HW9 Physics 311 Mechanics

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Contents

0.1 Problem 1

1. (5 points)
A rigid body of arbitrary shape rotates freely under zero torque. Use Euler's equations to show that the rotational kinetic energy and the magnitude of the angular momentum are constant.

SOLUTION:

Euler solid body rotation equations are

$$(I_2 - I_3)\,\omega_2\omega_3 - I_1\dot{\omega}_1 = 0 \tag{1}$$
$$(I_3 - I_1)\,\omega_3\omega_1 - I_2\dot{\omega}_2 = 0 \tag{2}$$
$$(I_1 - I_2)\,\omega_1\omega_2 - I_3\dot{\omega}_3 = 0 \tag{3}$$

Where I_1, I_2, I_3 are the body moments of inertia around the principal axes. Multiplying both sides of (1) by $I_1\omega_1$ and both sides of (2) by $I_2\omega_2$ and both sides of (3) by $I_3\omega_3$ gives

$$\omega_1\omega_2\omega_3 I_1 I_2 - \omega_1\omega_2\omega_3 I_1 I_3 - I_1^2\omega_1\dot{\omega}_1 = 0 \tag{1A}$$
$$\omega_1\omega_2\omega_3 I_2 I_3 - \omega_1\omega_2\omega_3 I_1 I_2 - I_2^2\omega_2\dot{\omega}_2 = 0 \tag{2A}$$
$$\omega_1\omega_2\omega_3 I_1 I_3 - \omega_1\omega_2\omega_3 I_2 I_3 - I_3^2\omega_3\dot{\omega}_3 = 0 \tag{3A}$$

Adding (1A,2A,3A) gives (lots of terms cancel, that has $\omega_1\omega_2\omega_3$ in them)

$$I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 = 0 \tag{4}$$

But (4) is the same thing as

$$\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 = 0$$

where $\boldsymbol{L}$ is the angular momentum vector

$$\boldsymbol{L} = \{I_1\omega_1, I_2\omega_2, I_3\omega_3\}$$

Hence

$$\boldsymbol{L}^2 = \boldsymbol{L}\cdot\boldsymbol{L} = \left\{I_1^2\omega_1^2, I_2^2\omega_2^2, I_3^2\omega_3^2\right\}$$

Therefore, and since the $I's$ are constant, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 &= \frac{1}{2}\left\{2I_1^2\omega_1\dot{\omega}_1, 2I_2^2\omega_2\dot{\omega}_2, 2I_3^2\omega_3\dot{\omega}_3\right\}\\ &= \left\{I_1^2\omega_1\dot{\omega}_1, I_2^2\omega_2\dot{\omega}_2, I_3^2\omega_3\dot{\omega}_3\right\}\end{aligned} \tag{5}$$

Comparing (5) and (4), we see they are the same. This means that $\frac{1}{2}\frac{d}{dt}\boldsymbol{L}^2 = 0$ or $\boldsymbol{L}^2$ is a constant. Which implies $\boldsymbol{L}$ or the angular momentum is a constant vector.

To show that rotational kinetic energy is constant, we need to show that $\frac{1}{2}(\omega\cdot\boldsymbol{L})$ (which is the kinetic energy) is constant, where $\boldsymbol{\omega} = \{\omega_1, \omega_2, \omega_3\}$ is the angular velocity vector. But

$$\frac{1}{2}\frac{d}{dt}(\omega\cdot\boldsymbol{L}) = \frac{1}{2}\left(\dot{\omega}\cdot\boldsymbol{L} + \omega\cdot\dot{\boldsymbol{L}}\right)$$

But we found that $\dot{\boldsymbol{L}} = 0$ since $\boldsymbol{L}$ is constant. Hence the above becomes

$$\frac{1}{2}\frac{d}{dt}(\omega \cdot \boldsymbol{L}) = \frac{1}{2}\dot{\omega} \cdot \boldsymbol{L} \tag{6}$$

