HW5, Math 322, Fall 2016

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Contents

1 HW 5

1.1 Problem 3.5.2

3.5.2. **(a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of x^2.**

(b) From part (a), determine the Fourier sine series of x^3.

1.1.1 Part a

Equation 3.3.11, page 100 is the Fourier sin series of x

$$x = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right) \qquad -L < x < L \tag{3.3.11}$$

Where

$$B_n = \frac{2L}{n\pi}(-1)^{n+1} \tag{3.3.12}$$

The goal is to find the Fourier $\cos$ series of x^2. Since $\int_0^x t dt = \frac{x^2}{2}$, then $x^2 = 2\int_0^x t dt$. Hence from 3.3.11

$$x^2 = 2\int_0^x \left[\sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}t\right)\right] dt$$

Interchanging the order of summation and integration the above becomes

$$\begin{aligned}
x^2 &= 2\sum_{n=1}^{\infty}\left(B_n \int_0^x \sin\left(n\frac{\pi}{L}t\right) dt\right) \\
&= 2\sum_{n=1}^{\infty} B_n \left(\frac{-\cos\left(n\frac{\pi}{L}t\right)}{n\frac{\pi}{L}}\right)_0^x \\
&= \sum_{n=1}^{\infty} \frac{-2L}{n\pi} B_n \left[\cos\left(n\frac{\pi}{L}t\right)\right]_0^x \\
&= \sum_{n=1}^{\infty} \frac{-2L}{n\pi} B_n \left[\cos\left(n\frac{\pi}{L}x\right) - 1\right] \\
&= \sum_{n=1}^{\infty} \left(\frac{-2L}{n\pi} B_n \cos\left(n\frac{\pi}{L}x\right) + \frac{2L}{n\pi} B_n\right) \\
&= \sum_{n=1}^{\infty} \frac{-2L}{n\pi} B_n \cos\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{\infty} B_n \frac{2L}{n\pi} \qquad (1)
\end{aligned}$$

But a Fourier $\cos$ series has the form

$$x^2 = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{\pi}{L}x\right) \tag{2}$$

Comparing (1) and (2) gives

$$A_n = \frac{-2L}{n\pi} B_n$$

Using 3.3.12 for B_n the above becomes

$$\begin{aligned} A_n &= \frac{-2L}{n\pi}\frac{2L}{n\pi}(-1)^{n+1} \\ &= (-1)^n\left(\frac{2L}{n\pi}\right)^2 \end{aligned}$$

And

$$\begin{aligned} A_0 &= \sum_{n=1}^{\infty} B_n \frac{2L}{n\pi} \\ &= \sum_{n=1}^{\infty}\left(\frac{2L}{n\pi}(-1)^{n+1}\right)\frac{2L}{n\pi} \\ &= \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} \end{aligned}$$

But $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{\pi^2}{12}$, hence the above becomes

$$\begin{aligned} A_0 &= \frac{4L^2}{\pi^2}\frac{\pi^2}{12} \\ &= \frac{L^2}{3} \end{aligned}$$

Summary The Fourier cos series of x^2 is

$$\begin{aligned} x^2 &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{\pi}{L}x\right) \\ &= \frac{L^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{2L}{n\pi}\right)^2\cos\left(n\frac{\pi}{L}x\right) \end{aligned}$$

1.1.2 Part (b)

Since

$$x^3 = 3\int_0^x t^2 dt$$

Then, using result from part (a) for Fourier cos series of t^2 results in

$$
\begin{aligned}
x^3 &= 3\int_0^x \left[A_0 + \sum_{n=1}^{\infty} A_n \cos\left(n\frac{\pi}{L}t\right)\right]dt \\
&= 3\int_0^x \frac{L^2}{3}dt + 3\int_0^x \sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \cos\left(n\frac{\pi}{L}t\right)dt \\
&= L^2 (t)_0^x + 3\sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \int_0^x \cos\left(n\frac{\pi}{L}t\right)dt \\
&= L^2 x + 3\sum_{n=1}^{\infty} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \left[\frac{\sin\left(n\frac{\pi}{L}t\right)}{n\frac{\pi}{L}}\right]_0^x \\
&= L^2 x + 3\sum_{n=1}^{\infty} \frac{L}{n\pi} (-1)^n \left(\frac{2L}{n\pi}\right)^2 \left[\sin\left(n\frac{\pi}{L}t\right)\right]_0^x \\
&= L^2 x + (3\cdot 4)\sum_{n=1}^{\infty} (-1)^n \left(\frac{L}{n\pi}\right)^3 \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Using 3.3.11 which is $x = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right)$, with $B_n = \frac{2L}{n\pi}(-1)^{n+1}$ the above becomes

$$
x^3 = L^2 \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1} \sin\left(n\frac{\pi}{L}x\right) + (3\cdot 4)\sum_{n=1}^{\infty} (-1)^n \left(\frac{L}{n\pi}\right)^3 \sin\left(n\frac{\pi}{L}x\right)
$$

