# HW 4, Math 319, Fall 2016

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## Contents

# 1 HW 4

## 1.1 Section 2.6 problem 19

Question Show that $x^2y^3 + x\left(1+y^2\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu\left(x,y\right) = \frac{1}{xy^3}$ and then solve.

Solution The first step is to apply theorem two and also check where the ODE is singular. Writing it as

$$\frac{dy}{dx} = f\left(x,y\right) = \frac{-x^2y^3}{x\left(1+y^2\right)}$$

This is non-linear first order ODE. There is a pole at $x = 0$. From theorem two, this says that unique solution is not guaranteed to exist since the first condition which says that $f\left(x,y\right)$ must be continuous, was not satisfied. Now the ODE is solved.

$$\overbrace{x^2y^3}^{M} + \overbrace{x\left(1+y^2\right)}^{N}y' = 0$$

Hence

$$M\left(x,y\right) = x^2y^3$$
$$N\left(x,y\right) = x\left(1+y^2\right)$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\frac{\partial M}{\partial y} = 3x^2y^2$$
$$\frac{\partial N}{\partial x} = 1+y^2$$

The above shows that that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\left(\mu x^2y^3\right) + \mu x\left(1+y^2\right)y' = 0$$
$$\frac{x^2y^3}{xy^3} + \frac{1}{xy^3}x\left(1+y^2\right)y' = 0$$
$$x + \frac{1}{y^3}\left(1+y^2\right)y' = 0$$

Now $\bar{M} = x$ and $\bar{N} = \frac{1}{y^3}\left(1+y^2\right)$. Checking that the new $\bar{M},\bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = 0$$
$$\frac{\partial \bar{N}}{\partial x} = 0$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = x \tag{1}$$
$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = \frac{1}{y^3}\left(1+y^2\right) \tag{2}$$

Integrating (1) w.r.t $x$ gives

$$\Psi = \frac{1}{2}x^2 + f\left(y\right) \tag{3}$$
$$\frac{\partial \Psi}{\partial y} = f'\left(y\right)$$

Comparing the above to (2) in order to solve for $f'\left(y\right)$ gives

$$f'\left(y\right) = \frac{1+y^2}{y^3}$$
$$f\left(y\right) = \int \frac{1+y^2}{y^3}dy + c \tag{4}$$

We need now to solve $\int \frac{1+y^2}{y^3} dy$

$$\begin{aligned}\int \frac{1+y^2}{y^3} dy &= \int \frac{1}{y^3} dy + \int \frac{y^2}{y^3} dy \\ &= -\frac{1}{2y^2} + \int \frac{1}{y} dy \\ &= -\frac{1}{2y^2} + \ln |y|\end{aligned}$$

Using the above solution in (4) gives

$$f(y) = -\frac{1}{2y^2} + \ln |y| + c$$

Using the above in (3) gives

$$\Psi = \frac{1}{2}x^2 - \frac{1}{2y^2} + \ln |y| + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln |y| = C \qquad x \neq 0$$

Checking $y = 0$ as solution, shows that putting $y = 0$ in $f(x, y) = \frac{-x^2 y^3}{x(1+y^2)} = 0$. Hence $y = 0$ is also a solution.

Summary The solutions are

$$\begin{aligned}\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln |y| = C \qquad & x \neq 0, y \neq 0 \\ y = 0 \qquad & x \neq 0\end{aligned}$$

## 1.2 Section 2.6 problem 20

Question Show that $\left(\frac{\sin y}{y} - 2e^{-x} \sin x\right) + \left(\frac{\cos y + 2e^{-x} \cos x}{y}\right) y' = 0$ is not exact, and then becomes exact when multiplied by $\mu(x, y) = ye^x$ and then solve.

