &= \begin{cases} t\alpha\left(x,y\right) & n = m = 0\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right) & \text{otherwise}\end{cases}\\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha\left(x,y\right)\cos\left(\frac{n\pi}{L}x\right)\cos\left(m\frac{\pi}{H}y\right)dxdy
\end{aligned}$$

1.4.3 Part (c)

Same problem, but using the following boundary conditions

$$
\begin{aligned}
u(0,y,t) &= 0\\
u(L,y,t) &= 0\\
u(x,0,t) &= 0\\
u(x,H,t) &= 0
\end{aligned}
$$

Since the boundary conditions are homogeneous Dirichlet then the $X(x)$ ODE solution is

$$
\begin{aligned}
X_n &= \sin\left(\frac{n\pi}{L}x\right)\\
\mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots
\end{aligned}
$$

And $Y(y)$ ODE solution is

$$
\begin{aligned}
Y_{nm}(y) &= \sin\left(\frac{m\pi}{H}y\right)\\
\lambda_{nm} &= \left(m\frac{\pi}{H}\right)^2 + \mu_n\\
&= \left(m\frac{\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad m = 1,2,3,\cdots, n = 1,2,3,\cdots
\end{aligned}
$$

And the time solution is

$$
T_{nm}(t) = A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right) + B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \qquad m = 1,2,3,\cdots, n = 1,2,3,\cdots
$$

Hence the total solution is

$$
\begin{aligned}
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} T_{nm}(t)\, X_{nm}(x)\, Y_{nm}(y)\\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)\\
&+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
\end{aligned}
$$

At $t = 0$

$$
0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
$$

Hence $A_{nm} = 0$ and the solution becomes

$$
u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
$$

Taking derivative

$$
\frac{\partial}{\partial t}u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}c\sqrt{\lambda_{nm}}\cos\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
$$

At $t = 0$

$$
\alpha(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}c\sqrt{\lambda_{nm}}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)
$$

Therefore, using orthogonality in 2D, we find

$$
B_{nm} = \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)dxdy
$$

Summary of solution

$$
\begin{aligned}
X_n(x) &= \sin\left(\frac{n\pi}{L}x\right) \qquad n = 1,2,3,\cdots \\
\mu_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots \\
Y_{nm}(y) &= \cos\left(m\frac{\pi}{H}y\right) \qquad m = 1,2,3,\cdots \\
\lambda_{nm} - \mu_n &= \left(m\frac{\pi}{H}\right)^2 \qquad m = 1,2,3,\cdots, n = 1,2,3,\cdots \\
T_{nm}(t) &= B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right) \\
u(x,y,t) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{nm}\sin\left(c\sqrt{\lambda_{nm}}t\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(m\frac{\pi}{H}y\right) \\
B_{nm} &= \frac{4}{LHc\sqrt{\lambda_{nm}}}\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{H}y\right)dxdy
\end{aligned}
$$

1.5 Problem 7.3.6

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$
\begin{aligned}
\frac{\partial}{\partial z}u(x,y,0) &= 0 \\
u(x,y,H) &= f(x,y)
\end{aligned}
$$

and $u = 0$ on the "lateral" sides.

(a) Separate the z-variable in general.

*(b) Solve for $u(x,y,z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

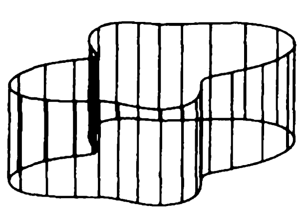

Figure 7.3.3

1.5.1 Part (a)

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Let $u = XYZ$ where $X \equiv X(x), Y \equiv Y(y), Z \equiv Z(z)$. Substituting this back in the above gives

$$X''YZ + Y''XZ + Z''XY = 0$$

Dividing by $XYZ \neq 0$ gives

$$
\begin{aligned}
\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} &= -\frac{Z''}{Z}
\end{aligned}
$$

Since the left side depends on x, y only and the right side depends on z only and they are equal, they must both be the same constant. Say $-\lambda$, and we write

$$
\begin{aligned}
\frac{X''}{X} + \frac{Y''}{Y} &= -\lambda \qquad (1) \\
\frac{Z''}{Z} &= \lambda
\end{aligned}
$$

The problem asks to separate the z variable, then the ODE for this variable is

$$Z'' - \lambda Z = 0 \tag{2}$$

With boundary conditions

$$\begin{aligned} Z'(0) &= 0 \\ Z(H) &= f(x,y) \end{aligned}$$

1.5.2 Part(b)

We will continue separation from part(a). From (1) in part (a)

$$\frac{X''}{X} + \frac{Y''}{Y} = -\lambda$$

We now need to separate X, Y. Therefore

$$\frac{X''}{X} = -\lambda - \frac{Y''}{Y}$$

As the left side depends on x only and right side depends on y only and both are equal, then they are equal to some constant, say $-\mu$

$$\begin{aligned} \frac{X''}{X} &= -\mu \\ -\lambda - \frac{Y''}{Y} &= -\mu \end{aligned}$$