If we can show that $\dot{\omega} \cdot \boldsymbol{L} = 0$ then we are done. To do this, we go back to Euler equations (1,2,3) and now instead of multiplying by $I_i\omega_i$ as before, we now multiply by just ω_i each equation. This gives

$$\omega_1\omega_2\omega_3 I_2 - \omega_1\omega_2\omega_3 I_3 - I_1\omega_1\dot{\omega}_1 = 0 \tag{1C}$$

$$\omega_1\omega_2\omega_3 I_3 - \omega_1\omega_2\omega_3 I_1 - I_2\omega_2\dot{\omega}_2 = 0 \tag{2C}$$

$$\omega_1\omega_2\omega_3 I_1 - \omega_1\omega_2\omega_3 I_2 - I_3\omega_3\dot{\omega}_3 = 0 \tag{3C}$$

Adding gives (lots of terms cancel, that has $\omega_1\omega_2\omega_3$ in them)

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0 \tag{7}$$

But the above is the same as (6), with a factor of $\frac{1}{2}$. This means $\dot{\omega} \cdot \boldsymbol{L} = 0$ or $\frac{d}{dt}(\omega \cdot \boldsymbol{L}) = 0$ or that the rotational kinetic energy is constant. Which is what we are asked to show.

0.2 Problem 2

2. (10 points)
A uniform block of mass m and dimensions a by $2a$ by $3a$ spins about a long diagonal with angular velocity $\vec{\omega}$.

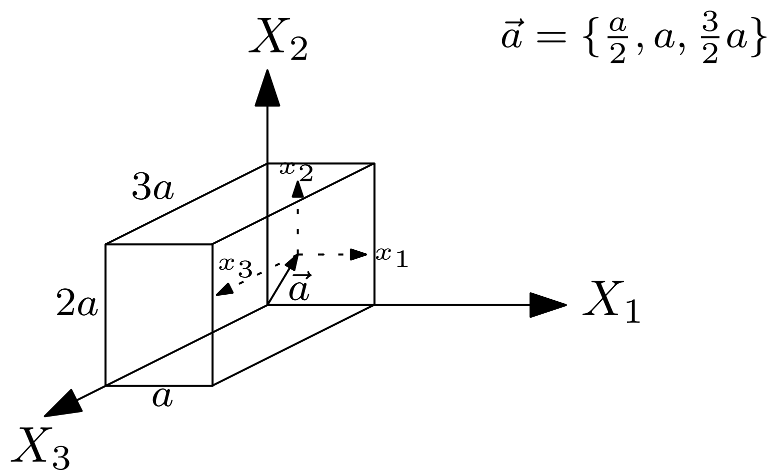
(1) Using a coordinate system with the origin at the center of the block, calculate the inertia tensor.

(2) Find the kinetic energy.

(3) Find the angle between the angular velocity $\vec{\omega}$ and the angular momentum $\vec{L}$.

(4) Find the magnitude of the torque that must be exerted on the block if $\vec{\omega}$ is constant.

SOLUTION:



0.2.1 Part(1)

We first find I (called J for now) around the origin of the inertial frame X_1, X_2, X_3 then use parallel axes theorem to find I at the center of the cube at $a = \left\{\frac{1}{2}a, a, \frac{3}{2}a\right\}$. The volume of the cube is $a\,(2a)\,(3a) = 6a^3$.

$$
\begin{aligned}
J_{11} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_2^2 + X_3^2\right) \\
&= \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 X_2^2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[a\,(3a) \int_0^{2a} dX_2 X_2^2\right] + \rho \left[a\,(2a) \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[a\,(3a) \left(\frac{X_2^3}{3}\right)_0^{2a}\right] + \rho \left[a\,(2a) \left(\frac{X_3^3}{3}\right)_0^{3a}\right] \\
&= \rho \left[3a^2 \frac{(2a)^3}{3}\right] + \rho \left[2a^2 \frac{(3a)^3}{3}\right] \\
&= \rho \left[3a^2 \frac{8a^3}{3}\right] + \rho \left[2a^2 \frac{27a^3}{3}\right] \\
&= \rho 8a^5 + \rho \frac{54a^5}{3} \\
&= 26a^5 \rho \\
&= \frac{26}{6} a^2 \left(6a^3 \rho\right) \\
&= \frac{13}{3} M a^2
\end{aligned}
$$