Combining all above terms

$$
x^3 = \sum_{n=1}^{\infty} \left[L^2 \frac{2L}{n\pi}(-1)^{n+1} + (3\cdot 4)(-1)^n \left(\frac{L}{n\pi}\right)^3\right] \sin\left(n\frac{\pi}{L}x\right)
$$

Will try to simplify more to obtain B_n

$$
\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} (-1)^n \frac{L^3}{n\pi}\left[-2 + (3\cdot 4)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} (-1)^n \frac{2L^3}{n\pi}\left[-1 + (3\times 2)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Comparing the above to the standard Fourier sin series $x^3 = \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right)$ then the above is the required sin series for x^3 with

$$
B_n = (-1)^n \frac{2L^3}{n\pi}\left[-1 + (3\times 2)\left(\frac{1}{n\pi}\right)^2\right] \sin\left(n\frac{\pi}{L}x\right)
$$

Expressing the above using B_n from x^1 to help find recursive relation for next problem.

Will now use the notation ${}^iB_n$ to mean the B_n for x^i. Then since ${}^1B_n = \frac{2L}{n\pi}(-1)^{n+1} = (-1)^n\left(-\frac{2L}{n\pi}\right)$ for x, then, using ${}^3B_n$ as the B_n for x^3, the series for x^3 can be written

$$
\begin{aligned}
x^3 &= \sum_{n=1}^{\infty} (-1)^n L^2 \left[-\frac{2L}{n\pi} + 6\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^1B_n + 6\left(2\frac{L}{n^2\pi^2}\right)\right] \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Where now

$$^3B_n = (-1)^n L^2 \left[B_n^1 + 6 \left(2 \frac{L}{n^2 \pi^2} \right) \right]$$

The above will help in the next problem in order to find recursive relation.

1.2 Problem 3.5.3

> **3.5.3. Generalize Exercise 3.5.2, in order to derive the Fourier sine series of x^m, m odd.**

Result from Last problem showed that

$$x = \sum_{n=1}^{\infty} B_n^1 \sin\left(n \frac{\pi}{L} x\right)$$

$$^1B_n = (-1)^n \left(-\frac{2L}{n\pi}\right)$$

And

$$x^3 = \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^1B_n + (3 \times 2) \left(2 \frac{L}{n^2 \pi^2} \right) \right] \sin\left(n \frac{\pi}{L} x\right)$$

This suggests that

$$x^5 = \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^3B_n + (5 \times 4 \times 3 \times 2) \left(2 \frac{L}{n^2 \pi^2} \right) \right] \sin\left(n \frac{\pi}{L} x\right)$$

$$^3B_n = (-1)^n L^2 \left[{}^1B_n + 6 \left(2 \frac{L}{n^2 \pi^2} \right) \right]$$

And in general

$$x^m = \sum_{n=1}^{\infty} (-1)^n L^2 \left[{}^{m-2}B_n + m! \left(2 \frac{L}{n^2 \pi^2} \right) \right] \sin\left(n \frac{\pi}{L} x\right)$$

Where

$$^{m-2}B_n = (-1)^n L^2 \left[{}^{m-4}B_n + (m-2)! \left(2 \frac{L}{n^2 \pi^2} \right) \right]$$

The above is a recursive definition to find x^m Fourier series for m odd.

1.3 Problem 3.5.7

> ***3.5.7. Evaluate**
>
> $$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$
>
> **using (3.5.6).**

Equation 3.5.6 is

$$\frac{x^2}{2} = \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3^3} + \frac{\sin\frac{5\pi x}{L}}{5^3} + \frac{\sin\frac{7\pi x}{L}}{7^3} + \cdots\right) \tag{3.5.6}$$

Letting $x = \frac{L}{2}$ in (3.5.6) gives

$$\begin{aligned}
\frac{L^2}{8} &= \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi\frac{L}{2}}{L} + \frac{\sin\frac{3\pi\frac{L}{2}}{L}}{3^3} + \frac{\sin\frac{5\pi\frac{L}{2}}{L}}{5^3} + \frac{\sin\frac{7\pi\frac{L}{2}}{L}}{7^3} + \cdots\right) \\
&= \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi}{2} + \frac{\sin 3\frac{\pi}{2}}{3^3} + \frac{\sin 5\frac{\pi}{2}}{5^3} + \frac{\sin 7\frac{\pi}{2}}{7^3} + \cdots\right) \\
&= \frac{L^2}{4} - \frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\cdots\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{L^2}{8} - \frac{L^2}{4} &= -\frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\cdots\right) \\
-\frac{L^2}{8} &= -\frac{4L^2}{\pi^3}\left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\cdots\right) \\
\frac{\pi^3}{4\times 8} &= \left(1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\cdots\right)
\end{aligned}$$

Or

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3}\cdots$$

1.4 Problem 3.6.1

***3.6.1.** Consider

$$f(x) = \begin{cases} 0 & x < x_0 \\ 1/\Delta & x_0 < x < x_0 + \Delta \\ 0 & x > x_0 + \Delta. \end{cases}$$

Assume that $x_0 > -L$ and $x_0 + \Delta < L$. Determine the complex Fourier coefficients c_n.