Solution First we will check where the ODE is singular. Writing it as

$$\frac{dy}{dx} = f(x, y) = \frac{\frac{\sin y}{y} - 2e^{-x} \sin x}{\frac{\cos y + 2e^{-x} \cos x}{y}}$$

This is non-linear first order ODE. We see a pole at $y = 0$. Hence $y \neq 0$. From theorem two, this says that that unique solution is not guaranteed since first condition which says that $f(x, y)$ must be continuous, was not satisfied.

$$\begin{aligned}M(x, y) &= \frac{\sin y}{y} - 2e^{-x} \sin x \\ N(x, y) &= \frac{\cos y + 2e^{-x} \cos x}{y}\end{aligned}$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\begin{aligned}\frac{\partial M}{\partial y} &= \ln y \sin y + \frac{1}{y} \cos y \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y} \cos y + \frac{1}{y} 2e^{-x} \cos x\right) = \frac{-1}{y} 2e^{-x} \cos x - \frac{1}{y} 2e^{-x} \sin x = \frac{-2e^{-x}}{y} (\cos x + \sin x)\end{aligned}$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\begin{aligned}\mu\left(\frac{\sin y}{y} - 2e^{-x} \sin x\right) + \mu\left(\frac{\cos y + 2e^{-x} \cos x}{y}\right) y' &= 0 \\ ye^x\left(\frac{\sin y}{y} - 2e^{-x} \sin x\right) + ye^x\left(\frac{\cos y + 2e^{-x} \cos x}{y}\right) y' &= 0 \\ \left(e^x \sin y - 2y \sin x\right) + \left(e^x \cos y + 2 \cos x\right) y' &= 0\end{aligned}$$

Now

$$\bar{M} = e^x \sin y - 2y \sin x$$
$$\bar{N} = e^x \cos y + 2\cos x$$

Checking now the new $\bar{M},\bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = e^x \cos y - 2\sin x$$
$$\frac{\partial \bar{N}}{\partial x} = e^x \cos y - 2\sin x$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi(x,y)}{\partial x} = \bar{M} = e^x \sin y - 2y \sin x \tag{1}$$
$$\frac{\partial \Psi(x,y)}{\partial y} = \bar{N} = e^x \cos y + 2\cos x \tag{2}$$

Integrating (1) w.r.t $x$ gives

$$\Psi = e^x \sin y + 2y\cos x + f(y) \tag{3}$$
$$\frac{\partial \Psi}{\partial y} = e^x \cos y + 2\cos x + f'(y)$$

Comparing the above to (2) in order to solve for $f'(y)$ gives

$$e^x \cos y + 2\cos x + f'(y) = e^x \cos y + 2\cos x$$
$$f'(y) = 0$$
$$f(y) = c \tag{4}$$

Substituting the above into (3) gives

$$\Psi = e^x \sin y + 2y\cos x + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$e^x \sin y + 2y \cos x = C \qquad y \neq 0$$

### 1.3 Section 2.6 problem 21

Question Show that $y + (2x - ye^y)\,y' = 0$ is not exact, and then becomes exact when multiplied by $\mu(x,y) = y$ and then solve.

Solution

$$M(x,y) = y$$
$$N(x,y) = 2x - ye^y$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\frac{\partial M}{\partial y} = 1$$
$$\frac{\partial N}{\partial x} = 2$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\mu y + \mu(2x - ye^y)\,y' = 0$$
$$y^2 + (2xy - y^2 e^y)\,y' = 0$$

Now

$$\bar{M} = y^2$$
$$\bar{N} = 2xy - y^2 e^y$$

Checking now the new $\bar{M},\bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = 2y$$
$$\frac{\partial \bar{N}}{\partial x} = 2y$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = y^2 \tag{1}$$

$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = 2xy - y^2 e^y \tag{2}$$

Integrating (1) w.r.t $x$ gives

$$\Psi = y^2 x + f\left(y\right) \tag{3}$$

$$\frac{\partial \Psi}{\partial y} = 2yx + f'\left(y\right)$$

Comparing the above to (2) in order to solve for $f'\left(y\right)$ gives

$$\begin{aligned} 2yx + f'\left(y\right) &= 2xy - y^2 e^y \\ f'\left(y\right) &= -y^2 e^y \\ f\left(y\right) &= -\int y^2 e^y dy + c \end{aligned} \tag{4}$$

