Lecture Nov 30, 2016, Math 319. Complete derivation of class example

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This is complete solution of class example (example 2). Math 319, lecture Nov. 30. 2016.

Solve the differential equation

$$2y''(t) + y'(t) + 2y(t) = g(t)$$
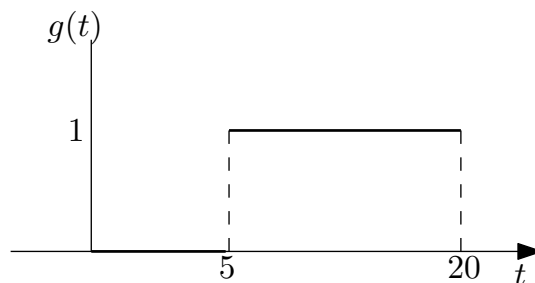$$y(0) = 0$$
$$y'(0) = 0$$

Where

$$g(t) = \begin{cases} 1 & 5 \leq t < 20 \\ 0 & \text{otherwise} \end{cases}$$

Using Laplace transform method.

Solution

The first step is to find the Laplace transform of the forcing function $g(t)$. The function $g(t)$ is

We now write $g(t)$ in terms of the unit step function $u_c(t)$ defined as $u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$ as follows

$$g(t) = u_5(t) - u_{20}(t) \tag{1}$$

Now we use the property that

$$\mathscr{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathscr{L}\{f(t)\}$$

To obtain the Laplace transform of $g(t)$ in (1) as follows

$$\mathscr{L}\{g(t)\} = \mathscr{L}\{u_5(t)\} - \mathscr{L}\{u_{20}(t)\}$$
$$= e^{-5s} \mathscr{L}\{1\} - e^{-20s} \mathscr{L}\{1\}$$

But $\mathscr{L}\{1\} = \frac{1}{s}$, hence the above becomes

$$\mathscr{L}\{g(t)\} = e^{-5s}\frac{1}{s} - e^{-20s}\frac{1}{s}$$
$$= \frac{e^{-5s} - e^{-20s}}{s}$$

Now that we found $\mathscr{L}\{g(t)\}$, we go back to the original ODE and take the Laplace transform of the ODE, which results in

$$\mathscr{L}\{2y''(t)\} + \mathscr{L}\{y'(t)\} + \mathscr{L}\{2y(t)\} = \mathscr{L}\{g(t)\}$$

Let $Y(s) = \mathscr{L}\{y(t)\}$, then the above becomes

$$2\left\{s^2 Y(s) - sy(0) - y'(0)\right\} + \left\{sY(s) - y(0)\right\} + 2Y(s) = \mathscr{L}\{g(t)\}$$

But $y(0) = y'(0) = 0$ and the above reduces to

$$2s^2 Y(s) + sY(s) + 2Y(s) = \frac{e^{-5s} - e^{-20s}}{s}$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-5s} - e^{-10s}}{s\left(2s^2 + s + 2\right)}$$

$$= \left(\frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} - \frac{e^{-20s}}{s\left(2s^2 + s + 2\right)}\right) \tag{2}$$

We now need to find the inverse Laplace transform of $Y(s)$. Looking at $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$, the first step is to use the property

$$u_c(t)\, f(t - c) \overset{\mathscr{L}}{\Longleftrightarrow} e^{-cs} F(s)$$

Comparing the expressions, we see that

$$u_5(t)\, f(t - c) \Longleftrightarrow \frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} \tag{3}$$

Where

$$f(t) \Longleftrightarrow \frac{1}{s\left(2s^2 + s + 2\right)} \tag{4}$$

Therefore, we just need to find inverse Laplace transform of $\frac{1}{s\left(2s^2+s+2\right)}$. Using partial fractions

$$\frac{1}{s\left(2s^2 + s + 2\right)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2} \tag{5}$$

$$1 = A\left(2s^2 + s + 2\right) + (Bs + C)\, s$$

$$1 = 2As^2 + As + 2A + Bs^2 + Cs$$

$$1 = 2A + s\left(A + C\right) + s^2\left(2A + B\right)$$

Therefore

$$A = \frac{1}{2}$$

$$A + C = 0$$

$$2A + B = 0$$

Hence from the second equation $C = -\frac{1}{2}$, and from the third equation $B = -1$ Therefore (5) becomes

$$\frac{1}{s\left(2s^2 + s + 2\right)} = \frac{1}{2}\frac{1}{s} + \frac{-s - \frac{1}{2}}{2s^2 + s + 2}$$

$$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{-1 - 2s}{2s^2 + s + 2}$$

$$= \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2 + s + 2} - \frac{1}{2}\frac{1}{2s^2 + s + 2} \tag{5A}$$

The first term above is easy, we know that

$$\frac{1}{2}\frac{1}{s} \Longleftrightarrow \frac{1}{2} \tag{6}$$

Now we will find inverse Laplace transform of second term in (5A) $\frac{s}{2s^2+s+2}$. For this we start by completing the squares in the denominator. Let

$$2s^2 + s + 2 = a\left(s + b\right)^2 + c$$

$$= a\left(s^2 + b^2 + 2bs\right) + c$$

$$= as^2 + ab^2 + 2bas + c$$

Hence $a = 2, 2ab = 1$ or $b = \frac{1}{4}$ and $ab^2 + c = 2$, hence $c = 2 - 2\left(\frac{1}{4}\right)^2 = 2 - 2\left(\frac{1}{16}\right) = 2 - \frac{1}{8} = \frac{15}{8}$, Therefore

$$2s^2 + s + 2 = 2\left(s + \frac{1}{4}\right)^2 + \frac{15}{8}$$

We now re-write second term in (5A), which is $\frac{s}{2s^2+s+2}$ as $\frac{s}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}}$. We did this because we wanted this to be in the form $\frac{s}{s^2+a}$, therefore

$$\frac{s}{2\left(s + \frac{1}{4}\right)^2 + \frac{15}{8}} = \frac{1}{2}\frac{s}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

