Lecture Nov 30, 2016, Math 319. Complete derivation of class example

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This is complete solution of class example (example 2). Math 319, lecture Nov. 30. 2016.

Solve the differential equation

$$\begin{aligned} 2y''(t) + y'(t) + 2y(t) &= g(t) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

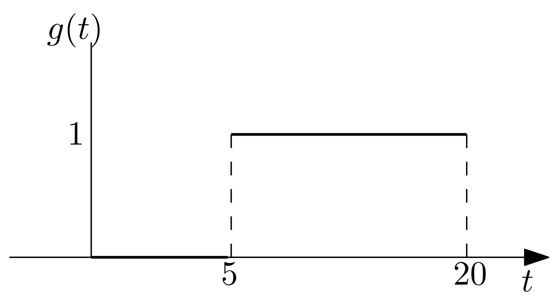
Where

$$g(t) = \begin{cases} 1 & 5 \leq t < 20 \\ 0 & \text{otherwise} \end{cases}$$

Using Laplace transform method.

Solution

The first step is to find the Laplace transform of the forcing function $g(t)$. The function $g(t)$ is



We now write $g(t)$ in terms of the unit step function $u_c(t)$ defined as $u_c = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$ as follows

$$g(t) = u_5(t) - u_{20}(t) \tag{1}$$

Now we use the property that

$$\mathscr{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs}\mathscr{L}\left\{f(t)\right\}$$

To obtain the Laplace transform of $g(t)$ in (1) as follows

$$\begin{aligned} \mathscr{L}\left\{g(t)\right\} &= \mathscr{L}\{u_5(t)\} - \mathscr{L}\{u_{20}(t)\} \\ &= e^{-5s}\mathscr{L}\{1\} - e^{-20s}\mathscr{L}\{1\} \end{aligned}$$

But $\mathscr{L}\{1\} = \frac{1}{s}$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{g(t)\right\} &= e^{-5s}\frac{1}{s} - e^{-20s}\frac{1}{s} \\ &= \frac{e^{-5s} - e^{-20s}}{s}\end{aligned}$$

Now that we found $\mathscr{L}\left\{g(t)\right\}$, we go back to the original ODE and take the Laplace transform of the ODE, which results in

$$\mathscr{L}\left\{2y''(t)\right\} + \mathscr{L}\left\{y'(t)\right\} + \mathscr{L}\left\{2y(t)\right\} = \mathscr{L}\left\{g(t)\right\}$$

Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$, then the above becomes

$$2\left\{s^2Y(s) - sy(0) - y'(0)\right\} + \left\{sY(s) - y(0)\right\} + 2Y(s) = \mathscr{L}\left\{g(t)\right\}$$

But $y(0) = y'(0) = 0$ and the above reduces to

$$2s^2Y(s) + sY(s) + 2Y(s) = \frac{e^{-5s} - e^{-20s}}{s}$$

Solving for $Y(s)$ gives

$$\begin{aligned}Y(s) &= \frac{e^{-5s} - e^{-10s}}{s\left(2s^2 + s + 2\right)} \\ &= \left(\frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} - \frac{e^{-20s}}{s\left(2s^2 + s + 2\right)}\right) \qquad (2)\end{aligned}$$

We now need to find the inverse Laplace transform of $Y(s)$. Looking at $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$,the first step is to use the property

$$u_c(t)\, f(t-c) \overset{\mathscr{L}}{\Longleftrightarrow} e^{-cs}F(s)$$

Comparing the expressions, we see that

$$u_5(t)\, f(t-c) \Longleftrightarrow \frac{e^{-5s}}{s\left(2s^2 + s + 2\right)} \qquad (3)$$

Where

$$f(t) \Longleftrightarrow \frac{1}{s\left(2s^2 + s + 2\right)} \qquad (4)$$

Therefore, we just need to find inverse Laplace transform of $\frac{1}{s\left(2s^2+s+2\right)}$. Using partial fractions

$$\begin{aligned}\frac{1}{s\left(2s^2 + s + 2\right)} &= \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2} \qquad (5) \\ 1 &= A\left(2s^2 + s + 2\right) + (Bs + C)s \\ 1 &= 2As^2 + As + 2A + Bs^2 + Cs \\ 1 &= 2A + s(A + C) + s^2(2A + B)\end{aligned}$$