The integral $\int y^2 e^y dy$ can be found using integration by parts. Let $u = y^2, dv = e^y \rightarrow du = 2y, v = e^y$, therefore

$$\begin{aligned} \int y^2 e^y dy &= \int u dv \\ &= uv - \int v du \\ &= y^2 e^y - 2\int y e^y dy \end{aligned}$$

Applying integration by parts again to $\int y e^y dy$, where now $u = y, dv = e^y \rightarrow du = 1, v = e^y$, the above becomes

$$\begin{aligned} \int y^2 e^y dy &= y^2 e^y - 2\left(y e^y - \int e^y dy\right) \\ &= y^2 e^y - 2\left(y e^y - e^y\right) \\ &= y^2 e^y - 2y e^y + 2e^y \\ &= e^y\left(y^2 - 2y + 2\right) \end{aligned}$$

Therefore from (4)

$$f\left(y\right) = -e^y\left(y^2 - 2y + 2\right) + c$$

Substituting the above into (3) gives

$$\Psi = y^2 x - e^y\left(y^2 - 2y + 2\right) + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore the above simplifies to, after collecting all constants to one

$$y^2 x - e^y\left(y^2 - 2y + 2\right) = C$$

### 1.4 Section 2.6 problem 22

Question Show that $(x+2)\sin y + \left(x\cos y\right)y' = 0$ is not exact, and then becomes exact when multiplied by $\mu\left(x,y\right) = xe^x$ and then solve.

Solution

$$\begin{aligned} M\left(x,y\right) &= (x+2)\sin y \\ N\left(x,y\right) &= x\cos y \end{aligned}$$

An ODE is exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. These are now calculated to see if the ODE is exact or not

$$\begin{aligned} \frac{\partial M}{\partial y} &= (x+2)\cos y \\ \frac{\partial N}{\partial x} &= \cos y \end{aligned}$$

From above we see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ therefore the ODE is not exact. Multiplying the original ODE by given integrating factor it becomes

$$\begin{aligned} \mu(x+2)\sin y + \mu\left(x\cos y\right)y' &= 0 \\ xe^x(x+2)\sin y + xe^x\left(x\cos y\right)y' &= 0 \end{aligned}$$

Now

$$\bar{M} = \left(x^2e^x + 2xe^x\right)\sin y$$
$$\bar{N} = x^2e^x\cos y$$

Checking now the new $\bar{M}, \bar{N}$ are indeed exact.

$$\frac{\partial \bar{M}}{\partial y} = \left(x^2e^x + 2xe^x\right)\cos y$$
$$\frac{\partial \bar{N}}{\partial x} = 2xe^x\cos y + x^2e^x\cos y = \left(x^2e^x + 2xe^x\right)\cos y$$

The new ODE is exact. Now the ODE is solved using the standard method.

$$\frac{\partial \Psi\left(x,y\right)}{\partial x} = \bar{M} = \left(x^2e^x + 2xe^x\right)\sin y \tag{1}$$
$$\frac{\partial \Psi\left(x,y\right)}{\partial y} = \bar{N} = x^2e^x\cos y \tag{2}$$

Integrating (2) w.r.t $y$ as it is simpler than integrating (1) w.r.t. $x$, gives

$$\Psi = \int x^2e^x\cos y dy = x^2e^x\sin y + f\left(x\right) \tag{3}$$
$$\frac{\partial \Psi}{\partial x} = 2xe^x\sin y + x^2e^x\sin y + f'\left(x\right)$$

Comparing the above to (1) in order to solve for $f'\left(x\right)$ gives

$$2xe^x\sin y + x^2e^x\sin y + f'\left(x\right) = \left(x^2e^x + 2xe^x\right)\sin y$$
$$f'\left(x\right) = 0$$
$$f\left(x\right) = c \tag{4}$$

Substituting the above into (3) gives

$$\Psi = x^2e^x\sin y + c$$

But $\frac{d\Psi}{dx} = c_0$, therefore $\Psi = c_1$ and the above simplifies to, after collecting all constants to one

$$x^2e^x\sin y = C$$

## 1.5 Section 2.6 problem 23

Question Show that if $\frac{N_x - M_y}{M} = Q$ where $Q$ is function of $y$ only, then $M + Ny' = 0$ has integrating factor of form $\mu\left(y\right) = e^{\int Q\left(y\right)dt}$