The function defined above is the Dirac delta function. (in the limit, as $\Delta \to 0$). Now

$$\begin{aligned}
c_n &= \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{in\frac{\pi}{L}x} dx \\
&= \frac{1}{2L}\int_{x_0}^{x_0+\Delta} \frac{1}{\Delta} e^{in\frac{\pi}{L}x} dx \\
&= \frac{1}{2L}\frac{1}{\Delta}\left[\frac{e^{in\frac{\pi}{L}x}}{in\frac{\pi}{L}}\right]_{x_0}^{x_0+\Delta} \\
&= \frac{1}{2L}\frac{L}{\Delta in\pi}\left[e^{in\frac{\pi}{L}x}\right]_{x_0}^{x_0+\Delta} \\
&= \frac{1}{i2n\Delta\pi}\left(e^{in\frac{\pi}{L}(x_0+\Delta)} - e^{in\frac{\pi}{L}x_0}\right)
\end{aligned}$$

Since $\frac{e^{iz}-e^{-iz}}{2i} = \sin z$. The denominator above has $2i$ in it. Factoring out $e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}$ from the above gives

$$\begin{aligned}
c_n &= \frac{1}{i2n\Delta\pi} e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}\left(e^{in\frac{\pi}{L}\frac{\Delta}{2}} - e^{-in\frac{\pi}{L}\frac{\Delta}{2}}\right) \\
&= \frac{1}{n\Delta\pi} e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}\frac{\left(e^{in\frac{\pi}{L}\frac{\Delta}{2}} - e^{-in\frac{\pi}{L}\frac{\Delta}{2}}\right)}{i2}
\end{aligned}$$

Now the form is $\sin(z)$ is obtained, hence it can be written as

$$c_n = \frac{e^{in\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)}}{n\Delta\pi}\sin\left(n\frac{\pi}{L}\frac{\Delta}{2}\right)$$

Or

$$c_n = \frac{\cos\left(n\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)\right) + i\sin\left(n\frac{\pi}{L}\left(x_0+\frac{\Delta}{2}\right)\right)}{\Delta n\pi}\sin\left(n\frac{\pi}{L}\frac{\Delta}{2}\right)$$

1.5 Problem 4.2.1

4.2.1. **(a)** **Using Equation (4.2.7), compute the sagged equilibrium position $u_E(x)$ if $Q(x,t) = -g$. The boundary conditions are $u(O) = 0$ and $u(L) = 0$.**

(b) **Show that $v(x,t) = u(x,t) - u_E(x)$ satisfies (4.2.9).**

1.5.1 Part (a)

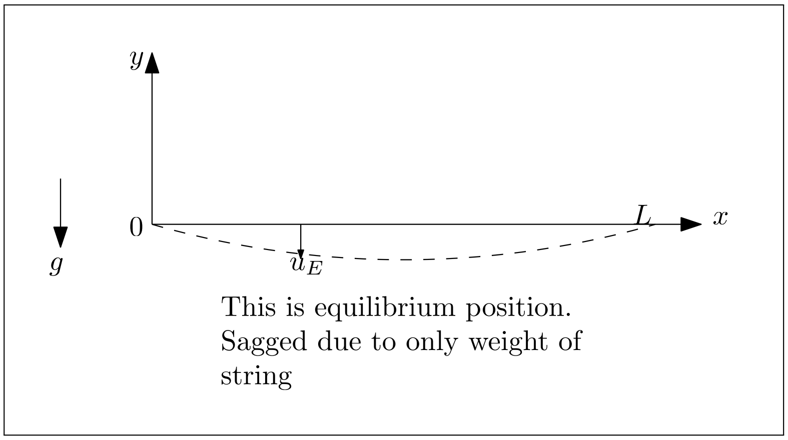
Equation 4.2.7 is

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho(x) \tag{4.2.7}$$

Replacing $Q(x,t)$ by $-g$

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} - g\rho(x)$$

At equilibrium, the string is sagged but is not moving.



Therefore $\frac{\partial^2 u_E}{\partial t^2} = 0$. The above becomes

$$0 = T_0 \frac{\partial^2 u_E}{\partial x^2} - g\rho(x)$$

This is now partial differential equation in only x. It becomes an ODE

$$\frac{d^2 u_E}{dx^2} = \frac{g\rho(x)}{T_0}$$

With boundary conditions $u_E(0) = 0, u_E(L) = 0$. By double integration the solution is found. Integrating once gives

$$\frac{du_E}{dx} = \int_0^x \frac{g\rho(s)}{T_0} ds + c_1$$

Integrating again

$$\begin{aligned} u_E &= \int_0^x \left(\int_0^s \frac{g\rho(z)}{T_0} dz + c_1 \right) ds + c_2 \\ &= \int_0^x \left(\int_0^s \frac{g\rho(z)}{T_0} dz \right) ds + \int_0^x c_1 ds + c_2 \\ &= \frac{g}{T_0} \int_0^x \int_0^s \rho(z)\, dz ds + c_1 x + c_2 \qquad (1) \end{aligned}$$