The x ODE becomes

$$\begin{aligned} X'' + \mu X &= 0 \qquad (1) \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

And the y ODE becomes

$$\begin{aligned} -\frac{Y''}{Y} &= -\mu + \lambda \\ Y'' + (\lambda - \mu) Y &= 0 \qquad (2) \end{aligned}$$

With B.C.

$$\begin{aligned} Y(0) &= 0 \\ Y(W) &= 0 \end{aligned}$$

Now that we have the three ODE's we start solving them. Starting with the x ODE (1). The solution is

$$\begin{aligned} X_n &= \sin\left(\frac{n\pi}{L}x\right) \\ \mu &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots \end{aligned}$$

For each n there is solution for the y ODE

$$\begin{aligned} Y_{nm} &= \sin\left(\frac{m\pi}{W}y\right) \\ \lambda_{nm} - \mu_n &= \left(\frac{m\pi}{W}\right)^2 \qquad m = 1,2,3,\cdots \end{aligned}$$

Or

$$\lambda_{nm} = \left(\frac{m\pi}{W}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots, m = 1,2,3,\cdots$$

And for each n and for each m there is a solution for the z ODE we found in part (a), which is

$$\begin{aligned} Z'' - \lambda_{nm} Z &= 0 \\ Z'(0) &= 0 \end{aligned}$$

The solution is, since $\lambda_{nm} > 0$ is

$$\begin{aligned} Z &= c_1 \cosh\left(\sqrt{\lambda_{nm}}z\right) + c_2 \sinh\left(\sqrt{\lambda_{nm}}z\right) \\ Z' &= c_1\sqrt{\lambda_{nm}} \sinh\left(\sqrt{\lambda_{nm}}z\right) + c_2\sqrt{\lambda_{nm}} \cosh\left(\sqrt{\lambda_{nm}}z\right) \end{aligned}$$

Applying B.C. $Z'(0) = 0$ gives

$$0 = c_2\sqrt{\lambda_{nm}}$$

Hence $c_2 = 0$ and the solution becomes

$$Z = c_{nm} \cosh\left(\sqrt{\lambda_{nm}}z\right)$$

Putting all these solutions together, we obtain

$$u\left(x,y,z\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}z\right)$$

Only now we apply the last boundary condition $u\left(x,y,H\right)=f\left(x,y\right)$ to find c_{nm}.

$$f\left(x,y\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}H\right)$$

Applying 2D orthogonality gives

$$\begin{aligned}\int_0^L\int_0^W f\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy &= c_{nm}\cosh\left(\sqrt{\lambda_{nm}}H\right)\int_0^L\int_0^W \sin^2\left(\frac{n\pi}{L}x\right)\sin^2\left(\frac{m\pi}{W}y\right)dxdy\\ &= c_{nm}\cosh\left(\sqrt{\lambda_{nm}}H\right)\left(\frac{L}{2}\right)\left(\frac{W}{2}\right)\end{aligned}$$

Hence

$$\begin{aligned}c_{nm} &= \frac{\int_0^L\int_0^W f\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy}{\cosh\left(\sqrt{\lambda_{nm}}H\right)\left(\frac{L}{2}\right)\left(\frac{W}{2}\right)}\\ &= \frac{4}{LW\cosh\left(\sqrt{\lambda_{nm}}H\right)}\int_0^L\int_0^W f\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy\end{aligned}$$

Summary of solution

$$\begin{aligned}X_n &= \sin\left(\frac{n\pi}{L}x\right)\\ Y_{nm} &= \sin\left(\frac{m\pi}{W}y\right)\\ \lambda_{nm} &= \left(\frac{m\pi}{W}\right)^2+\left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,3,\cdots,m=1,2,3,\cdots\\ u\left(x,y,z\right) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}c_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)\cosh\left(\sqrt{\lambda_{nm}}z\right)\\ c_{nm} &= \frac{4}{LW\cosh\left(\sqrt{\lambda_{nm}}H\right)}\int_0^L\int_0^W f\left(x,y\right)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{W}y\right)dxdy\end{aligned}$$

1.6 Problem 7.4.2

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

Equation 7.4.7 is

$$\begin{aligned}\nabla^2\phi+\lambda\phi &= 0\\ \phi\left(0,y\right) &= 0\\ \phi\left(L,y\right) &= 0\\ \phi\left(x,0\right) &= 0\\ \phi\left(x,H\right) &= 0\end{aligned}$$

And 7.4.6 is

$$\lambda=\frac{-\oint\phi\nabla\phi\cdot\hat{n}ds+\iint_R\left|\nabla\phi\right|^2dxdy}{\iint_R\phi^2dxdy}$$

$\oint\phi\nabla\phi\cdot\hat{n}ds=0$ as we are told $\phi=0$ on the boundary and this integration is for the boundary only. Hence λ simplifies to

$$\lambda=\frac{\iint_R\left|\nabla\phi\right|^2dxdy}{\iint_R\phi^2dxdy}$$

The numerator can not be negative, since the integrand $\left|\nabla\phi\right|^2$ is not negative. Similarly, the denominator has positive integrand, because ϕ can not be identically zero, as it is an eigenfunction. Hence we conclude that $\lambda \geq 0$.