Solution Given the differential equation

$$M\left(x,y\right) + N\left(x,y\right)\frac{dy\left(x\right)}{dx} = 0$$

Multiplying by $\mu\left(y\right)$ results in

$$\mu M + \mu Ny' = 0$$

The above is exact if

$$\frac{\partial\left(\mu M\right)}{\partial y} = \frac{\partial\left(\mu N\right)}{\partial x}$$

Performing the above, taking into account that $\mu$ depends on $y$ only, results in

$$\frac{d\mu}{dy}M + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}$$

The above is first order ODE in $\mu$

$$\frac{d\mu}{dy}M = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$
$$\frac{d\mu}{dy} = \mu\left(\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\right)$$

Let $Q=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$. If $Q$ depends on $y$ only, then the above ODE is separable. Hence

$$\begin{aligned}\frac{d\mu}{dy}&=\mu Q\left(y\right)\\ \frac{d\mu}{\mu}&=Q\left(y\right)dy\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\ln\left|\mu\right|&=\int Q\left(y\right)dy+C\\ \left|\mu\right|&=e^{\int Q\left(y\right)dy+C}\\ \mu\left(y\right)&=Ae^{\int Q\left(y\right)dy}\end{aligned}$$

Where $A$ is some constant, which can be taken to be 1 leading to the result required to show. The above procedure works only when $Q=\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}$ happened to be function of $y$ only. This complete the proof.

## 1.6 Section 2.6 problem 24

Question Show that if $\frac{N_x-M_y}{xM-yN}=R$ where $R$ is function of $xy$ only, then $M+Ny'=0$ has integrating factor of form $\mu\left(x,y\right)$. Find the general formula for $\mu$.

Solution Given the differential equation

$$M\left(x,y\right)+N\left(x,y\right)\frac{dy\left(x\right)}{dx}=0$$

Let $\mu\left(t\right)$ where $t=xy$. Multiplying the above with $\mu\left(t\right)$ gives

$$\mu\left(t\right)M\left(x,y\right)+\mu\left(t\right)N\left(x,y\right)\frac{dy\left(x\right)}{dx}=0$$

The above is exact when

$$\frac{\partial\mu M}{\partial y}=\frac{\partial\mu N}{\partial x}$$

Hence

$$\frac{\partial\mu}{\partial y}M+\mu\frac{\partial M}{\partial y}=\frac{\partial\mu}{\partial x}N+\mu\frac{\partial N}{\partial x}\tag{1}$$

However,

$$\frac{\partial\mu}{\partial y}=\frac{d\mu}{dt}\frac{\partial t}{dy}=\frac{d\mu}{dt}x\tag{2}$$

And

$$\frac{\partial\mu}{\partial x}=\frac{d\mu}{dt}\frac{\partial t}{dx}=\frac{d\mu}{dt}y\tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned}\frac{d\mu}{dt}xM+\mu\frac{\partial M}{\partial y}&=\frac{d\mu}{dt}yN+\mu\frac{\partial N}{\partial x}\\ \frac{d\mu}{dt}\left(xM-yN\right)&=\mu\frac{\partial N}{\partial x}-\mu\frac{\partial M}{\partial y}\\ \frac{d\mu\left(t\right)}{dt}&=\mu\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}\end{aligned}$$

In the above, $\mu$ depends on $t$ only, where $t$ is function of $xy$ only. If $\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}$ depends on $t$ only, then the above can be considered a separable first order ODE in $\mu$. Let $R\left(t\right)=\frac{\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)}{\left(xM-yN\right)}$ and the above can be written as

$$\frac{d\mu\left(t\right)}{dt}=\mu R\left(t\right)$$

Since separable, then

$$\begin{aligned}\frac{d\mu(t)}{\mu} &= R(t)\,dt\\ \int\frac{d\mu}{\mu} &= \int Rdt\\ \ln|\mu| &= \int Rdt + C\\ |\mu| &= e^{\int Rdt+C}\\ \mu &= Ae^{\int Rdt}\end{aligned}$$

Where $A$ is constant of integration which can be taken to be 1. Hence $\mu = e^{\int Rdt}$. This works only if $R$ is function of $t$ only.

