HW 5, Math 319, Fall 2016

Nasser M. Abbasi (Discussion section 44272, Th 4:35PM-5:25 PM)

December 30, 2019

Contents

1 HW 5

1.1 Section 3.1 problem 9

Find the solution to $y'' + y' - 2y = 0; y(0) = 1, y'(0) = 1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} r^2 + r - 2 &= 0 \\ (r+2)(r-1) &= 0 \end{aligned}$$

Hence the roots are $r_1 = -2, r_2 = 1$. Roots are real and distinct. The two solutions are

$$\begin{aligned} y_1 &= e^{-2t} \\ y_2 &= e^{t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-2t} + c_2 e^{t} \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(0) = 1)$ to the general solution gives

$$1 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t) = -2c_1 e^{-2t} + c_2 e^{t}$. Applying second initial condition to this results in

$$1 = -2c_1 + c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1 = 1 - c_2$. Substituting this into (2) gives

$$\begin{aligned} 1 &= -2(1 - c_2) + c_2 \\ &= -2 + 2c_2 + c_2 \\ &= -2 + 3c_2 \end{aligned}$$

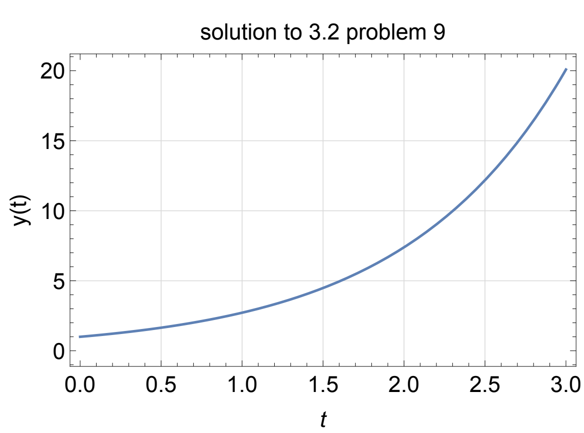
Hence $c_2 = \frac{1+2}{3} = 1$. Therefore $c_1 = 1 - 1 = 0$. Hence

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 1 \end{aligned}$$

Substituting these back into the general solution gives

$$y(t) = e^{t}$$

Since the solution is exponential, it will grow in time and blows up. Here is sketch of the solution.



1.2 Section 3.1 problem 10

Find the solution to $y'' + 4y' + 3y = 0; y(0) = 2, y'(0) = -1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} r^2 + 4r + 3 &= 0 \\ (r+3)(r+1) &= 0 \end{aligned}$$

Hence the roots are $r_1 = -3, r_2 = -1$. Roots are real and distinct. The two solutions are

$$y_1 = e^{-3t}$$
$$y_2 = e^{-t}$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-3t} + c_2 e^{-t} \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 2$) to the general solution gives

$$2 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t) = -3c_1 e^{-3t} - c_2 e^{-t}$. Applying second initial condition to this results in

$$-1 = -3c_1 - c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1 = 2 - c_2$. Substituting this into (2) gives

$$\begin{aligned} -1 &= -3(2 - c_2) - c_2 \\ &= -6 + 3c_2 - c_2 \\ &= -6 + 2c_2 \end{aligned}$$

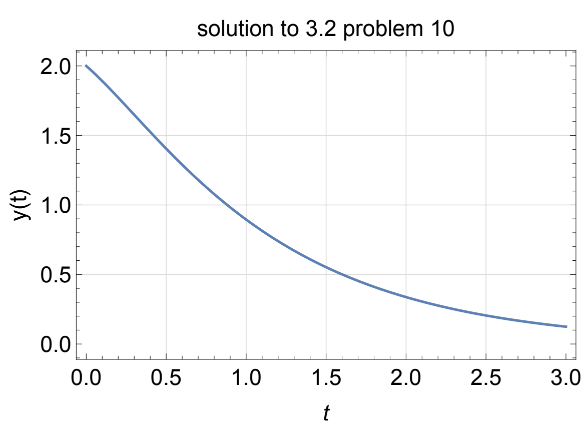
Hence $c_2 = \frac{-1+6}{2} = 2.5$. Therefore $c_1 = 2 - 2.5 = 0.5$. Hence

$$c_1 = 0.5$$
$$c_2 = 2.5$$

Substituting these back into the general solution gives

$$y(t) = 0.5e^{-3t} + 2.5e^{-t}$$

At t becomes large, both solutions decay to zero. So we expect the general solution to go to zero very fast. Here is a sketch.