Equation (1) is the solution. Applying B.C. to find c_1, c_2. At $x = 0$ the above gives

$$0 = c_2$$

The solution (1) becomes

$$u_E = \frac{g}{T_0} \int_0^x \int_0^s \rho(z)\, dz ds + c_1 x \qquad (2)$$

And at $x = L$ the above becomes

$$\begin{aligned} 0 &= \frac{g}{T_0} \int_0^L \int_0^s \rho(z)\, dz ds + c_1 L \\ c_1 &= \frac{-g}{L T_0} \int_0^L \int_0^s \rho(z)\, dz ds \end{aligned}$$

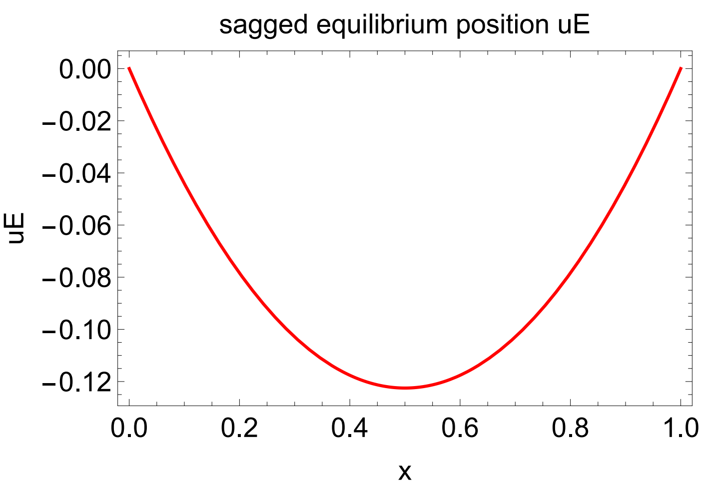
Substituting this into (2) gives the final solution

$$u_E = \frac{g}{T_0}\int_0^x \left(\int_0^s \rho(z)\,dz\right) ds + \left(\frac{-g}{LT_0}\int_0^L \left(\int_0^s \rho(z)\,dz\right) ds\right) x \tag{3}$$

If the density was constant, (3) reduces to

$$\begin{aligned} u_E &= \frac{g\rho}{T_0}\int_0^x s\,ds + \left(\frac{-g\rho}{LT_0}\int_0^L s\,ds\right) x \\ &= \frac{g\rho}{T_0}\frac{x^2}{2} - \frac{g\rho}{LT_0}\frac{L^2}{2}x \\ &= \frac{g\rho}{T_0}\left(\frac{x^2}{2} - \frac{L}{2}x\right) \end{aligned}$$

Here is a plot of the above function for $g = 9.8, L = 1, T_0 = 1, \rho = 0.1$ for verification.



1.5.2 Part (b)

Equation 4.2.9 is

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho(x)}\frac{\partial^2 u}{\partial x^2} \tag{4.2.9}$$

Since

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho(x) \tag{1}$$

And

$$\rho(x)\frac{\partial^2 u_E}{\partial t^2} = T_0\frac{\partial^2 u_E}{\partial x^2} + Q(x,t)\,\rho(x) \tag{2}$$

Then by subtracting (2) from (1)

$$\begin{aligned} \rho(x)\frac{\partial^2 u}{\partial t^2} - \rho(x)\frac{\partial^2 u_E}{\partial t^2} &= T_0\frac{\partial^2 u}{\partial x^2} + Q(x,t)\,\rho(x) - T_0\frac{\partial^2 u_E}{\partial x^2} - Q(x,t)\,\rho(x) \\ \rho(x)\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_E}{\partial t^2}\right) &= T_0\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2}\right) \end{aligned}$$

Since $v(x,t) = u(x,t) - u_E(x,t)$ then $\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u_E}{\partial t^2}$ and $\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2}$, therefore the above equation becomes

$$\begin{aligned}\rho(x)\frac{\partial^2 v}{\partial t^2} &= T_0\frac{\partial^2 v}{\partial x^2}\\ \frac{\partial^2 v}{\partial t^2} &= \frac{T_0}{\rho(x)}\frac{\partial^2 v}{\partial x^2}\\ &= c^2\frac{\partial^2 v}{\partial x^2}\end{aligned}$$

Which is 4.2.9. QED.

1.6 Problem 4.2.5

4.2.5. Derive the partial differential equation for a vibrating string in the simplest possible manner. You may assume the string has constant mass density ρ_0, you may assume the tension T_0 is constant, and you may assume small displacements (with small slopes).

