HW 6, Math 319, Fall 2016

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1 HW 6

1.1 Section 3.4 problem 1

Find the general solution of $y'' - 2y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 - 2r + 1 = 0$$
$$(r-1)(r-1) = 0$$

Hence $r = 1$ double root. Therefore the two solutions are

$$y_1 = e^t$$
$$y_2 = te^t$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^t + c_2 te^t$$

1.2 Section 3.4 problem 2

Find the general solution of $9y'' + 6y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$9r^2 + 6r + 1 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 + \sqrt{36 - 36}}{18} = -\frac{1}{3}$$

Hence $r = -\frac{1}{3}$ double root. Therefore the two solutions are

$$y_1 = e^{\frac{-1}{3}t}$$
$$y_2 = te^{\frac{-1}{3}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{-1}{3}t} + c_2 te^{\frac{-1}{3}t}$$

1.3 Section 3.4 problem 3

Find the general solution of $4y'' - 4y' - 3y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 - 4r - 3 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 + 48}}{8} = \frac{4 \pm 8}{8} = \frac{1 \pm 2}{2} = \frac{1}{2} \pm 1$$

Hence $r_1 = \frac{3}{2}, r_2 = -\frac{1}{2}$. Therefore the two solutions are

$$y_1 = e^{\frac{3}{2}t}$$
$$y_2 = e^{-\frac{1}{2}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{3}{2}t} + c_2 e^{-\frac{1}{2}t}$$

1.4 Section 3.4 problem 4

Find the general solution of $4y'' + 12y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 + 12r + 9 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12 \pm \sqrt{144 - 144}}{8} = \frac{-3}{2}$$

Hence $r = \frac{-3}{2}$ double root. Therefore the two solutions are

$$y_1 = e^{\frac{-3}{2}t}$$
$$y_2 = te^{\frac{-3}{2}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{-3}{2}t} + c_2 te^{\frac{-3}{2}t}$$

1.5 Section 3.4 problem 5

Find the general solution of $y'' - 2y' + 10y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 - 2r + 10 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

Hence $r_1 = 1 + 3i, r_2 = 1 - 3i$. Therefore the two solutions are

$$y_1 = e^{(1+3i)t} = e^t e^{i3t}$$
$$y_2 = e^t e^{i3t}$$

And the general solution is linear combination of the above solutions, the complex exponential can be converted to trig functions $\cos, \sin$ using the standard Euler identities, resulting in

$$y = e^t \left(c_1 \cos 3t + c_2 \sin 3t\right)$$

1.6 Section 3.4 problem 6

Find the general solution of $y'' - 6y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 - 6r + 9 = 0$$
$$(r-3)^2 = 0$$

Hence $r = 3$. Double root. Therefore the two solutions are

$$y_1 = e^{3t}$$
$$y_2 = te^{3t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{3t} + c_2 t e^{3t}$$

1.7 Section 3.4 problem 7

Find the general solution of $4y'' + 17y' + 4y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 + 17r + 4 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-17 \pm \sqrt{289 - 64}}{8} = \frac{-17 \pm \sqrt{225}}{8} = \frac{-17 \pm 15}{8}$$

Hence $r_1 = \frac{-17-15}{8} = -4, r_2 = \frac{-17+15}{8} = -\frac{1}{4}$. Therefore the two solutions are

$$y_1 = e^{-4t}$$
$$y_2 = e^{-\frac{1}{4}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{-4t} + c_2 e^{-\frac{1}{4}t}$$

1.8 Section 3.4 problem 8

Find the general solution of $16y'' + 24y' + 9y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$16r^2 + 24r + 9 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-24 \pm \sqrt{576 - 4(16)(9)}}{32} = \frac{-24}{32} = -\frac{3}{4}$$

Hence $r=-\frac{3}{4}$. Double root. Therefore the two solutions are

$$y_1 = e^{-\frac{3}{4}t}$$
$$y_2 = te^{-\frac{3}{4}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{-\frac{3}{4}t} + c_2 t e^{-\frac{3}{4}t}$$