And

$$
\begin{aligned}
J_{12} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \,(X_1 X_2) \\
&= -\rho \int_0^a X_1 dX_1 \int_0^{2a} X_2 dX_2 \int_0^{3a} dX_3 \\
&= -\rho \left(\frac{X_1^2}{2}\right)_0^a \left(\frac{X_2^2}{2}\right)_0^{2a} 3a \\
&= -\rho \left(\frac{a^2}{2}\right) \left(\frac{4a^2}{2}\right) 3a \\
&= -3a^5 \rho \\
&= -\frac{3}{6} a^2 \left(6a^3 \rho\right) \\
&= -\frac{1}{2} M a^2
\end{aligned}
$$

And

$$
\begin{aligned}
J_{13} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1 X_3\right) \\
&= -\rho \int_0^a X_1 dX_1 \int_0^{2a} X_2 \int_0^{3a} X_3 dX_3 \\
&= -\rho \left(\frac{X_1^2}{2}\right)_0^a 2a \left(\frac{X_3^2}{2}\right)_0^{3a} \\
&= -\rho \frac{a^2}{2} 2a \frac{9a^2}{2} \\
&= -\frac{9}{2} a^5 \rho \\
&= -\frac{9}{2\left(6\right)} a^2 \left(6a^3 \rho\right) \\
&= -\frac{3}{4} M a^2
\end{aligned}
$$

And $J_{21} = J_{12}$ and

$$
\begin{aligned}
J_{22} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1^2 + X_3^2\right) \\
&= \rho \left[\int_0^a X_1^2 dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 X_3^2\right] \\
&= \rho \left[\left(\frac{X_1^3}{3}\right)_0^a \left(2a\right)\left(3a\right)\right] + \rho \left[a\left(2a\right)\left(\frac{X_3^3}{3}\right)_0^{3a}\right] \\
&= \rho \left[\frac{a^3}{3}\left(2a\right)\left(3a\right)\right] + \rho \left[a\left(2a\right)\frac{\left(3a\right)^3}{3}\right] \\
&= \rho \left[\frac{6a^5}{3}\right] + \rho \left[2a^2 \frac{27a^3}{3}\right] \\
&= \rho 2a^5 + 18a^5 \rho \\
&= 20a^5 \rho \\
&= \frac{20}{6} a^2 \left(6a^3 \rho\right) \\
&= M \frac{20}{6} a^2
\end{aligned}
$$

And

$$
\begin{aligned}
J_{23} &= -\rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3\,(X_2X_3) \\
&= -\rho \int_0^a X_1 \int_0^{2a} X_2 dX_2 \int_0^{3a} X_3 dX_3 \\
&= -\rho a \left(\frac{X_2^2}{2}\right)_0^{2a} \left(\frac{X_3^2}{2}\right)_0^{3a} \\
&= -\rho a \left(\frac{4a^2}{2}\right)\left(\frac{9a^2}{2}\right) \\
&= -9a^5\rho \\
&= -\frac{9}{6}a^2\left(6a^3\rho\right) \\
&= -\frac{9}{6}Ma^2
\end{aligned}
$$

And $J_{31} = J_{13}$ and $J_{32} = J_{23}$ and

$$
\begin{aligned}
J_{33} &= \rho \int_0^a dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3 \left(X_1^2 + X_2^2\right) \\
&= \rho \left[\int_0^a X_1^2 dX_1 \int_0^{2a} dX_2 \int_0^{3a} dX_3\right] + \rho \left[\int_0^a dX_1 \int_0^{2a} X_2^2 dX_2 \int_0^{3a} dX_3\right] \\
&= \rho \left[\left(\frac{X_1^3}{3}\right)_0^a (2a)(3a)\right] + \rho \left[a\left(\frac{X_2^3}{3}\right)_0^{2a} 3a\right] \\
&= \rho \left[\frac{a^3}{3}(2a)(3a)\right] + \rho \left[a\left(\frac{8a^3}{3}\right)3a\right] \\
&= \rho 2a^5 + \rho 8a^5 \\
&= 10a^5\rho \\
&= \frac{10}{6}a^2\left(6a^3\rho\right) \\
&= M\frac{10}{6}a^2
\end{aligned}
$$