## 1.7 Section 2.7 problem 20

20. **Convergence of Euler's Method.** It can be shown that under suitable conditions on $f$, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y),\ y(t_0) = y_0$ converges to the exact solution as the step size $h$ decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \qquad y(t_0) = y_0.$$

(a) Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.
(b) Using the Euler formula, show that

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, \ldots.$$

(c) Noting that $y_1 = (1+h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1+h)^n(y_0 - t_0) + t_n \tag{i}$$

for each positive integer $n$.
(d) Consider a fixed point $t > t_0$ and for a given $n$ choose $h = (t - t_0)/n$. Then $t_n = t$ for every $n$. Note also that $h \to 0$ as $n \to \infty$. By substituting for $h$ in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.
*Hint:* $\lim_{n\to\infty}(1 + a/n)^n = e^a$.

### 1.7.1 part a

$$\begin{aligned}y' &= 1 - t + y\\ y(t_0) &= y_0\end{aligned}$$

This is linear first order ODE. Writing it as $y' - y = 1 - t$, then the integrating factor is $\mu = e^{-\int dt} = e^{-t}$ and the ODE becomes

$$\frac{d}{dt}\left(ye^{-t}\right) = e^{-t}(1-t)$$

Integrating both sides

$$\begin{aligned}ye^{-t} &= \int e^{-t}(1-t)\,dt + c\\ &= \int e^{-t}dt - \int te^{-t}dt + c \qquad (1)\end{aligned}$$

But $\int te^{-t}dt = \int udv$ where $u = t, dv = e^{-t} \to du = 1, v = -e^{-t}$, hence

$$\begin{aligned}\int te^{-t}dt &= uv - \int vdu\\ &= -te^{-t} + \int e^{-t}dt\\ &= -te^{-t} - e^{-t}\end{aligned}$$

Putting this result in (1) gives

$$
\begin{aligned}
ye^{-t} &= -e^{-t} - \left(-te^{-t} - e^{-t}\right) + c \\
&= -e^{-t} + te^{-t} + e^{-t} + c \\
&= te^{-t} + c
\end{aligned}
$$

Therefore solving for $y$ gives

$$
y = t + ce^{t} \tag{2}
$$

The constant $c$ is now found from initial conditions.

$$
\begin{aligned}
y_0 &= t_0 + ce^{t_0} \\
c &= \left(y_0 - t_0\right) e^{-t_0}
\end{aligned}
$$

Substituting $c$ found back into (2) gives the final solution

$$
\begin{aligned}
y &= t + \left(y_0 - t_0\right) e^{-t_0} e^{t} \\
&= \left(y_0 - t_0\right) e^{t-t_0} + t
\end{aligned} \tag{3}
$$

### 1.7.2 Part b

Euler formula is

$$
y_k = hf\left(t_{k-1}, y_{k-1}\right) + y_{k-1} \qquad k = 1, 2, 3, \cdots \tag{1}
$$

Where in this problem $f\left(t_{k-1}, y_{k-1}\right)$ is the RHS of $y' = 1 - t + y$ but evaluated at $t_{k-1}$. Hence

$$
f\left(t_{k-1}, y_{k-1}\right) = 1 - t_{k-1} + y_{k-1}
$$

Substituting this into (1) gives

$$
\begin{aligned}
y_k &= h\left(1 - t_{k-1} + y_{k-1}\right) + y_{k-1} \\
&= h - ht_{k-1} + hy_{k-1} + y_{k-1} \\
&= (1 + h)\, y_{k-1} + h - ht_{k-1} \qquad k = 1, 2, 3, \cdots
\end{aligned}
$$

Which is the required formula asked to derive.