1.3 Section 3.1 problem 11

Find the solution to $6y'' - 5y' + y = 0; y(0) = 4, y'(0) = 0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$6r^2 - 5r + 1 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 25 - (4)(6) = 1$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$\begin{aligned} r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{5}{12} \pm \frac{1}{12} \end{aligned}$$

Hence the roots are $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$. Roots are real and distinct. The two solutions are

$$y_1 = e^{\frac{1}{2}t}$$
$$y_2 = e^{\frac{1}{3}t}$$

The general solution is linear combination of the above two solutions

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{3}t} \end{aligned}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 4$) to the general solution gives

$$4 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives $y'(t) = \frac{1}{2}c_1 e^{\frac{1}{2}t} + \frac{1}{3}c_2 e^{\frac{1}{3}t}$. Applying second initial condition to this results in

$$0 = \frac{1}{2}c_1 + \frac{1}{3}c_2 \tag{2}$$

Equation (1,2) are now solved for c_1, c_2. From (1), $c_1 = 4 - c_2$. Substituting this into (2) gives

$$\begin{aligned} 0 &= \frac{1}{2}(4 - c_2) + \frac{1}{3}c_2 \\ &= 2 - \frac{1}{2}c_2 + \frac{1}{3}c_2 \\ &= 2 - \frac{1}{6}c_2 \end{aligned}$$

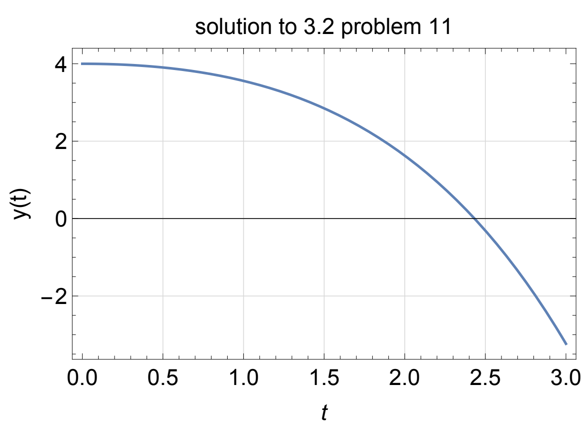
Hence $c_2 = 12$. Therefore $c_1 = 4 - 12 = -8$. Hence

$$\begin{aligned} c_1 &= -8 \\ c_2 &= 12 \end{aligned}$$

Substituting these back into the general solution gives

$$y(t) = -8e^{\frac{1}{2}t} + 12e^{\frac{1}{3}t}$$

Since $e^{\frac{1}{2}t}$ grows faster than $e^{\frac{1}{3}t}$ and since $e^{\frac{1}{2}t}$ has negative coefficient, then the solution will go to $-\infty$ as t increases. Here is sketch of the solution



1.4 Section 3.1 problem 12

Find the solution to $y'' + 3y' = 0; y(0) = -2, y'(0) = 3$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$\begin{aligned} r^2 + 3r &= 0 \\ r(r + 3) &= 0 \end{aligned}$$