Now we let $\tilde{s} = s + \frac{1}{4}$, therefore the above becomes

$$\frac{1}{2}\frac{\tilde{s} - \frac{1}{4}}{\tilde{s}^2 + \frac{15}{16}} = \frac{1}{2}\left(\frac{\tilde{s}}{\tilde{s}^2 + \frac{15}{16}} - \frac{1}{4}\frac{1}{\tilde{s}^2 + \frac{15}{16}}\right) \tag{7}$$

Using $\frac{s}{s^2+a^2} \Leftrightarrow \cos(at)$ then

$$\frac{\tilde{s}}{\tilde{s}^2 + \frac{15}{16}} \Leftrightarrow e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right)$$

The reason for $e^{-\frac{t}{4}}$ being there, is because we evaluated $F(s)$ at $F\left(s+\frac{1}{4}\right)$. This used the shift property

$$F(s+a) = e^{-at}f(t)$$

Therefore $F\left(s+\frac{1}{4}\right) = e^{-\frac{t}{4}}f(t)$. Now we do the second term in (7). Since $\frac{1}{\tilde{s}^2+\frac{15}{16}} = \frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}}$, then, now using $\frac{a}{s^2+a^2} \Leftrightarrow \sin(at)$ we obtain

$$\frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}} \Leftrightarrow \frac{1}{\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right)$$

And we remember to add $e^{-\frac{t}{4}}$ again, due to the shift in s. Therefore (7) becomes

$$\frac{s}{2s^2+s+2} \Leftrightarrow \frac{1}{2}\left(e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right) - \frac{1}{4}e^{-\frac{t}{4}}\frac{1}{\sqrt{\frac{15}{16}}}\sin\left(\sqrt{\frac{15}{16}}t\right)\right)$$

$$= \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) \tag{8}$$

This complete the second term in (5A). Now we will do the third term in (5A) which is $\frac{1}{2s^2+s+2}$ which is

$$\frac{1}{2s^2+s+2} = \frac{1}{2\left(s+\frac{1}{4}\right)^2 + \frac{15}{8}}$$

$$= \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

$$= \frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

Hence

$$\frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \Longleftrightarrow \frac{1}{2\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right)$$

$$= \frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right) \tag{9}$$

Now we put all the results back together.

$$\frac{1}{s\left(2s^2+s+2\right)} = \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2}$$

$$\Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) - \frac{1}{2}\left(\frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$

We can simplify this more

$$\frac{1}{s\left(2s^2+s+2\right)} \Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right) + 2\sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$

$$= \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) + \sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$

Using this back in (2), where we want to evaluates $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$, gives

$$\frac{e^{-5s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_5(t)f(t-5)$$

Where

$$f(t-5) = \frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)$$

The above complete the first term in (2). The second term in (2) is the same, but the delay now is 20 instead of 5. Hence

$$\frac{e^{-20s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_{20}(t)f(t-20)$$

With the same function $f(t)$ found above. Therefore, the final inverse transform now is

$$y(t) \Longleftrightarrow \left(\frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} - \frac{e^{-20s}}{s\left(2s^2 + s + 2\right)} \right)$$

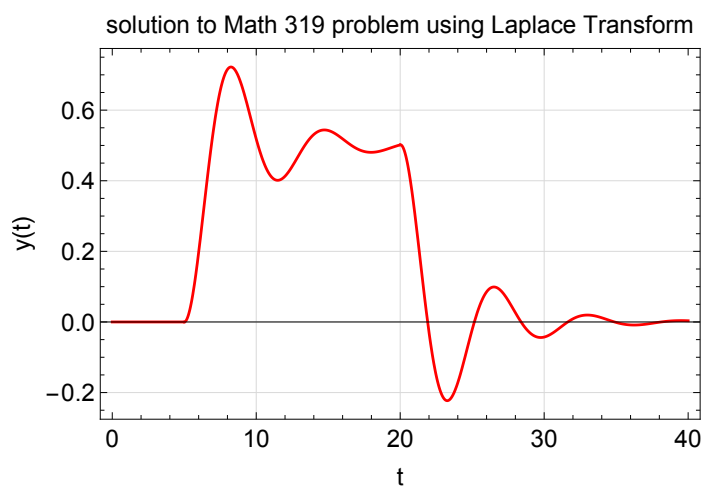$$= \left(u_5(t) f(t-5) - u_{20}(t) f(t-20) \right)$$

Where

$$f(t-20) = \frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}} \left(\sqrt{15} \cos\left(\frac{\sqrt{15}}{4}(t-20) \right) + \sin\left(\frac{\sqrt{15}}{4}(t-20) \right) \right)$$

This complete the solution. The final solution is

$$y(t) = u_5(t) \left[\frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}} \left(\sqrt{15} \cos\left(\frac{\sqrt{15}}{4}(t-5) \right) + \sin\left(\frac{\sqrt{15}}{4}(t-5) \right) \right) \right]$$

$$- u_{20}(t) \left[\frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}} \left(\sqrt{15} \cos\left(\frac{\sqrt{15}}{4}(t-20) \right) + \sin\left(\frac{\sqrt{15}}{4}(t-20) \right) \right) \right]$$

Here is a plot of the above solution

solution to Math 319 problem using Laplace Transform

References

1. Lecture notes Nov. 30, 2016 by Professor Minh-Binh Tran. Math dept. Univ. Of Wisconsin Madison.

2. Wikipedia web page on Laplace transform properties.