Let us consider a small segment of the string of length Δx from x to $x+\Delta x$. The mass of this segment is $\rho\Delta x$, where ρ is density of the string per unit length, assumed here to be constant. Let the angle that the string makes with the horizontal at x and at $x+\Delta x$ be $\theta(x,t)$ and $\theta(x+\Delta x,t)$ respectively. Since we are only interested in the vertical displacement $u(x,t)$ of the string, the vertical force on this segment consists of two parts: Its weight (acting downwards) and the net tension resolved in the vertical direction. Let the total vertical force be F_y. Therefore

$$F_y = \overbrace{-\rho\Delta x g}^{\text{weight}} + \overbrace{\left(T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)\right)}^{\text{net tension on segment in vertical direction}}$$

Applying Newton's second law in the vertical direction $F_y = ma_y$ where $a_y = \frac{\partial^2 u(x,t)}{\partial t^2}$ and $m = \rho\Delta x$, gives the equation of motion of the string segment in the vertical direction

$$\rho\Delta x\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho\Delta x g + \left(T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)\right)$$

Dividing both sides by Δx

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\left(T(x+\Delta x)\sin\theta(x+\Delta x,t) - T(x)\sin\theta(x,t)\right)}{\Delta x}$$

Taking the limit $\Delta x \to 0$

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\partial}{\partial x}\left(T(x,t)\sin\theta(x,t)\right)$$

Assuming small angles then $\frac{\partial u}{\partial x} = \tan\theta = \frac{\sin\theta}{\cos\theta} \approx \sin\theta$, then we can replace $\sin\theta$ in the above with $\frac{\partial u}{\partial x}$ giving

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + \frac{\partial}{\partial x}\left(T(x,t)\frac{\partial u(x,t)}{\partial x}\right)$$

Assuming tension $T(x,t)$ is constant, say T_0 then the above becomes

$$\rho\frac{\partial^2 u(x,t)}{\partial t^2} = -\rho g + T_0\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\right)$$
$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{T_0}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} - \rho g$$

Setting $\frac{T_0}{\rho} = c^2$ then the above becomes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2} - \rho g$$

Note: In the above g (gravity acceleration) was used instead of $Q(x,t)$ as in the book to represent the body forces. In other words, the above can also be written as

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2} + \rho Q(x,t)$$

This is the required PDE, assuming constant density, constant tension, small angles and small vertical displacement.

1.7 Problem 4.4.1

4.4.1. Consider vibrating strings of uniform density ρ_0 and tension T_0.

***(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?**

***(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.4.1.**

(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, \ldots$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

1.7.1 Part (a)

The natural frequencies of vibrating string of length L with fixed ends, is given by equation 4.4.11 in the book, which is the solution to the string wave equation

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(n\frac{\pi}{L}x\right)\left(A_n\cos\left(n\frac{\pi c}{L}t\right) + B_n\sin\left(n\frac{\pi c}{L}t\right)\right)$$

The frequency of the time solution part of the PDE is given by the arguments of eigenfucntions $A_n\cos\left(n\frac{\pi c}{L}t\right) + B_n\sin\left(n\frac{\pi c}{L}t\right)$. Therefore $n\frac{\pi c}{L}$ represents the circular frequency ω_n. Comparing general form of $\cos\omega t$ with $\cos\left(n\frac{\pi c}{L}t\right)$ we see that each mode n has circular frequency given by

$$\omega_n \equiv n\frac{\pi c}{L}$$

For $n = 1, 2, 3, \cdots$. In cycles per seconds (Hertz), and since $\omega = 2\pi f$, then $2\pi f = n\frac{\pi c}{L}$. Solving for f gives

$$\begin{aligned} f_n &= n\frac{\pi c}{2\pi L} \\ &= n\frac{c}{2L} \end{aligned}$$

Where $c = \sqrt{\frac{T_0}{\rho_0}}$ in all of the above.

1.7.2 Part (b)

Equation 4.4.11 above was for a string with fixed ends. Now the B.C. are different, so we need to solve the spatial equation again to find the new eigenvalues. Starting with $u = X(x)T(t)$ and substituting this in the PDE $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ with $0 < x < H$ gives

$$\begin{aligned} T''X &= c^2 TX'' \\ \frac{1}{c^2}\frac{T''}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Where both sides are set equal to some constant $-\lambda$. We now obtain the two ODE's to solve. The spatial ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(H) &= 0 \end{aligned}$$

And the time ODE is

$$T'' + \lambda c^2 T = 0$$

The eigenvalues will always be positive for the wave equation. Taking $\lambda > 0$ the solution to the space ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Applying first B.C. gives

$$0 = A$$

Hence $X(x) = B\sin\left(\sqrt{\lambda}x\right)$ and $X'(x) = -B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$. Applying second B.C. gives

$$0 = -B\sqrt{\lambda}\cos\left(\sqrt{\lambda}H\right)$$

Therefore for non-trivial solution, we want $\sqrt{\lambda}H = \frac{n}{2}\pi$ for $n = 1, 3, 5, \cdots$ or written another way

$$\sqrt{\lambda}H = \left(n - \frac{1}{2}\right)\pi \qquad n = 1, 2, 3, \cdots$$

Therefore

$$\lambda_n = \left(\left(n - \frac{1}{2}\right)\frac{\pi}{H}\right)^2 \qquad n = 1, 2, 3, \cdots$$

These are the eigenvalues. Now that we know what λ_n is, we go back to the solution found before, which is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\sqrt{\lambda_n}x\right)\left(A_n \cos\left(\sqrt{\lambda_n}ct\right) + B_n \sin\left(\sqrt{\lambda_n}ct\right)\right)$$