Therefore

$$\begin{aligned}A &= \frac{1}{2} \\ A + C &= 0 \\ 2A + B &= 0\end{aligned}$$

Hence from the second equation $C=-\frac{1}{2}$, and from the third equation $B=-1$ Therefore (5) becomes

$$\begin{aligned}\frac{1}{s\left(2s^2+s+2\right)}&=\frac{1}{2}\frac{1}{s}+\frac{-s-\frac{1}{2}}{2s^2+s+2}\\&=\frac{1}{2}\frac{1}{s}+\frac{1}{2}\frac{-1-2s}{2s^2+s+2}\\&=\frac{1}{2}\frac{1}{s}-\frac{s}{2s^2+s+2}-\frac{1}{2}\frac{1}{2s^2+s+2}\end{aligned}\tag{5A}$$

The first term above is easy, we know that

$$\frac{1}{2}\frac{1}{s}\Longleftrightarrow\frac{1}{2}\tag{6}$$

Now we will find inverse Laplace transform of second term in (5A) $\frac{s}{2s^2+s+2}$. For this we start by completing the squares in the denominator. Let

$$\begin{aligned}2s^2+s+2&=a(s+b)^2+c\\&=a\left(s^2+b^2+2bs\right)+c\\&=as^2+ab^2+2bas+c\end{aligned}$$

Hence $a=2,2ab=1$ or $b=\frac{1}{4}$ and $ab^2+c=2$, hence $c=2-2\left(\frac{1}{4}\right)^2=2-2\left(\frac{1}{16}\right)=2-\frac{1}{8}=\frac{15}{8}$, Therefore

$$2s^2+s+2=2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}$$

We now re-write second term in (5A), which is $\frac{s}{2s^2+s+2}$ as $\frac{s}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}}$. We did this because we wanted this to be in the form $\frac{s}{s^2+a}$, therefore

$$\frac{s}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}}=\frac{1}{2}\frac{s}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

Now we let $\tilde{s}=s+\frac{1}{4}$, therefore the above becomes

$$\frac{1}{2}\frac{\tilde{s}-\frac{1}{4}}{\tilde{s}^2+\frac{15}{16}}=\frac{1}{2}\left(\frac{\tilde{s}}{\tilde{s}^2+\frac{15}{16}}-\frac{1}{4}\frac{1}{\tilde{s}^2+\frac{15}{16}}\right)\tag{7}$$

Using $\frac{s}{s^2+a^2}\Leftrightarrow\cos(at)$ then

$$\frac{\tilde{s}}{\tilde{s}^2+\frac{15}{16}}\Leftrightarrow e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right)$$

The reason for $e^{-\frac{t}{4}}$ being there, is because we evaluated $F(s)$ at $F\left(s+\frac{1}{4}\right)$. This used the shift property

$$F(s+a)=e^{-at}f(t)$$

Therefore $F\left(s+\frac{1}{4}\right)=e^{-\frac{t}{4}}f(t)$. Now we do the second term in (7). Since $\frac{1}{\tilde{s}^2+\frac{15}{16}}=\frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}}$, then, now using $\frac{a}{s^2+a^2}\Leftrightarrow\sin(at)$ we obtain

$$\frac{1}{\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\tilde{s}^2+\frac{15}{16}}\Leftrightarrow\frac{1}{\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right)$$

And we remember to add $e^{-\frac{t}{4}}$ again, due to the shift in s. Therefore (7) becomes

$$\frac{s}{2s^2+s+2} \Leftrightarrow \frac{1}{2}\left(e^{-\frac{t}{4}}\cos\left(\sqrt{\frac{15}{16}}t\right) - \frac{1}{4}e^{-\frac{t}{4}}\frac{1}{\sqrt{\frac{15}{16}}}\sin\left(\sqrt{\frac{15}{16}}t\right)\right)$$
$$= \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) \tag{8}$$