Hence the roots are $r_1 = 0, r_2 = -3$. Roots are real and distinct. The two solutions are

$$\begin{aligned} y_1 &= 1 \\ y_2 &= e^{-3t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$y = c_1 + c_2 e^{-3t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = -2$) to the general solution gives

$$-2 = c_1 + c_2 \tag{1}$$

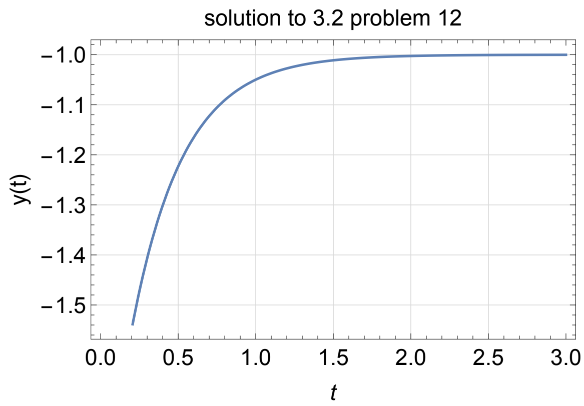
Taking time derivative of the general solution gives $y'(t) = -3c_2e^{-3t}$. Applying second initial condition to this results in

$$3 = -3c_2 \tag{2}$$

Hence $c_2 = -1$. Therefore $c_1 = -1$. Substituting these back into the general solution gives

$$y(t) = -1 - e^{-3t}$$

As $t \to \infty$, the term $e^{-3t} \to 0$ and we are left with -1. Hence $\lim_{t-\infty} y(t) = -1$. Here is sketch of the solution



1.5 Section 3.1 problem 13

Find the solution to $y'' + 5y' + 3y = 0; y(0) = 1, y'(0) = 0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 + 5r + 3 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 25 - (4)(3) = 13$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$\begin{aligned} r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5}{2} \pm \frac{\sqrt{13}}{2} \end{aligned}$$

Hence the roots are $r_1 = \frac{-5}{2} + \frac{\sqrt{13}}{2}, r_2 = \frac{-5}{2} - \frac{\sqrt{13}}{2}$. The two solutions are

$$\begin{aligned} y_1 &= e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t} \\ y_2 &= e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t} \end{aligned}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t} + c_2 e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 1$) to the general solution gives

$$1 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)e^{\left(\frac{-5}{2}+\frac{\sqrt{13}}{2}\right)t} + c_2\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)e^{\left(\frac{-5}{2}-\frac{\sqrt{13}}{2}\right)t}$$

Applying second initial condition to this results in

$$0 = c_1\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right) + c_2\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right) \tag{2}$$

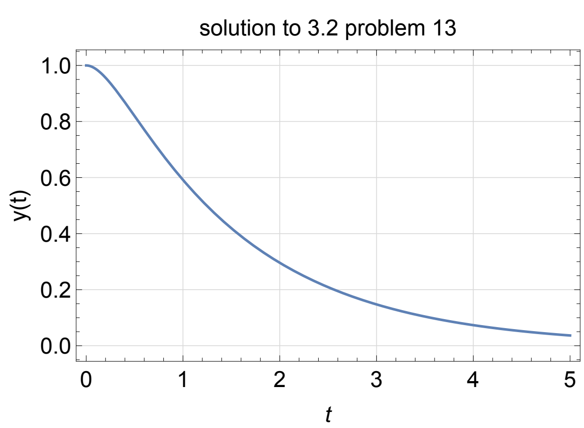
From (1), $c_1 = 1 - c_2$ and from (2)

$$
\begin{aligned}
0 &= (1 - c_2)\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right) + c_2\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right) \\
&= \frac{-5}{2} + \frac{\sqrt{13}}{2} + \frac{5}{2}c_2 - \frac{\sqrt{13}}{2}c_2 - \frac{5}{2}c_2 - \frac{\sqrt{13}}{2}c_2 \\
&= -\frac{5}{2} + \frac{\sqrt{13}}{2} - \sqrt{13}c_2 \\
c_2 &= \frac{-5}{2\sqrt{13}} + \frac{1}{2} \\
&= \frac{-5 + \sqrt{13}}{2\sqrt{13}}
\end{aligned}
$$