### 1.7.3 Part c

The formula given $y_1 = (1 + h)\left(y_0 - t_0\right) + t_1$ can be found as follows. Since

$$
\begin{aligned}
y_1 &= y_0 + hf\left(t_0, y_0\right) \\
&= y_0 + h\left(1 - t_0 + y_0\right) \\
&= y_0 + h - ht_1 + hy_0
\end{aligned}
$$

Adding $t_0 - t_0$ to the above will not changed anything, hence

$$
y_1 = y_0 + h - ht_1 + hy_0 + t_0 - t_0
$$

But $t_1 = t_0 + h$ by definition, hence the above becomes, by replacing $t_0 + h$ above with $t_1$

$$
y_1 = y_0 + t_1 - ht_1 + hy_0 - t_0
$$

Simplifying

$$
\begin{aligned}
y_1 &= \left(y_0 - t_0\right) + h\left(y_0 - t_0\right) + t_1 \\
&= (1 + h)\left(y_0 - t_0\right) + t_1
\end{aligned}
$$

Now the question will be answered. Need to show that $y_n = (1 + h)^n \left(y_0 - t_0\right) + t_n$ is true, using induction. This is true for $k = 1$ as shown above. Now assuming it is true for $k$, we then need to show it is true for $k + 1$.

By assumption, it is true for $k$, hence

$$
y_k = (1 + h)^k \left(y_0 - t_0\right) + t_k \tag{1}
$$

But using Euler formula

$$
\begin{aligned}
y_{k+1} &= y_k + hf\left(t_k, y_k\right) \\
&= y_k + h\left(1 - t_k + y_k\right)
\end{aligned} \tag{2}
$$

Substituting (1) into RHS of (2)

$$\begin{aligned}
y_{k+1} &= \left((1+h)^k \left(y_0 - t_0\right) + t_k\right) + h\left(1 - t_k + \left((1+h)^k \left(y_0 - t_0\right) + t_k\right)\right) \\
&= (1+h)^k \left(y_0 - t_0\right) + t_k + h - ht_k + h\left((1+h)^k \left(y_0 - t_0\right) + t_k\right) \\
&= (1+h)^k \left(y_0 - t_0\right) + t_k + h - ht_k + h(1+h)^k \left(y_0 - t_0\right) + ht_k \\
&= (1+h)^k \left(y_0 - t_0\right) + t_k + h + h(1+h)^k \left(y_0 - t_0\right)
\end{aligned}$$

But $t_k + h = t_{k+1}$ by definition, hence

$$\begin{aligned}
y_{k+1} &= (1+h)^k \left(y_0 - t_0\right) + t_{k+1} + h(1+h)^k \left(y_0 - t_0\right) \\
&= (1+h)^k \left(y_0 - t_0\right)(1+h) + t_{k+1} \\
&= (1+h)^{k+1} \left(y_0 - t_0\right) + t_{k+1}
\end{aligned}$$

The above shows it is true for $k+1$ given it is true for $k$. Therefore, it is true for any positive integer $n$.

### 1.7.4 Part d

Using

$$y_n = (1+h)^n \left(y_0 - t_0\right) + t_n$$

Replacing $h = \frac{t_n - t_0}{n}$ in the above gives

$$y_n = \left(1 + \left(\frac{t_n - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + t_n$$

Taking the limit

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \left(\frac{t_n - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + \lim_{n\to\infty} t_n$$

But $\lim_{n\to\infty} t_n = t$, hence replacing all $t_n$ with $t$ in the above gives

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \left(\frac{t - t_0}{n}\right)\right)^n \left(y_0 - t_0\right) + t$$

Using hint that $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$ the above simplifies to

$$\begin{aligned}
y(t) &= \lim_{n\to\infty} y_n \\
&= e^{(t-t_0)} \left(y_0 - t_0\right) + t
\end{aligned}$$

Which is the analytical solution found in part (a).

## 1.8 Section 3.1 problem 1

Find the general solution to $y'' + 2y' - 3y = 0$.