And see now that the circular frequency ω_n is given by

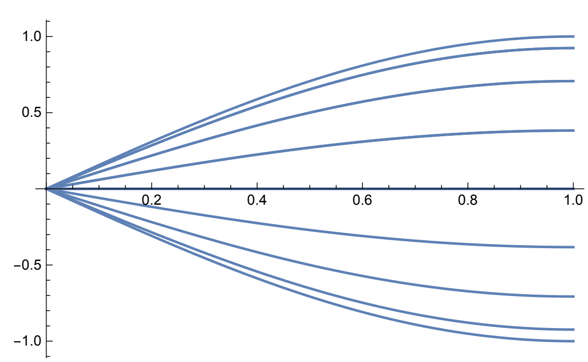
$$\begin{aligned}\omega_n &= \sqrt{\lambda_n}c \\ &= \frac{\left(n - \frac{1}{2}\right)\pi}{H}c \qquad n = 1,2,3,\cdots\end{aligned}$$

In cycles per second, since $\omega = 2\pi f$ then

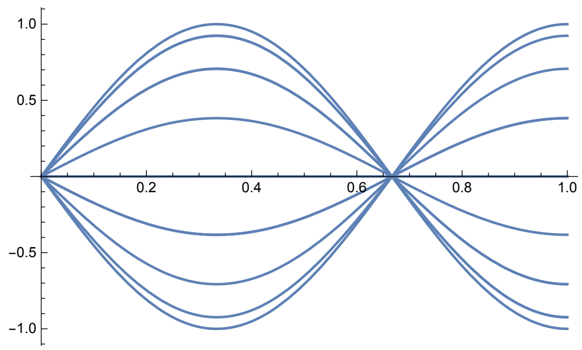
$$\begin{aligned}2\pi f_n &= \frac{\left(n - \frac{1}{2}\right)\pi}{H}c \\ f_n &= \frac{\left(n - \frac{1}{2}\right)}{2H}c \qquad n = 1,2,3,\cdots\end{aligned}$$

The following are plots for $n = 1,2,3,4,5$ for $t = 0\cdots 3$ seconds by small time increments.

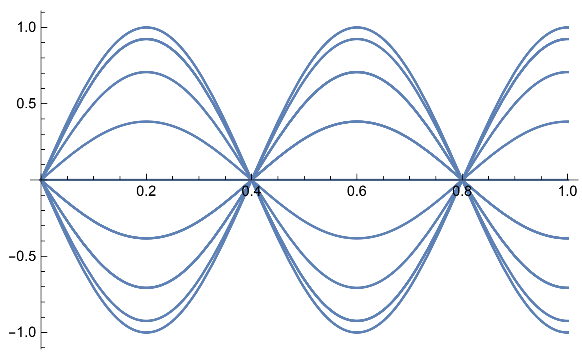
```
(*solution for HW 5, problem 4.4.1*)
f[x_, n_, t_] := Module[{H0 = 1, c = 1, lam},
lam = ((n - 1/2) Pi/H0);
Sin[lam x] (Sin[lam c t])
] ;
Table[Plot[f[x, 1, t], {x, 0, 1}, AxesOrigin -> {0, 0}], {t, 0,3, .25}];
p = Labeled[Show[
```



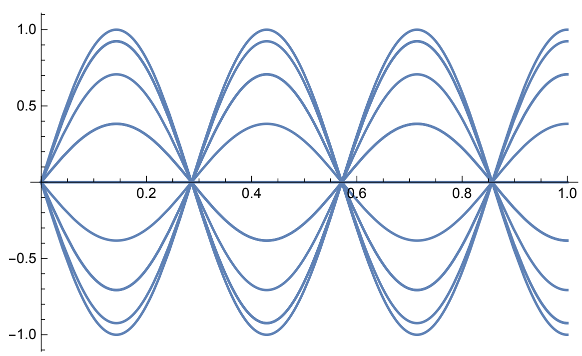
Mode n=1 vibration, from t=0 to t=3 seconds



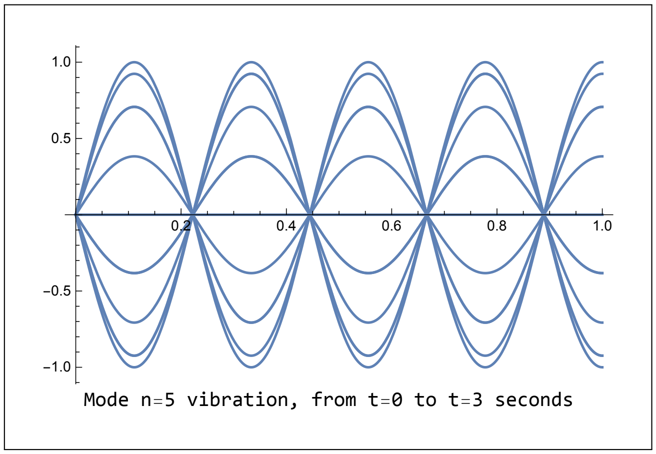
Mode n=2 vibration, from t=0 to t=3 seconds



Mode n=3 vibration, from t=0 to t=3 seconds



Mode n=4 vibration, from t=0 to t=3 seconds



1.7.3 Part (c)

For part (a), the harmonics had circular frequency $\omega_n = \frac{n\pi}{L}c$. Hence for odd n, these will generate

$$\frac{\pi}{L}c, 3\frac{\pi}{L}c, 5\frac{\pi}{L}c, 7\frac{\pi}{L}c, \cdots \tag{1}$$