Therefore $c_1 = 1 + \frac{5-\sqrt{13}}{2\sqrt{13}}$ and the solution becomes

$$
\begin{aligned}
y(t) &= \left(1 + \frac{5 - \sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \left(\frac{-5 + \sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} \\
&= e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \frac{5 - \sqrt{13}}{2\sqrt{13}}e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \left(\frac{-5 + \sqrt{13}}{2\sqrt{13}}\right)e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} \\
&= e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \frac{5\sqrt{13} - 13}{26}e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \left(\frac{-5\sqrt{13} + 13}{26}\right)e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} \\
&= \frac{1}{26}\left(26e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \left(5\sqrt{13} - 13\right)e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + \left(-5\sqrt{13} + 13\right)e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t}\right) \\
&= \frac{1}{26}\left(26e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + 5\sqrt{13}e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} - 13e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} - 5\sqrt{13}e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} + 13e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t}\right) \\
&= \frac{1}{26}\left(13e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} + 5\sqrt{13}e^{\left(\frac{-5}{2} + \frac{\sqrt{13}}{2}\right)t} - 5\sqrt{13}e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t} + 13e^{\left(\frac{-5}{2} - \frac{\sqrt{13}}{2}\right)t}\right)
\end{aligned}
$$

Here is sketch of the solution showing that $y \to 0$ as $t \to \infty$



1.6 Section 3.1 problem 14

Find the solution to $2y'' + y' - 4y = 0; y(0) = 0, y'(0) = 1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$2r^2 + r - 4 = 0$$

Hence $r_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$, where $\Delta = b^2 - 4ac = 1 - (4)(2)(-4) = 33$. Since $\Delta > 0$, the roots will be real and distinct. The roots are

$$
\begin{aligned}
r_{1,2} &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-1}{4} \pm \frac{\sqrt{33}}{4}
\end{aligned}
$$

Hence the roots are $r_1 = \frac{1}{4} + \frac{\sqrt{33}}{4}, r_2 = \frac{1}{4} - \frac{\sqrt{33}}{4}$. The two solutions are

$$y_1 = e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t}$$
$$y_2 = e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} + c_2 e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(0) = 0$) to the general solution gives

$$0 = c_1 + c_2 \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) e^{\left(\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) e^{\left(\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

Applying second initial condition to this results in

$$1 = c_1 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) \tag{2}$$

From (1), $c_1 = -c_2$ and from (2)

$$\begin{aligned} 1 &= -c_2 \left(-\frac{1}{4} + \frac{\sqrt{33}}{4}\right) + c_2 \left(-\frac{1}{4} - \frac{\sqrt{33}}{4}\right) \\ &= \frac{1}{4}c_2 - \frac{\sqrt{33}}{4}c_2 - \frac{1}{4}c_2 - \frac{\sqrt{33}}{4}c_2 \\ &= \frac{-\sqrt{33}}{2}c_2 \\ c_2 &= \frac{-2}{\sqrt{33}} \end{aligned}$$

Therefore $c_1 = \frac{2}{\sqrt{33}}$ and the solution becomes

$$y = \frac{2}{\sqrt{33}} e^{\left(-\frac{1}{4}+\frac{\sqrt{33}}{4}\right)t} - \frac{2}{\sqrt{33}} e^{\left(-\frac{1}{4}-\frac{\sqrt{33}}{4}\right)t}$$

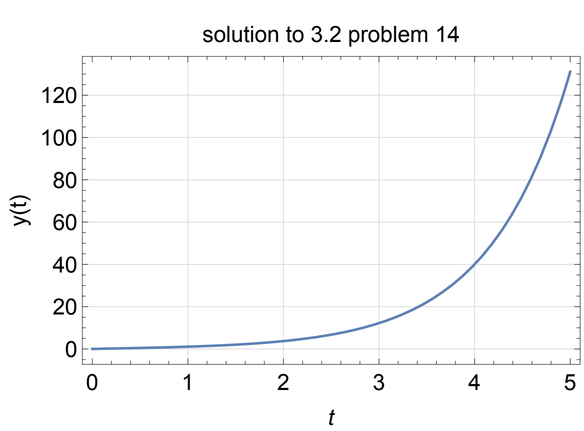
Since $-\frac{1}{4} + \frac{\sqrt{33}}{4} = 1.186$ and $-\frac{1}{4} - \frac{\sqrt{33}}{4} = -1.686$ then the above can be written as

$$y = \frac{2}{\sqrt{33}} e^{1.186t} - \frac{2}{\sqrt{33}} e^{-1.186t}$$

Then we see that as $t \to \infty$ the second term $e^{-1.186t} \to 0$ and we are left with $e^{1.186t}$ which will go to ∞ for large t. Hence

$$\lim_{t\to\infty} y(t) = \infty$$

Here is sketch of the solution



1.7 Section 3.1 problem 15

Find the solution to $y'' + 8y' - 9y = 0; y(1) = 1, y'(1) = 0$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$r^2 + 8r - 9 = 0$$
$$(r-1)(r+9) = 0$$