This is second order, linear, constant coefficient ODE. Letting $y = e^{rt}$ and replacing this into the ODE gives

$$e^{rt}\left(r^2 + 2r - 3\right) = 0$$

Since $e^{rt} \neq 0$, the above reduces to what is called the characteristic equation of the ODE

$$\boxed{r^2 + 2r - 3 = 0}$$

Which can be written as $(r-1)(r+3) = 0$. Hence $r_1 = 1, r_2 = -3$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^t + c_2 e^{-3t}$$

## 1.9 Section 3.1 problem 2

Find the general solution to $y'' + 3y' + 2y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{r^2 + 3r + 2 = 0}$$

Which can be written as $(r+1)(r+2)=0$. Hence $r_1=-1, r_2=-2$. Therefore the solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t)=c_1e^{-t}+c_2e^{-2t}$$

## 1.10 Section 3.1 problem 3

Find the general solution to $6y''-y'-y=0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{6r^2-r-1=0}$$

Hence $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{1\pm\sqrt{1-4(6)(-1)}}{12}=\frac{1\pm\sqrt{1+24}}{12}=\frac{1\pm5}{12}$. Hence $r_1=\frac{1}{2}, r_2=\frac{-1}{3}$. Therefore the solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t)=c_1e^{\frac{1}{2}t}+c_2e^{\frac{-1}{3}t}$$

## 1.11 Section 3.1 problem 4

Find the general solution to $2y''-3y'+y=0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{2r^2-3r+1=0}$$

Hence $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{3\pm\sqrt{9-4(2)(1)}}{4}=\frac{3\pm1}{4}$. Hence $r_1=1, r_2=\frac{1}{2}$. Therefore the solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t)=c_1e^{t}+c_2e^{\frac{1}{2}t}$$

## 1.12 Section 3.1 problem 5

Find the general solution to $y''+5y'=0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{r^2+5r=0}$$

Which can be written as $r(r+5)=0$, hence $r_1=0, r_2=-5$.Therefore the solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t)=c_1+c_2e^{-5t}$$

## 1.13 Section 3.1 problem 6

Find the general solution to $4y''-9y=0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE

$$\boxed{4r^2-9=0}$$

Therefore $r^2=\frac{9}{4}$ or $r=\pm\sqrt{\frac{9}{4}}=\pm\frac{3}{2}$. Hence $r_1=\frac{3}{2}, r_2=-\frac{3}{2}$.Therefore the solution is

$$y(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t)=c_1e^{\frac{3}{2}t}+c_2e^{-\frac{3}{2}t}$$

## 1.14 Section 3.1 problem 7

Find the general solution to $y''-9y'+9y=0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE is

$$\boxed{r^2 - 9r + 9 = 0}$$

Hence $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{9\pm\sqrt{81-4(1)(9)}}{2} = \frac{9\pm\sqrt{81-36}}{2} = \frac{9\pm\sqrt{45}}{2} = \frac{9\pm3\sqrt{5}}{2}$. Hence $r_1 = \frac{9+3\sqrt{5}}{2}, r_2 = \frac{9-3\sqrt{5}}{2}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\frac{9+3\sqrt{5}}{2}t} + c_2 e^{\frac{9-3\sqrt{5}}{2}t}$$

## 1.15 Section 3.1 problem 8

Find the general solution to $y'' - 2y' - 2y = 0$.

This is second order, linear, constant coefficient ODE. The characteristic equation of the ODE is

$$\boxed{r^2 - 2r - 2 = 0}$$

Hence $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{2\pm\sqrt{4-4(1)(-2)}}{2} = \frac{2\pm\sqrt{4+8}}{2} = \frac{2\pm\sqrt{12}}{2} = \frac{2\pm2\sqrt{3}}{2} = 1 \pm \sqrt{3}$. Hence $r_1 = 1+\sqrt{3}, r_2 = 1-\sqrt{3}$. Therefore the solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Where $c_1, c_2$ are constants which can be found from initial conditions. Hence the general solution is

$$y(t) = c_1 e^{\left(1+\sqrt{3}\right)t} + c_2 e^{\left(1-\sqrt{3}\right)t}$$