For part (b), $\omega_n = \frac{\left(n-\frac{1}{2}\right)\pi}{H}c$. When $H = \frac{L}{2}$, this becomes $\omega_n = \frac{2\left(n-\frac{1}{2}\right)\pi}{L}c$. Looking at the first few modes gives

$$\begin{gathered}\frac{2\left(1-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(2-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(3-\frac{1}{2}\right)\pi}{L}c, \frac{2\left(4-\frac{1}{2}\right)\pi}{L}c, \cdots\\ \frac{\pi}{L}c, \frac{3\pi}{L}c, \frac{5\pi}{L}c, \frac{7\pi}{L}c, \cdots\end{gathered} \tag{2}$$

Comparing (1) and (2) we see they are the same. Which is what we asked to show.

1.8 Problem 4.4.3

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small $(\beta^2 < 4\pi^2\rho_0 T_0/L^2)$.

1.8.1 Part (a)

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

The term $-\beta \frac{\partial u}{\partial t}$ is the force that acts on the spring segment due to damping. This is the Viscous damping force which is proportional to speed, where β represents viscous damping coefficient. This damping force always opposes the direction of the motion. Hence if $\frac{\partial u}{\partial t} > 0$ then $-\beta \frac{\partial u}{\partial t}$ should come out to be negative. This occurs if $\beta > 0$. On the other hand, if $\frac{\partial u}{\partial t} < 0$ then $-\beta \frac{\partial u}{\partial t}$ should now be positive. Which means again that β must be positive quantity. Hence only case were the damping force always opposes the motion of the string is when $\beta > 0$.

1.8.2 Part (b)

Starting with $u = X(x)T(t)$ and substituting this in the above PDE with $0 < x < L$ gives

$$\begin{aligned} \rho_0 T'' X &= T_0 T X'' - \beta T' X \\ \frac{\rho_0}{T_0}\frac{T''}{T} + \frac{\beta}{T_0}\frac{T'}{T} &= \frac{X''}{X} = -\lambda \end{aligned}$$

Hence we obtain two ODE's. The space ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

And the time ODE is

$$\begin{aligned} T'' + c^2 \beta T' + c^2 \lambda T &= 0 \\ T(0) &= f(x) \\ T'(0) &= g(x) \end{aligned}$$

The eigenvalues will always be positive for the wave equation. Hence taking $\lambda > 0$ the solution to the space ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Applying first B.C. gives

$$0 = A$$

Hence $X = B\sin\left(\sqrt{\lambda}x\right)$. Applying the second B.C. gives

$$0 = B\sin\left(\sqrt{\lambda}L\right)$$

Therefore

$$\begin{aligned} \sqrt{\lambda}L &= n\pi \qquad n = 1,2,3,\cdots \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots \end{aligned}$$

Hence the space solution is

$$X = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \tag{1}$$

Now we solve the time ODE. This is second order ODE, linear, with constant coefficients.

$$\frac{\rho_0}{T_0}\frac{T''}{T} + \frac{\beta}{T_0}\frac{T'}{T} = -\lambda$$
$$\frac{\rho_0}{T_0}T'' + \frac{\beta}{T_0}T' + \lambda T = 0$$
$$T'' + \frac{\beta}{\rho_0}T' + \frac{T_0}{\rho_0}\lambda T = 0$$

Where in the above $\lambda \equiv \lambda_n$ for $n = 1,2,3,\cdots$. The characteristic equation is $r^2 + c^2\beta r + c^2\lambda = 0$. The roots are found from the quadratic formula

$$\begin{aligned} r_{1,2} &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{-\frac{\beta}{\rho_0} \pm \sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\lambda}}{2} \\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\lambda} \end{aligned}$$

Replacing $\lambda = \left(\frac{n\pi}{L}\right)^2$, gives

$$\begin{aligned} r_{1,2} &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\left(\frac{\beta}{\rho_0}\right)^2 - 4\frac{T_0}{\rho_0}\left(\frac{n\pi}{L}\right)^2} \\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2}\sqrt{\frac{\beta^2}{\rho_0^2} - 4\frac{T_0}{\rho_0}\frac{n^2\pi^2}{L^2}} \\ &= -\frac{\beta}{2\rho_0} \pm \frac{1}{2\rho_0}\sqrt{\beta^2 - n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right)} \end{aligned}$$

We are told that $\beta^2 < 4\rho_0 T_0\frac{\pi^2}{L^2}$, what this means is that $\beta^2 - n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right) < 0$, since $n^2 > 0$. This means we will get complex roots. Let

$$\Delta = n^2\left(4\rho_0 T_0\frac{\pi^2}{L^2}\right) - \beta^2$$

Hence the roots can now be written as

$$r_{1,2} = -\frac{\beta}{2\rho_0} \pm \frac{i\sqrt{\Delta}}{2\rho_0}$$

Therefore the time solution is

$$T_n(t) = e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right) + B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)$$