Therefore

$$
J = Ma^2 \begin{pmatrix} \frac{13}{3} & -\frac{1}{2} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{20}{6} & -\frac{9}{6} \\ -\frac{3}{4} & -\frac{9}{6} & \frac{10}{6} \end{pmatrix}
$$

We now find I around the center of the cube where the position vector of the center is $\vec{a} = \left\{\frac{1}{2}a, a, \frac{3}{2}a\right\}$. Therefore

$$\begin{aligned}
I_{11} &= J_{11} - M\left(\vec{a}^2 - a_1^2\right) \\
&= Ma^2\frac{13}{3} - M\left(a_2^2 + a_3^2\right) \\
&= Ma^2\frac{13}{3} - M\left(a^2 + \left(\frac{3}{2}a\right)^2\right) \\
&= \frac{13}{12}Ma^2
\end{aligned}$$

And

$$\begin{aligned}
I_{12} &= J_{12} - M\left(-a_1 a_2\right) \\
&= -Ma^2\frac{1}{2} - M\left(-\left(\frac{1}{2}a\right)a\right) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
I_{13} &= J_{13} - M\left(-a_1 a_3\right) \\
&= -Ma^2\frac{3}{4} - M\left(-\left(\frac{1}{2}a\right)\frac{3}{2}a\right) \\
&= 0
\end{aligned}$$

And $I_{21} = I_{12}$ And

$$\begin{aligned}
I_{22} &= J_{22} - M\left(\vec{a}^2 - a_2^2\right) \\
&= Ma^2\frac{20}{6} - M\left(a_1^2 + a_3^2\right) \\
&= Ma^2\frac{20}{6} - M\left(\left(\frac{1}{2}a\right)^2 + \left(\frac{3}{2}a\right)^2\right) \\
&= \frac{5}{6}Ma^2
\end{aligned}$$

And

$$\begin{aligned}
I_{23} &= J_{23} - M\left(-a_2 a_3\right) \\
&= -Ma^2\frac{9}{6} - M\left(-(a)\frac{3}{2}a\right) \\
&= 0
\end{aligned}$$

And $I_{31}=I_{31}$ and $I_{32}=I_{23}$ and

$$\begin{aligned}I_{33} &= J_{33}-M\left(\vec{a}^2-a_3^2\right)\\ &= Ma^2\frac{10}{6}-M\left(a_1^2+a_2^2\right)\\ &= Ma^2\frac{10}{6}-M\left(\left(\frac{1}{2}a\right)^2+a^2\right)\\ &= \frac{5}{12}Ma^2\end{aligned}$$

Therefore the moment of inertia tensor around the center of mass is

$$I=Ma^2\begin{pmatrix}\frac{13}{12} & 0 & 0\\ 0 & \frac{10}{12} & 0\\ 0 & 0 & \frac{5}{12}\end{pmatrix}$$

0.2.2 Part(2)

The kinetic energy is $\frac{1}{2}\omega\cdot L$ where $\omega=\{\omega_1,\omega_2,\omega_3\}$ and

$$\begin{aligned}L &= I\omega\\ &= Ma^2\begin{pmatrix}\frac{13}{12} & 0 & 0\\ 0 & \frac{10}{12} & 0\\ 0 & 0 & \frac{5}{12}\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix}\\ &= \begin{pmatrix}\frac{13}{12}Ma^2\omega_1\\ \frac{10}{12}Ma^2\omega_2\\ \frac{5}{12}Ma^2\omega_3\end{pmatrix}\end{aligned}$$

Hence

$$\begin{aligned}T=\frac{1}{2}\omega\cdot L &= \frac{1}{2}\left(\frac{13}{12}Ma^2\omega_1^2+\frac{10}{12}Ma^2\omega_2^2+\frac{5}{12}Ma^2\omega_3^2\right)\\ &= \frac{1}{24}Ma^2\left(13\omega_1^2+10\omega_2^2+5\omega_3^2\right)\end{aligned}$$