Hence the roots are $r_1 = 1, r_2 = -9$. The two solutions are

$$y_1 = e^t$$
$$y_2 = e^{-9t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^t + c_2 e^{-9t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition $(y(1) = 1)$ to the general solution gives

$$1 = c_1 e^1 + c_2 e^{-9} \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = c_1 e^t - 9c_2 e^{-9t}$$

Applying second initial condition to this results in

$$0 = c_1 e^1 - 9c_2 e^{-9} \tag{2}$$

From (1), $c_1 = \frac{1-c_2 e^{-9}}{e^1} = e^{-1} - c_2 e^{-10}$ and from (2)

$$\begin{aligned} 0 &= \left(e^{-1} - c_2 e^{-10}\right) e^1 - 9c_2 e^{-9} \\ &= 1 - c_2 e^{-9} - 9c_2 e^{-9} \\ &= 1 + c_2\left(-e^{-9} - 9e^{-9}\right) \\ 0 &= 1 + c_2\left(-10e^{-9}\right) \end{aligned}$$

Hence

$$c_2 = \frac{1}{10} e^9$$

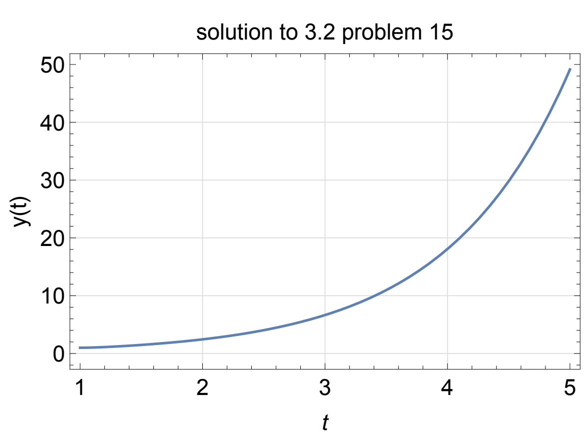
Therefore $c_1 = e^{-1} - c_2 e^{-10} = e^{-1} - \frac{1}{10} e^9 e^{-10} = e^{-1} - \frac{1}{10} e^{-1} = \frac{9}{10} e^{-1}$ and the solution becomes

$$\begin{aligned} y &= \frac{9}{10} e^{-1} e^t + \frac{1}{10} e^9 e^{-9t} \\ &= \frac{9}{10} e^{t-1} + \frac{1}{10} e^{9-9t} \end{aligned}$$

Then we see that as $t \to \infty$ the second term $e^{9-9t} \to 0$ and we are left with e^{t-1} which will go to ∞ for large t. Hence

$$\lim_{t\to\infty} y(t) = \infty$$

Here is sketch of the solution.



1.8 Section 3.1 problem 16

Find the solution to $4y'' - y = 0; y(-2) = 1, y'(-2) = -1$ and sketch the solution and describe its behavior as t increases.

solution

The characteristic equation is found by substituting $y = e^{rt}$ into the ODE and simplifying, giving

$$4r^2 - 1 = 0$$

Hence the roots are $r_1 = \pm\frac{1}{2}$. The two solutions are

$$y_1 = e^{\frac{1}{2}t}$$
$$y_2 = e^{-\frac{1}{2}t}$$

The general solution is linear combination of the above two solutions

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t}$$

Now c_1, c_2 are found from initial conditions. Applying first initial condition ($y(-2) = 1$) to the general solution gives

$$1 = c_1 e^{-1} + c_2 e \tag{1}$$

Taking time derivative of the general solution gives

$$y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} - \frac{1}{2} c_2 e^{-\frac{1}{2}t}$$