1.9 Section 3.4 problem 9

Find the general solution of $25y'' - 20y' + 4y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$25r^2 - 20r + 4 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{20 \pm \sqrt{400 - 4(25)(4)}}{50} = \frac{20}{50} = \frac{2}{5}$$

Hence $r = \frac{2}{5}$. Double root. Therefore the two solutions are

$$y_1 = e^{\frac{2}{5}t}$$
$$y_2 = te^{\frac{2}{5}t}$$

And the general solution is linear combination of the above solutions

$$y = c_1 e^{\frac{2}{5}t} + c_2 t e^{\frac{2}{5}t}$$

1.10 Section 3.4 problem 10

Find the general solution of $2y'' + 2y' + y = 0$

Solution:

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$2r^2 + 2r + 1 = 0$$
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{4} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = \frac{-1}{2} \pm \frac{i}{2}$$

Hence $r_1 = \frac{-1}{2} + \frac{i}{2}, r_2 = \frac{-1}{2} - \frac{i}{2}$. Therefore the two solutions are

$$y_1 = e^{\left(\frac{-1}{2} - \frac{i}{2}\right)t} = e^{\frac{-1}{2}t} e^{\frac{-i}{2}t}$$
$$y_2 = e^{\left(\frac{-1}{2} + \frac{i}{2}\right)t} = e^{\frac{-1}{2}t} e^{\frac{i}{2}t}$$

And the general solution is linear combination of the above solutions, the complex exponential can be converted to trig functions $\cos, \sin$ using the standard Euler identities, resulting in

$$y = e^{\frac{-1}{2}t}\left(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}\right)$$

1.11 Section 3.5 problem 1

Find the general solution of $y'' - 2y' - 3y = 3e^{2t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' - 2y' - 3y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 - 2r - 3 &= 0 \\ (r+1)(r-3) &= 0 \end{aligned}$$

Hence $r_1 = -1, r_2 = 3$. Therefore the two solution are

$$\begin{aligned} y_1 &= e^{-t} \\ y_2 &= e^{3t} \end{aligned}$$

And the homogeneous solution is linear combination of the above solutions

$$y_h = c_1 e^{-t} + c_2 e^{3t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' - 2y' - 3y = 3e^{2t}$$

We guess $y_p = Ae^{2t}$. Hence

$$\begin{aligned} y_p' &= 2Ae^{2t} \\ y_p'' &= 4Ae^{2t} \end{aligned}$$

Substituting this into the original ODE in order to solve for A gives

$$\begin{aligned} 4Ae^{2t} - 2\left(2Ae^{2t}\right) - 3\left(Ae^{2t}\right) &= 3e^{2t} \\ -3Ae^{2t} &= 3e^{2t} \end{aligned}$$

Hence $A = -1$ and therefore

$$y_p = -e^{2t}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{3t} - e^{2t} \end{aligned}$$

1.12 Section 3.5 problem 2

Find the general solution of $y'' + 2y' + 5y = 3\sin 2t$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + 2y' + 5y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 + 2r + 5 &= 0 \\ r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i \end{aligned}$$

From before, we know the solution is of the form

$$y_h = e^{-t}\left(c_1 \cos 2t + \sin 2t\right)$$

Where

$$\begin{aligned} y_1 &= e^{-t} \cos 2t \\ y_2 &= e^{-t} \sin 2t \end{aligned}$$

Finding y_p

Now we need to find one particular solution to

$$y'' + 2y' + 5y = 3 \sin 2t$$

We guess $y_p = A \cos 2t + B \sin 2t$ hence

$$\begin{aligned} y_p' &= -2A \sin 2t + 2B \cos 2t \\ y_p'' &= -4A \cos 2t - 4B \sin 2t \end{aligned}$$