This complete the second term in (5A). Now we will do the third term in (5A) which is $\frac{1}{2s^2+s+2}$ which is

$$\frac{1}{2s^2+s+2} = \frac{1}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{8}}$$
$$= \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$
$$= \frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

Hence

$$\frac{1}{2\sqrt{\frac{15}{16}}}\frac{\sqrt{\frac{15}{16}}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \Longleftrightarrow \frac{1}{2\sqrt{\frac{15}{16}}}e^{-\frac{t}{4}}\sin\left(\sqrt{\frac{15}{16}}t\right)$$
$$= \frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right) \tag{9}$$

Now we put all the results back together.

$$\frac{1}{s\left(2s^2+s+2\right)} = \frac{1}{2}\frac{1}{s} - \frac{s}{2s^2+s+2} - \frac{1}{2}\frac{1}{2s^2+s+2}$$
$$\Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right)\right) - \frac{1}{2}\left(\frac{2}{\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$

We can simplify this more

$$\frac{1}{s\left(2s^2+s+2\right)} \Longleftrightarrow \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) - \sin\left(\frac{\sqrt{15}}{4}t\right) + 2\sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$
$$= \frac{1}{2} - \frac{e^{-\frac{t}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}t\right) + \sin\left(\frac{\sqrt{15}}{4}t\right)\right)$$

Using this back in (2), where we want to evaluates $\frac{e^{-5s}}{s\left(2s^2+s+2\right)}$, gives

$$\frac{e^{-5s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_5(t)\, f(t-5)$$

Where

$$f(t-5) = \frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)$$

The above complete the first term in (2). The second term in (2) is the same, but the delay now is 20 instead of 5. Hence

$$\frac{e^{-20s}}{s\left(2s^2+s+2\right)} \Longleftrightarrow u_{20}(t)\, f(t-20)$$

With the same function $f(t)$ found above. Therefore, the final inverse transform now is

$$\begin{aligned} y(t) &\Longleftrightarrow \left(\frac{e^{-5s}}{s\left(2s^2+s+2\right)} - \frac{e^{-20s}}{s\left(2s^2+s+2\right)}\right) \\ &= \left(u_5(t)\, f(t-5) - u_{20}(t)\, f(t-20)\right) \end{aligned}$$

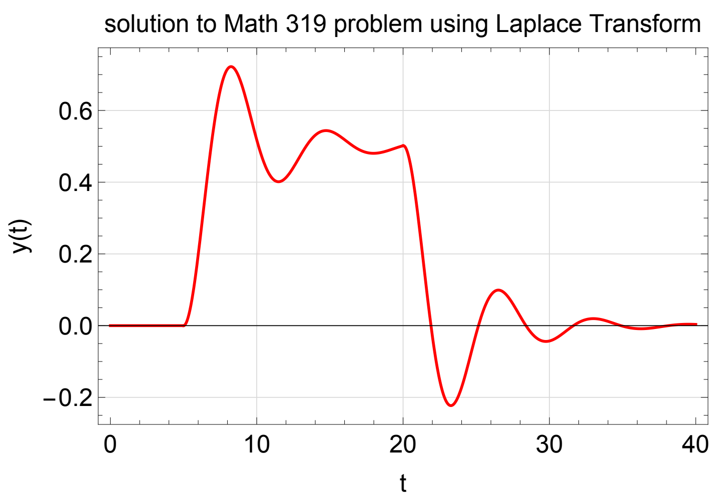
Where

$$f(t-20) = \frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right)$$

This complete the solution. The final solution is

$$\begin{aligned} y(t) = u_5(t)&\left(\frac{1}{2} - \frac{e^{\frac{-(t-5)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right)\right) \\ - u_{20}(t)&\left(\frac{1}{2} - \frac{e^{\frac{-(t-20)}{4}}}{2\sqrt{15}}\left(\sqrt{15}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) + \sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right)\right) \end{aligned}$$

Here is a plot of the above solution



References

1. Lecture notes Nov. 30, 2016 by Professor Minh-Binh Tran. Math dept. Univ. Of Wisconsin Madison.
2. Wikipedia web page on Laplace transform properties.