Applying second initial condition to this results in

$$-1 = \frac{1}{2} c_1 e^{-1} - \frac{1}{2} c_2 e \tag{2}$$

From (1), $c_1 = \frac{1-c_2 e}{e^{-1}} = e - c_2 e^2$ and from (2)

$$\begin{aligned} -1 &= \frac{1}{2}\left(e - c_2 e^2\right) e^{-1} - \frac{1}{2} c_2 e \\ &= \frac{1}{2} - \frac{1}{2} c_2 e - \frac{1}{2} c_2 e \\ &= \frac{1}{2} - c_2 e \end{aligned}$$

Hence

$$c_2 = \frac{1}{2} e^{-1} + e^{-1} = \frac{3}{2} e^{-1}$$

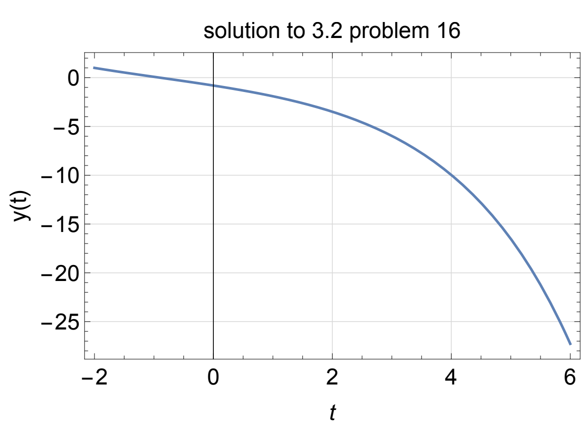
Therefore $c_1 = e - \left(\frac{3}{2} e^{-1}\right) e^2 = e - \frac{3}{2} e = -\frac{1}{2} e$ and the solution becomes

$$\begin{aligned} y &= c_1 e^{\frac{1}{2}t} + c_2 e^{-\frac{1}{2}t} \\ &= -\frac{1}{2} e e^{\frac{1}{2}t} + \frac{3}{2} e^{-1} e^{\frac{-1}{2}t} \\ &= -\frac{1}{2} e^{1+\frac{t}{2}} + \frac{3}{2} e^{-1-\frac{t}{2}} \end{aligned}$$

Then we see that as $t \to \infty$ the second term $e^{-1-\frac{t}{2}} \to 0$ and we are left with $-\frac{1}{2} e^{1+\frac{t}{2}}$ which will go to $-\infty$ for large t. Hence

$$\lim_{t\to\infty} y(t) = -\infty$$

Here is sketch of the solution.



1.9 Section 3.2 problem 1

Find the Wronskian of the given pair of functions $e^{2t}, e^{-\frac{3t}{2}}$

solution

We are given $y_1(t) = e^{2t}, y_2(t) = e^{\frac{-3}{2}t}$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} e^{2t} & e^{-\frac{3t}{2}} \\ 2e^{2t} & -\frac{2}{3}e^{-\frac{3t}{2}} \end{vmatrix} \\ &= \frac{-3}{2}e^{\frac{t}{2}} - 2e^{\frac{t}{2}} \\ &= \frac{-7}{2}e^{\frac{t}{2}} \end{aligned}$$

1.10 Section 3.2 problem 2

Find the Wronskian of the given pair of functions $\cos t, \sin t$

solution

We are given $y_1(t) = \cos t, y_2(t) = \sin t$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= \cos^2 t + \sin^2 t \\ &= 1 \end{aligned}$$

1.11 Section 3.2 problem 3

Find the Wronskian of the given pair of functions e^{-2t}, te^{-2t}

solution

We are given $y_1(t) = e^{-2t}, y_2(t) = te^{-2t}$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} \\ &= \left(e^{-2t}\right)\left(e^{-2t} - 2te^{-2t}\right) + 2e^{-2t}te^{-2t} \\ &= e^{-4t} - 2te^{-4t} + 2te^{-4t} \\ &= e^{-4t} \end{aligned}$$