Substituting these back into the original ODE in order to solve for A, B gives

$$\begin{aligned} y_p'' + 2y_p' + 5y_p &= 3 \sin 2t \\ -4A \cos 2t - 4B \sin 2t + 2\left(-2A \sin 2t + 2B \cos 2t\right) + 5\left(A \cos 2t + B \sin 2t\right) &= 3 \sin 2t \\ -4A \cos 2t - 4B \sin 2t - 4A \sin 2t + 4B \cos 2t + 5A \cos 2t + 5B \sin 2t &= 3 \sin 2t \\ (A + 4B) \cos 2t + (B - 4A) \sin 2t &= 3 \sin 2t \end{aligned}$$

Hence

$$\begin{aligned} A + 4B &= 0 \\ B - 4A &= 3 \end{aligned}$$

From first equation, $A = -4B$, and the second equation becomes $B - 4(-4B) = 3$ or $B + 16B = 3$ or $B = \frac{3}{17}$, hence $A = \frac{-12}{17}$, therefore

$$y_p = \frac{-12}{17} \cos 2t + \frac{3}{17} \sin 2t$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= e^{-t}\left(c_1 \cos 2t + \sin 2t\right) - \frac{12}{17} \cos 2t + \frac{3}{17} \sin 2t \end{aligned}$$

1.13 Section 3.5 problem 3

Find the general solution of $y'' - y' - 2y = -2t + 4t^2$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' - y' - 2y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 - r - 2 &= 0 \\ (r+1)(r-2) &= 0 \end{aligned}$$

Hence $r_1 = -1, r_2 = 2$ and therefore

$$y_h = c_1 e^{-t} + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' - y' - 2y = -2t + 4t^2$$

We guess $y_p = A_0 + A_1 t + A_2 t^2$. Therefore

$$\begin{aligned} y_p' &= A_1 + 2A_2 t \\ y_p'' &= 2A_2 \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} 2A_2 - (A_1 + 2A_2 t) - 2\left(A_0 + A_1 t + A_2 t^2\right) &= -2t + 4t^2 \\ t^0 (2A_2 - A_1 - 2A_0) + t(-2A_2 - 2A_1) + t^2(-2A_2) &= -2t + 4t^2 \end{aligned}$$

Hence

$$\begin{aligned} 2A_2 - A_1 - 2A_0 &= 0 \\ -2A_2 - 2A_1 &= -2 \\ -2A_2 &= 4 \end{aligned}$$

From the last equation, $A_2 = -2$, and from the second equation $A_1 = \frac{-2+2(-2)}{-2} = 3$ and from the first equation $2(-2) - 3 - 2A_0 = 0$ hence $A_0 = \frac{4+3}{-2} = -\frac{7}{2}$, Therefore

$$\begin{aligned} y_p &= A_0 + A_1 t + A_2 t^2 \\ &= -\frac{7}{2} + 3t - 2t^2 \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{7}{2} + 3t - 2t^2 \end{aligned}$$

1.14 Section 3.5 problem 4

Find the general solution of $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + y' - 6y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 + r - 6 &= 0 \\ (r+3)(r-2) &= 0 \end{aligned}$$

Hence $r_1 = -3, r_2 = 2$ and therefore

$$y_h = c_1 e^{-3t} + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$$

We guess $y_p = Ae^{3t} + Be^{-2t}$. Therefore

$$\begin{aligned} y_p' &= 3Ae^{3t} - 2Be^{-2t} \\ y_p'' &= 9Ae^{3t} + 4Be^{-2t} \end{aligned}$$

Substituting these back into the original ODE gives

$$\begin{aligned} y_p'' + y_p' - 6y_p &= 12e^{3t} + 12e^{-2t} \\ 9Ae^{3t} + 4Be^{-2t} + 3Ae^{3t} - 2Be^{-2t} - 6\left(Ae^{3t} + Be^{-2t}\right) &= 12e^{3t} + 12e^{-2t} \\ e^{3t}(9A + 3A - 6A) + e^{-2t}(4B - 2B - 6B) &= 12e^{3t} + 12e^{-2t} \\ 6Ae^{3t} - 4Be^{-2t} &= 12e^{3t} + 12e^{-2t} \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} A &= 2 \\ B &= -3 \end{aligned}$$

Hence

$$y_p = 2e^{3t} - 3e^{-2t}$$

And the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-3t} + c_2 e^{2t} + 2e^{3t} - 3e^{-2t} \end{aligned}$$