1.7 Problem 7.4.3

7.4.3. If necessary, see Sec. 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u)\cdot \hat{n}\, ds$.

(b) From part (a), derive (7.4.5).

1.7.1 part (a)

$$\nabla \cdot (u\nabla v) = u\nabla^2 v + \nabla u \cdot \nabla v \tag{1}$$

$$\nabla \cdot (v\nabla u) = v\nabla^2 u + \nabla v \cdot \nabla u \tag{2}$$

Equation (1)-(2) leads to

$$\nabla \cdot (u\nabla v) - \nabla \cdot (v\nabla u) = \left(u\nabla^2 v + \nabla u \cdot \nabla v\right) - \left(v\nabla^2 u + \nabla v \cdot \nabla u\right)$$

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u + \nabla u \cdot \nabla v - \nabla v \cdot \nabla u$$

But $\nabla u \cdot \nabla v = \nabla v \cdot \nabla u$ so the above reduces to

$$\nabla \cdot (u\nabla v - v\nabla u) = u\nabla^2 v - v\nabla^2 u$$

Therefore

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \iint \nabla \cdot (u\nabla v - v\nabla u)\, dxdy \tag{3}$$

But the RHS of the above is of the form $\iint (\nabla \cdot A)\, dxdy$ where $A = (u\nabla v - v\nabla u)$ here. Which we can apply divergence theorem on it and obtain $\oint (A \cdot \hat{n})\, ds$. Therefore, using divergence theorem on the RHS of (3), then (3) can be written as

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \oint (u\nabla v - v\nabla u)\cdot \hat{n} ds$$

Which is what is required to show.

1.7.2 Part(b)

Equation 7.4.5 is

$$\iint_R \phi_{\lambda_1}\phi_{\lambda_2} dxdy = 0 \qquad \text{if } \lambda_1 \neq \lambda_2 \tag{7.4.5}$$

From part (a), we found

$$\iint \left(u\nabla^2 v - v\nabla^2 u\right) dxdy = \oint (u\nabla v - v\nabla u)\cdot \hat{n} ds \tag{1}$$

But we know that, since both u, v satisfy the multidimensional eigenvalue problem on same domain, then

$$\nabla^2 v + \lambda_v v = 0 \tag{2}$$

$$\beta_1 v + \beta_2 (\nabla v \cdot \hat{n}) = 0 \tag{3}$$

And similarly

$$\nabla^2 u + \lambda_u u = 0 \tag{4}$$

$$\beta_1 u + \beta_2 (\nabla u \cdot \hat{n}) = 0 \tag{5}$$

Now we will use (2,3,4,5) into (1) to obtain 7.4.5. From (2), we see that $\nabla^2 v = -\lambda_v v$ and from (4) $\nabla^2 u = -\lambda_u u$ and from (3) $\nabla v \cdot \hat{n} = -\frac{\beta_1}{\beta_2} v$ and from (5) $\nabla u \cdot \hat{n} = -\frac{\beta_1}{\beta_2} u$. Substituting all of these back into (1) gives

$$\iint (u(-\lambda_v v) - v(-\lambda_u u))\ dxdy = \oint u(\nabla v \cdot \hat{n}) - v(\nabla u \cdot \hat{n})\, ds$$

$$\iint (-\lambda_v uv + \lambda_u vu)\ dxdy = \oint u\left(-\frac{\beta_1}{\beta_2} v\right) - v\left(-\frac{\beta_1}{\beta_2} u\right) ds$$

$$\iint (\lambda_u - \lambda_v)\, uv\ dxdy = \oint \frac{\beta_1}{\beta_2}\left[-uv + uv\right] ds$$

$$(\lambda_u - \lambda_v)\iint uv\ dxdy = 0 \tag{6}$$

We now use (6) the above to show that 7.4.5 is correct. In (6), if we replace $u = \phi_{\lambda_1}, v = \phi_{\lambda 1}$ and $\lambda_u = \lambda_1, \lambda_v = \lambda_2$ then (6) becomes

$$(\lambda_1 - \lambda_2) \iint \left(\phi_{\lambda_1} \phi_{\lambda 1}\right) \, dxdy = 0$$

We see now that for $\lambda_1 \neq \lambda_2$, then $\iint \left(\phi_{\lambda_1} \phi_{\lambda 1}\right) \, dxdy = 0$. Which is what we asked to show.