Since body is rotating around the long diagonal. The long diagonal has length $\sqrt{a^2+(2a)^2+(3a)^2}=\sqrt{14}a$, therefore

$$\omega=\frac{\omega}{\sqrt{14}a}\{a,2a,3a\}=\frac{\omega}{\sqrt{14}}\{1,2,3\}$$

and the above becomes

$$\begin{aligned}T &= \frac{1}{24}Ma^2\omega^2\left(\frac{13}{14}+10\left(\frac{4}{14}\right)+5\left(\frac{9}{14}\right)\right)\\ &= \frac{7}{24}Ma^2\omega^2\end{aligned}$$

0.2.3 Part(3)

Using

$$\begin{aligned}
\omega \cdot L &= |\omega|\,|L|\cos\theta \\
\cos\theta &= \frac{\omega \cdot L}{|\omega|\,|L|} \\
&= \frac{\frac{14}{24}Ma^2\omega^2}{\sqrt{\omega_1^2+\omega_2^2+\omega_3^2}\sqrt{\left(\frac{13}{12}Ma^2\omega_1\right)^2+\left(\frac{10}{12}Ma^2\omega_2\right)^2+\left(\frac{5}{12}Ma^2\omega_3\right)^2}} \\
&= \frac{\frac{14}{24}Ma^2\omega^2}{\sqrt{\left(\frac{\omega}{\sqrt{14}}\right)^2+\left(\frac{2\omega}{\sqrt{14}}\right)^2+\left(\frac{3\omega}{\sqrt{14}}\right)^2}\sqrt{\left(\frac{13}{12}Ma^2\frac{\omega}{\sqrt{14}}\right)^2+\left(\frac{10}{12}Ma^2\frac{2\omega}{\sqrt{14}}\right)^2+\left(\frac{5}{12}Ma^2\frac{3\omega}{\sqrt{14}}\right)^2}} \\
&= \frac{\frac{14}{24}Ma^2\omega^2}{\sqrt{\omega^2}\sqrt{\frac{397}{1008}M^2a^4\omega^2}} \\
&= \frac{\frac{14}{24}}{\sqrt{\frac{397}{1008}}} \\
&= 0.92951
\end{aligned}$$

Hence

$$\boxed{\theta = 21.64^0}$$

0.2.4 Part(4)

Since

$$\begin{aligned}
\tau_{external} &= \frac{d}{dt}\left(\boldsymbol{L}\right)_{inertial} \\
&= \frac{d}{dt}\left(\boldsymbol{L}\right)_{body} + \omega \times \boldsymbol{L}
\end{aligned}$$

But $\frac{d}{dt}(\boldsymbol{L})_{body} = 0$ since $\boldsymbol{L} = I\omega$ and I is constant and ω is constant. Therefore

$$
\begin{aligned}
\tau &= \omega \times \boldsymbol{L} \\
&= \omega \times I\omega \\
&= \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \\
&= \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix} \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_1\omega_1 & I_2\omega_2 & I_3\omega_3 \end{vmatrix} \\
&= \boldsymbol{i}\,(I_3\omega_2\omega_3 - I_2\omega_2\omega_3) - \boldsymbol{j}\,(I_3\omega_3\omega_1 - I_1\omega_1\omega_3) + \boldsymbol{k}\,(I_2\omega_2\omega_1 - I_1\omega_1\omega_2) \\
&= \begin{pmatrix} \omega_2\omega_3\,(I_3 - I_2) \\ \omega_3\omega_1\,(I_1 - I_3) \\ \omega_2\omega_1\,(I_2 - I_1) \end{pmatrix}
\end{aligned}
$$

The above are Euler equations for constant ω, and could have been written down directly from Euler equations by setting all the $\dot{\omega}_i = 0$ also.

