# Math 319, Puzzle Challenge, Fall 2016

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1. Let $y_1, y_2, ..., y_n$ be differentiable (real-valued) solutions of the following system of differential equations

$$\begin{aligned}
\frac{dy_1}{dt} &= a_{11}y_1 + \cdots + a_{1n}y_n, \\
\frac{dy_2}{dt} &= a_{21}y_1 + \cdots + a_{2n}y_n, \\
&\cdots \\
\frac{dy_n}{dt} &= a_{n1}y_1 + \cdots + a_{nn}y_n,
\end{aligned}$$

for some constant $a_{ij} > 0$. Suppose that

$$y_i(t) \to 0,$$

as $t \to \infty, \forall i = 1, \cdots, n$. Are the functions $y_1, y_2, ..., y_n$ necessarily linearly dependent?

What we know (given): We have state space representation of a system in the form $Y' = AY$, where $y_1, y_2, \cdots, y_n$ are the states, and we are told the system goes to stable equilibrium $Y = 0$, as $t \to \infty$ when starting from any initial point in the $n$ dimensions state space. The original system is described by a single $n^{th}$ degree one differential equation, and is broken down to $n$ first order differential equation. These are $y_1', y_2', \cdots, y_n'$. The system is coupled, since each $y_i'(t)$ depends on all other $y_i(t)$.

Solution The only way I can see to answer this question in concrete way, is to resort to using the Wronskian. Writing down the Wronskian $W(t)$ of the functions $y_1(t), y_2(t), \cdots, y_n(t)$ we obtain

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

In the limit, as $t \to \infty$, since we are told $y_i \to 0$, then the above becomes

$$\lim_{t\to\infty} W(t) = \begin{vmatrix} 0 & 0 & \cdots & 0 \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ y_1''(t) & y_2''(t) & \cdots & y_n''(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Since, at least, one row becomes all zero, then the determinant above is zero (from linear algebra). Therefore

$$\lim_{t\to\infty} W(t) = 0$$

We could conclude now that $y_1, y_2, \cdots, y_n$ are therefore linearly dependent functions since we found that $W(t) = 0$ at some point. However, the Wronskian being zero at some point does not necessarily imply that the functions are linearly dependent. So the Wronskian test is not conclusive when it gives zero when evaluated at one point, and we need another test to do. The following are the important facts about using the Wronskian

1. If $W(t) \neq 0$ at any point $t$ (in the interval of interest) $\Rightarrow y_1, y_2, \cdots, y_n$ are linearly independent (in that interval).
2. If $y_1, y_2, \cdots, y_n$ are analytic (differentiable) functions and linearly dependent (in the interval of interest) $\Rightarrow W(t) = 0$ at every point $t$ in the interval.
3. If $W(t) = 0$ at every point $t$ (in the interval of interest) and $y_1(t), y_2(t), \cdots, y_n(r)$ are all analytic functions $\Rightarrow y_1(t), y_2(t), \cdots, y_n(r)$ are linearly dependent in that interval.
4. If $W(t) = 0$ at one point $t$ (or at countable number of points) in the interval of interest $\Rightarrow$ test is not conclusive.

The above are results from Linear algebra. We see from the above, that $W(t) = 0$ at $t = \infty$ does not imply that the functions are necessarily linearly dependent. In this case, we would use a different test if we are given the functions, by writing

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

And then we would try to find constants $c_1, c_2, \cdots, c_n$, not all zero, which would satisfy the above. If we can find such constants, only then we can conclude that $y_1, y_2, \cdots, y_n$ are linearly dependent since If the functions are linearly independent, then $c_i = 0$ will be the only possible solution.

In conclusion The functions $y_1(t), y_2(t), \cdots, y_n(t)$ are not necessarily linearly dependent, even though $W(t) = 0$ in the limit as $t \to \infty$.