This is sinusoidal damped oscillation. Therefore

$$T(t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right) + B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right) \tag{2}$$

Combining (1) and (2), gives the total solution

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right) \tag{3}$$

Where b_n constants for space ODE merged with the constants A_n, B_n for the time solution. Now we are ready to find A_n, B_n from initial conditions. At $t=0$

$$f(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)A_n$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx=\int_0^L\sum_{n=1}^{\infty}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)A_n dx$$

Changing the order of integration and summation

$$\begin{aligned}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx&=\sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx\\&=A_m\frac{L}{2}\end{aligned}$$

Hence

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

To find B_n, we first take time derivative of the solution above in (3) which gives

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t)=&\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(-\frac{\sqrt{\Delta}}{2\rho_0}A_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\frac{\sqrt{\Delta}}{2\rho_0}\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)\\&-\frac{\beta}{2\rho_0}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{\beta}{2\rho_0}t}\left(A_n\cos\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)+B_n\sin\left(\frac{\sqrt{\Delta}}{2\rho_0}t\right)\right)\end{aligned}$$

At $t=0$, using the second initial condition gives

$$g(x)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{L}x\right)B_n\frac{\sqrt{\Delta}}{2\rho_0}-\frac{\beta}{2\rho_0}A_n\sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx=\int_0^L\sum_{n=1}^{\infty}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)B_n\frac{\sqrt{\Delta}}{2\rho_0}dx-\sum_{n=1}^{\infty}\frac{\beta}{2\rho_0}A_n\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)$$

Changing the order of integration and summation

$$\begin{aligned}\int_0^L g(x)\sin\left(\frac{m\pi}{L}x\right)dx&=\sum_{n=1}^{\infty}B_n\frac{\sqrt{\Delta}}{2\rho_0}\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx-\sum_{n=1}^{\infty}\frac{\beta}{2\rho_0}A_n\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\\&=B_m\frac{\sqrt{\Delta}}{2\rho_0}\frac{L}{2}-\frac{\beta}{2\rho_0}A_n\frac{L}{2}\\&=\frac{L}{2}\left(B_m\frac{\sqrt{\Delta}}{2\rho_0}-\frac{\beta}{2\rho_0}A_n\right)\end{aligned}$$

Hence

$$B_m \frac{\sqrt{\Delta}}{2\rho_0} - \frac{\beta}{2\rho_0} A_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{m\pi}{L} x\right) dx$$

$$B_m = \left(\frac{2}{L} \int_0^L g(x) \sin\left(\frac{m\pi}{L} x\right) dx + \frac{\beta}{2\rho_0} A_n\right) \frac{2\rho_0}{\sqrt{\Delta}}$$

This completes the solution. Summary of solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) e^{-\frac{\beta}{2\rho_0} t} \left(A_n \cos\left(\frac{\sqrt{\Delta}}{2\rho_0} t\right) + B_n \sin\left(\frac{\sqrt{\Delta}}{2\rho_0} t\right)\right)$$

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$B_n = \left(\frac{2}{L} \int_0^L g(x) \sin\left(\frac{m\pi}{L} x\right) dx + \frac{\beta}{2\rho_0} A_n\right) \frac{2\rho_0}{\sqrt{\Delta}}$$

$$\Delta = n^2 \left(4\rho_0 T_0 \frac{\pi^2}{L^2}\right) - \beta^2$$

1.9 Problem 4.4.9

4.4.9 **From (4.4.1), derive conservation of energy for a vibrating string,**

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}\Big|_0^L, \qquad (4.4.15)$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x}\right)^2 dx$.

$$E = \frac{1}{2} \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

Hence

$$\frac{dE}{dt} = \frac{1}{2} \frac{d}{dt} \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2} \frac{d}{dt} \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$$

Moving $\frac{d}{dt}$ inside the integral, it becomes partial derivative

$$\frac{dE}{dt} = \frac{1}{2} \int_0^L \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2} \int_0^L \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x}\right)^2 dx \qquad (1)$$

But

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \frac{\partial u}{\partial t}\right) = \frac{\partial^2 u}{\partial t^2} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2} = 2\left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2}\right) \qquad (2)$$

And

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x \partial t} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} = 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x \partial t} \qquad (3)$$

Substituting (2,3) into (1) gives

$$\begin{aligned}\frac{dE}{dt} &= \frac{1}{2}\int_0^L 2\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right)dx + \frac{c^2}{2}\int_0^L 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= \int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx\end{aligned}$$

But $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ then the above becomes

$$\begin{aligned}\frac{dE}{dt} &= \int_0^L \left(\frac{\partial u}{\partial t}\left[c^2\frac{\partial^2 u}{\partial x^2}\right]\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right)dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}dx \\ &= c^2\int_0^L \left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}\right) + \left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\right)dx \qquad (4)\end{aligned}$$

But since the integrand in (4) can also be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x\partial t}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x^2}$$

Then (4) becomes

$$\begin{aligned}\frac{dE}{dt} &= c^2\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)dx \\ &= c^2\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right)_0^L\end{aligned}$$

Which is what we are asked to show. QED.