Now, since $\omega = \frac{\omega}{\sqrt{14}}\{1,2,3\}$ and $I_1 = \frac{13}{12}Ma^2, I_2 = \frac{10}{12}Ma^2, I_3 = \frac{5}{12}Ma^2$, Therefore the above torque becomes

$$
\begin{aligned}
\tau &= \frac{\omega^2}{14}Ma^2 \begin{pmatrix} 6\left(\frac{5}{12} - \frac{10}{12}\right) \\ 3\left(\frac{13}{12} - \frac{5}{12}\right) \\ 2\left(\frac{10}{12} - \frac{13}{12}\right) \end{pmatrix} \\
&= \frac{\omega^2}{14}Ma^2 \begin{pmatrix} -\frac{5}{2} \\ 2 \\ -\frac{1}{2} \end{pmatrix} \\
&= \omega^2 Ma^2 \begin{pmatrix} -\frac{5}{28} \\ \frac{1}{7} \\ -\frac{1}{28} \end{pmatrix} \\
&= \omega^2 Ma^2 \begin{pmatrix} -0.1786 \\ 0.1429 \\ -0.0357 \end{pmatrix}
\end{aligned}
$$

Units check: $\frac{1}{T^2}ML^2 = [N]\,[L]$ units of torque. OK. The above is the external torque exerted on the block.

0.3 Problem 3

3. (10 points)
Consider a simple top consisting of a heavy circular disc of mass m and radius a mounted at the center of a thin rod of mass $m/2$ and length a. The top is set spinning at a rate S with the axis at an angle 45° with the vertical.

(1) Show that there are two possible values of the precession rate $\dot{\phi}$ such that the top precesses steadily at a constant value of $\theta = 45°$.

(2) Calculate the numerical values for $\dot{\phi}$ if $S = 900\,\text{rpm}$ and $a = 10\,\text{cm}$.

(3) If a top is set spinning sufficiently fast and is started in a vertical position, the axis remains steady in the upright position. This is called a "sleeping top." How fast must the top spin to sleep in the vertical position?

SOLUTION:

0.3.1 Part(1)

Starting with the Euler equations for Gyroscope precession, equations 9.71. in textbook, page 371, Analytical mechanics, 6th edition, by Fowles and Cassiday

$$\begin{aligned} Mgl\sin\theta &= I_x\ddot{\theta} + I_z S\dot{\phi}\sin\theta - I_y\dot{\phi}^2\cos\theta\sin\theta \\ 0 &= I_y\frac{d}{dt}\left(\dot{\phi}\sin\theta\right) - I_z S\dot{\theta} + I_x\dot{\theta}\dot{\phi}\cos\theta \\ 0 &= I_z\dot{S} \end{aligned} \tag{1}$$

Where the spin of the disk S around its own z body axis is

$$S = \dot{\psi} + \dot{\phi}\cos\theta$$

Instead of drawing this again, which would take sometime, I am showing the diagram from the book above, page 371 for illustration

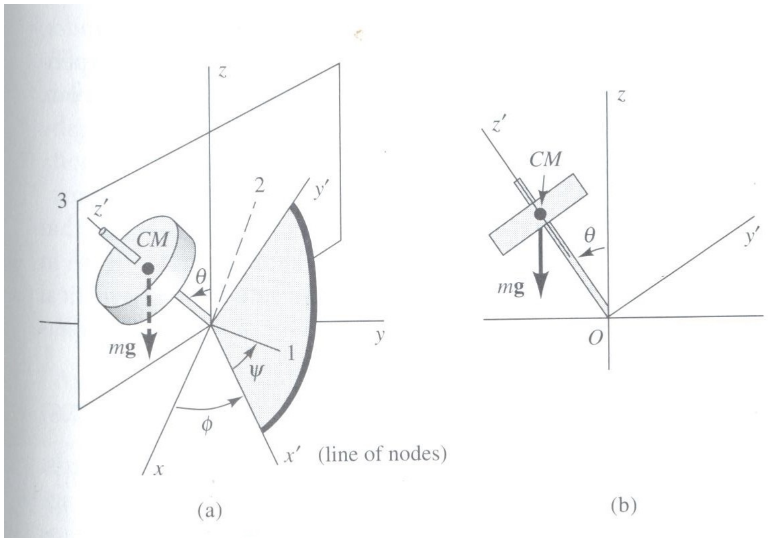


Figure 9.7.1 The simple gyroscope.