1.12 Section 3.2 problem 4

Find the Wronskian of the given pair of functions x, xe^x

solution

We are given $y_1(x) = x, y_2(x) = xe^x$, hence by definition, the Wronskian is

$$\begin{aligned} W &= \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= (x)(e^x + xe^x) - xe^x \\ &= xe^x + x^2e^x - xe^x \\ &= x^2e^x \end{aligned}$$

1.13 Section 3.2 problem 5

Find the Wronskian of the given pair of functions $e^t \sin t, e^t \cos t$

solution

We are given $y_1(t) = e^t \sin t, y_2(t) = e^t \cos t$, hence by definition, the Wronskian is

$$\begin{aligned}
W &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\
&= \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \sin t + e^t \cos t & e^t \cos t - e^t \sin t \end{vmatrix} \\
&= \left(e^t \sin t\right)\left(e^t \cos t - e^t \sin t\right) - e^t \cos t \left(e^t \sin t + e^t \cos t\right) \\
&= e^{2t} \sin t \cos t - e^{2t} \sin^2 t - e^{2t} \cos t \sin t - e^{2t} \cos^2 t \\
&= -e^{2t} \sin^2 t - e^{2t} \cos^2 t \\
&= -2e^{2t} \left(\sin^2 t + \cos^2 t\right) \\
&= -2e^{2t}
\end{aligned}$$

1.14 Section 3.2 problem 6

Find the Wronskian of the given pair of functions $\cos^2 \theta, 1 + \cos 2\theta$

solution

We are given $y_1(\theta) = \cos^2 \theta, y_2(\theta) = 1 + \cos 2\theta$, hence by definition, the Wronskian is

$$\begin{aligned}
W &= \begin{vmatrix} y_1(\theta) & y_2(\theta) \\ y_1'(\theta) & y_2'(\theta) \end{vmatrix} \\
&= \begin{vmatrix} \cos^2 \theta & 1 + \cos 2\theta \\ -2 \cos \theta \sin \theta & -2 \sin 2\theta \end{vmatrix} \\
&= -2 \cos^2 \theta \sin 2\theta - (1 + \cos 2\theta)(-2 \cos \theta \sin \theta) \\
&= -2 \cos^2 \theta \sin 2\theta - (-2 \cos \theta \sin \theta - 2 \cos \theta \sin \theta \cos 2\theta) \\
&= -2 \cos^2 \theta \sin 2\theta + 2 \cos \theta \sin \theta + 2 \cos \theta \sin \theta \cos 2\theta
\end{aligned}$$

Using $\cos 2\theta = 2 \cos^2 \theta - 1$ And $\sin 2\theta = 2 \sin \theta \cos \theta$ the above becomes

$$\begin{aligned}
W &= -2 \cos^2 \theta (2 \sin \theta \cos \theta) + 2 \cos \theta \sin \theta + 2 \cos \theta \sin \theta \left(2 \cos^2 \theta - 1\right) \\
&= -4 \cos^3 \theta \sin \theta + 2 \cos \theta \sin \theta + 4 \cos^3 \theta \sin \theta - 2 \cos \theta \sin \theta \\
&= -4 \cos^3 \theta \sin \theta + 4 \cos^3 \theta \sin \theta \\
&= 0
\end{aligned}$$

We could also see that $W = 0$ more directly, by noticing that $y_1 = \cos^2 \theta = 1 - \sin^2 \theta$ and since $\sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$ then

$$\begin{aligned}
y_1 &= \cos^2 \theta \\
&= 1 - \left(\frac{1}{2} - \frac{1}{2} \cos 2\theta\right) \\
&= \frac{1}{2} + \frac{1}{2} \cos 2\theta \\
&= \frac{1}{2}(1 + \cos 2\theta)
\end{aligned}$$

Therefore, $y_1 = \frac{1}{2} y_2$. Hence y_2 is just a scaled version of y_1 and so these are two solutions are not linearly independent functions, (parallel to each others in vector space view) and so we expect that the Wronskian to be zero.