1.15 Section 3.5 problem 5

Find the general solution of $y'' - 2y' - 3y = -3te^{-t}$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' - 2y' - 3y = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 - 2r - 3 &= 0 \\ (r-3)(r+1) &= 0 \end{aligned}$$

Hence $r_1 = 3, r_2 = -1$ and therefore

$$y_h = c_1 e^{3t} + c_2 e^{-t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' - 2y' - 3y = -3te^{-t}$$

Guess for t is $A_0 + B_0 t$ and the guess for e^{-t} is Cte^{-t} (where we multiplied by t since e^{-t} shows up in the homogenous solution. Therefore the product is

$$\begin{aligned} y_p &= (A_0 + B_0 t)\, Cte^{-t} \\ &= A_0 Cte^{-t} + CB_0 t^2 e^{-t} \end{aligned}$$

Let $A_0 C = A, CB_0 = B$, and the above becomes

$$\begin{aligned} y_p &= Ate^{-t} + Bt^2 e^{-t} \\ &= (A + Bt)\, te^{-t} \end{aligned}$$

Substituting these back into the ODE and solving for A, B gives $B = \frac{3}{8}$ and $A = \frac{3}{16}$, hence

$$\begin{aligned} y_p &= \left(At + Bt^2\right) e^{-t} \\ &= \left(\frac{3}{16} t + \frac{3}{8} t^2\right) e^{-t} \end{aligned}$$

And the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{3t} + c_2 e^{-t} + \left(\frac{3}{16} t + \frac{3}{8} t^2\right) e^{-t} \end{aligned}$$

1.16 Section 3.5 problem 6

Find the general solution of $y'' + 2y' = 3 + 4 \sin 2t$

Solution:

The first step is to solve the homogenous ODE and find y_h, then find a particular solution y_p to the inhomogeneous ODE, then add both solutions $y_h + y_p$ in order to find the complete solution.

Finding y_h

We need to solve homogenous ODE

$$y'' + 2y' = 0$$

The characteristic equation is found by substituting $y = e^{rt}$ into the above ODE and simplifying, giving

$$\begin{aligned} r^2 + 2r &= 0 \\ r(r+2) &= 0 \end{aligned}$$

Hence $r_1 = 0, r_2 = -2$ and therefore

$$y_h = c_1 + c_2 e^{2t}$$

Finding y_p

Now we need to find one particular solution to

$$y'' + 2y' = 3 + 4\sin 2t$$

Guess that $y_p = At + B\cos 2t + C\sin 2t$, hence

$$\begin{aligned} y_p' &= A - 2B\sin 2t + 2C\cos 2t \\ y_p'' &= -4B\cos 2t - 4C\sin 2t \end{aligned}$$

Substituting back into

$$\begin{aligned} y_p'' + 2y_p' &= 3 + 4\sin 2t \\ -4B\cos 2t - 4C\sin 2t + 2(A - 2B\sin 2t + 2C\cos 2t) &= 3 + 4\sin 2t \\ -4B\cos 2t - 4C\sin 2t + 2A - 4B\sin 2t + 4C\cos 2t &= 3 + 4\sin 2t \\ (-4B + 4C)\cos 2t + 2A + (-4C - 4B)\sin 2t &= 3 + 4\sin 2t \end{aligned}$$

Hence

$$\begin{aligned} 2A &= 3 \\ 1 &= -C - B \\ 0 &= -B + C \end{aligned}$$

From first equation, $A = \frac{3}{2}$, From third equation, $B = C$ and from the second equation $1 = -2B$ or $B = \frac{-1}{2}$, hence $C = \frac{-1}{2}$, and the particular solution is

$$y_p = \frac{3}{2}t + \frac{-1}{2}\cos 2t - \frac{1}{2}\sin 2t$$

Hence the complete solution is

$$y = c_1 + c_2 e^{2t} + \frac{3}{2}t - \frac{1}{2}\cos 2t - \frac{1}{2}\sin 2t$$