HW 12, (wave PDE) Math 322, Fall 2016

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1 HW 12

1.1 Problem 12.2.1

Show that the wave equation can be considered as the following system of two coupled first-order PDE

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w \tag{1}$$

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \tag{2}$$

Answer

The wave PDE in 1D is $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$. Taking time derivative of equation (1) gives (assuming c is constant)

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial w}{\partial t} \tag{3}$$

Taking space derivative of equation (1) gives (assuming c is constant)

$$\frac{\partial^2 u}{\partial t \partial x} - c\frac{\partial^2 u}{\partial x^2} = \frac{\partial w}{\partial x} \tag{4}$$

Multiplying (4) by c

$$c\frac{\partial^2 u}{\partial t \partial x} - c^2\frac{\partial^2 u}{\partial x^2} = c\frac{\partial w}{\partial x} \tag{5}$$

Adding (3)+(5) gives

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial t \partial x} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x}$$

But the RHS of the above is zero, since it is equation (2). Therefore the above reduces to

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

Which is the wave PDE.

1.2 Problem 12.2.2

Solve

$$\frac{\partial w}{\partial t} - 3\frac{\partial w}{\partial x} = 0 \tag{1}$$

with $w(x, 0) = \cos x$

Answer

Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = -3$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 0$. Therefore, $w(x(t), t)$ is constant. At time $t = 0$, we are given that

$$w(x(0), t) = \cos x(0) \qquad t = 0 \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt} = -3$, which has the solution $x = x(0) - 3t$. Hence $x(0) = x + 3t$. Therefore (3) becomes

$$w(x(t), t) = \cos(x + 3t)$$

1.3 Problem 12.2.3

Solve

$$\frac{\partial w}{\partial t} + 4\frac{\partial w}{\partial x} = 0 \tag{1}$$

with $w(0,t) = \sin 3t$

Answer

Let

$$w \equiv w(x, t(x))$$

Hence

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\frac{dt}{dx} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dt}{dx} = \frac{1}{4}$ in (2), then we obtain (1). Hence we conclude that $\frac{dw}{dx} = 0$. Therefore, $w(x, t(x))$ is constant. At $x = 0$, we are given that

$$w(x, t(0)) = \sin(3t(0)) \qquad x = 0 \tag{3}$$

We just now need to determine $t(0)$. This is found from $\frac{dt}{dx} = \frac{1}{4}$, which has the solution $t(x) = t(0) + \frac{1}{4}x$. Hence $t(0) = t(x) - \frac{1}{4}x$. Therefore (3) becomes

$$\begin{aligned} w(x, t(x)) &= \sin\left(3\left(t(x) - \frac{1}{4}x\right)\right) \\ &= \sin\left(3t - \frac{3}{4}x\right) \end{aligned}$$

1.4 Problem 12.2.4

Solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \tag{1}$$

with $c > 0$ and

$$\begin{aligned} w(x,0) &= f(x) \qquad x > 0 \\ w(0,t) &= h(t) \qquad t > 0 \end{aligned}$$

Answer

Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = c$ in (2), then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 0$. Therefore, $w(x(t), t)$ is constant. At $t = 0$, we are given that

$$w(x(t), t) = f(x(0)) \qquad t = 0 \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt} = c$, which has the solution $x(t) = x(0) + ct$. Hence $x(0) = x(t) - ct$. Therefore (3) becomes

$$w(x,t) = f(x - ct)$$

This is valid for $x > ct$. We now start all over again, and look at Let

$$w \equiv w(x, t(x))$$

Hence

$$\frac{dw}{dx} = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial t}\frac{dt}{dx} \tag{4}$$

Comparing (4) and (1), we see that if we let $\frac{dt}{dx} = \frac{1}{c}$ in (4), then we obtain (1). Hence we conclude that $\frac{dw}{dx} = 0$. Therefore, $w(x, t(x))$ is constant. At $x = 0$, we are given that

$$w(x, t(x)) = h(t(0)) \qquad x = 0 \tag{5}$$

We just now need to determine $t(0)$. This is found from $\frac{dt}{dx} = \frac{1}{c}$, which has the solution $t(x) = t(0) + \frac{1}{c}x$. Hence $t(0) = t(x) - \frac{1}{c}x$. Therefore (5) becomes

$$w(x,t) = h\left(t - \frac{1}{c}x\right)$$

Valid for $t > \frac{x}{c}$ or $x < ct$. Therefore, the solution is

$$w(x,t) = \begin{cases} f(x-ct) & x > ct \\ h\left(t - \frac{1}{c}x\right) & x < ct \end{cases}$$

1.5 Problem 12.2.5

1.5.1 Part (a)

Solve

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = e^{2x} \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = c$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = e^{2x}$. Hence

$$w = w(0) + te^{2x}$$

At $t = 0$, $w(0) = f(x(0))$, hence

$$w = f(x(0)) + te^{2x} \tag{3}$$

We just now need to determine $x(0)$. This is found from $\frac{dx}{dt} = c$, which has the solution $x = x(0) + ct$. Hence $x(0) = x - ct$. Therefore (3) becomes

$$w(x(t), t) = f(x - ct) + te^{2x}$$

1.5.2 Part (b)

Solve

$$\frac{\partial w}{\partial t} + x\frac{\partial w}{\partial x} = 1 \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = x$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 1$. Hence

$$w = w(0) + t$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0)) + t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = x$, the solution is $\ln|x| = t + x(0)$ or $x = x(0)e^t$. Hence $x(0) = xe^{-t}$ and the above becomes

$$w = f\left(xe^{-t}\right) + t$$

1.5.3 Part (c)

Solve

$$\frac{\partial w}{\partial t} + t\frac{\partial w}{\partial x} = 1 \tag{1}$$

with $w(x,0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = t$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = 1$. Hence

$$w = w(0) + t$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0)) + t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = t$, the solution is $x = x(0) + \frac{t^2}{2}$. Hence $x(0) = x - \frac{t^2}{2}$ and the above becomes

$$w = f\left(x - \frac{t^2}{2}\right) + t$$

1.5.4 Part (d)

Solve

$$\frac{\partial w}{\partial t} + 3t\frac{\partial w}{\partial x} = w \tag{1}$$

with $w(x, 0) = f(x)$

Answer Let

$$w \equiv w(x(t), t)$$

Hence

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x}\frac{dx}{dt} \tag{2}$$

Comparing (2) and (1), we see that if we let $\frac{dx}{dt} = 3t$ in the above, then we obtain (1). Hence we conclude that $\frac{dw}{dt} = w$. Hence

$$\ln|w| = w(0) + t$$
$$w = w(0)e^t$$

At $t = 0$, $w(0) = f(x(0))$, hence the above becomes

$$w = f(x(0))e^t$$

We now need to find $x(0)$. From $\frac{dx}{dt} = 3t$, the solution is $x = x(0) + \frac{3t^2}{2}$. Hence $x(0) = x - \frac{3t^2}{2}$ and the above becomes

$$w = f\left(x - \frac{3t^2}{2}\right)e^t$$