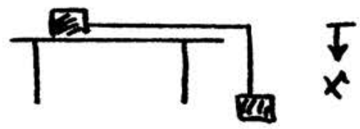
Mechanics
Physics 311 - Fall 2012
Midterm 1 - Solutions

Problem 1:



(a) one coordinate describes the system, for example x (as shown)

(b)
$$T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2$$
$$= M\dot{x}^2 + \frac{1}{2} m \dot{x}^2$$

$$U = -Mgx - \frac{m}{\ell} x g \frac{x}{2}$$

fraction of cord mass hanging over the edge:
$$\frac{m}{\ell} \cdot x$$
center of mass is at $\frac{x}{2}$ for the overhanging part

$$\Rightarrow \boxed{L = \left(M + \frac{m}{2}\right) \dot{x}^2 + Mgx + \frac{mg}{2\ell} x^2}$$

(c)
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \Rightarrow \quad Mg + \frac{mg}{\ell} x - 2\left(M + \frac{m}{2}\right)\ddot{x} = 0$$

$$\Rightarrow (2M+m)\ddot{x} = \frac{g}{\ell}(M\ell + mx)$$

$$\Leftrightarrow \boxed{\ddot{x} = \frac{g}{\ell}\left(\frac{M\ell + mx}{2M+m}\right)}$$

(d) for $m=0$, the acceleration of the blocks is

$$\ddot{x} = \frac{g}{\ell}\frac{M\ell}{2M} \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{g}{2}}$$

Problem 2:

(a) Since the object can move in r and θ, we need two generalized coordinates

(b) $T = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$

$U = Mgr$ ($U=0$ for $r=0$)

$$\Rightarrow \boxed{L = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - Mgr}$$

(c) in θ: $\frac{\partial L}{\partial \theta} = 0 \Rightarrow$ Lagrangian does not explicitly depend on θ, so there is an integral of the motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 = \frac{d}{dt}\underbrace{(m r^2 \dot{\theta})}_{\text{angular momentum}}$$

$m r^2 \dot{\theta} = \text{const.}$

in r: $\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$

$$\Rightarrow m r \dot{\theta}^2 - Mg - (m+M)\ddot{r} = 0$$

$$\Leftrightarrow \boxed{(m+M)\ddot{r} - m r \dot{\theta}^2 + Mg = 0}$$

(d) $r = \text{const.} \Rightarrow \ddot{r} = 0 \Rightarrow m r \dot{\theta}^2 = Mg$

so $\dot{\theta} = \omega = \text{const}!$

$\Rightarrow m r \omega^2 = Mg$

or $$\boxed{\frac{m v^2}{r} = Mg}$$

so Mg is the force responsible for the centripetal acceleration

(e) there are two integrals of the motion,

(i) $mr^2\dot{\theta}$ (angular momentum)

(ii) mechanical energy $T+U$, since the Lagrangian does not explicitly depend on time