In (1), the length l is the distance from center of mass of the combined disc and rod, to the origin of the inertial frame. This will be $l=\frac{a}{2}$. M is the total mass of both the disc and the rod, which will be $M=\frac{3}{2}m$.

We are told that $\theta(t)$ is constant. Hence $\ddot{\theta}=0$ and first equation in (1) becomes

$$
\begin{aligned}
Mgl\sin\theta &= I_zS\dot{\phi}\sin\theta - I_y\dot{\phi}^2\cos\theta\sin\theta\\
Mgl &= I_zS\dot{\phi} - I_y\dot{\phi}^2\cos\theta
\end{aligned}
$$

This is quadratic in $\dot{\phi}$. Solving gives

$$
\begin{aligned}
I_y\dot{\phi}^2\cos\theta - I_zS\dot{\phi} + Mgl = 0\\
\dot{\phi} &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
&= \frac{I_zS\pm\sqrt{I_z^2S^2-4I_y\cos\theta Mgl}}{2\cos\theta I_y} \qquad (2)
\end{aligned}
$$

The only thing left is to calculate I_z and I_y for the disc and the rod about the mass center, then use parallel axes theorem to move this to the pivot, which is the origin of the inertial frame.

Due to symmetry, the center of mass for both disk and rod is located distance $\frac{a}{2}$ from pivot. Hence $l=\frac{a}{2}$. For the disc, its moment of inertial around the spin axes at its center of mass is

$$
(I_z)_{disk} = m\frac{a^2}{2}
$$

And along the y axis $I_y=m\frac{a^2}{4}$. Since the distance of the center of mass from the pivot is $\frac{a}{2}$, we need to adjust I_y by this distance using parallel axes. Hence

$$\begin{aligned}\left(I_y\right)_{disk} &= m\frac{a^2}{4} + m\left(\frac{a}{2}\right)^2\\ &= \frac{1}{2}a^2 m\end{aligned}$$

For the rod, it only has moment of inertial around y at the end of the rod. From tables $\left(I_y\right)_{rod} = \left(\frac{m}{2}\right)\left(\frac{a^2}{3}\right)$. Therefore

$$\begin{aligned}I_z &= m\frac{a^2}{2}\\ I_y &= \left(I_y\right)_{disk} + \left(I_y\right)_{rod} = \frac{1}{2}a^2 m + \frac{m}{2}\frac{a^2}{3}\\ &= \frac{2}{3}a^2 m\end{aligned}$$

From (2), and using $\theta = 45^0$ we find, using $M = m + \frac{m}{2} = \frac{3}{2}m$ and $l = \frac{a}{2}$

$$\dot{\phi} = \frac{I_z S \pm \sqrt{I_z^2 S^2 - 4 I_y \cos\theta M g l}}{2\cos\theta I_y} \tag{3}$$

0.3.2 Part(2)

For $\theta = 45^0$ and $S = 900$ rpm, which is 94.248 rad/sec. $a = 0.1$ meter and $l = \frac{a}{2} = 0.05$ meter (3) becomes

$$\begin{aligned}\dot{\phi} &= \frac{\left(m\frac{a^2}{2}\right)(94.248) \pm \sqrt{\left(m\frac{a^2}{2}\right)^2 (94.248)^2 - 4\left(\frac{2}{3}a^2 m\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}m\right)(9.8)(0.05)}}{2\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{2}{3}a^2 m\right)}\\ &= \frac{\left(m\frac{(0.1)^2}{2}\right)(94.248) \pm m\sqrt{\left(\frac{(0.1)^2}{2}\right)^2 (94.248)^2 - 4\left(\frac{2}{3}(0.1)^2\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}\right)(9.8)(0.05)}}{2\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{2}{3}(0.1)^2 m\right)}\\ &= \frac{3}{4}\frac{\left(\frac{(0.1)^2}{2}\right)(94.248)}{\cos\left(45\left(\frac{\pi}{180}\right)\right)(0.1)^2} \pm \frac{3}{4}\frac{\sqrt{\left(\frac{(0.1)^2}{2}\right)^2 (94.248)^2 - 4\left(\frac{2}{3}(0.1)^2\right)\cos\left(45\left(\frac{\pi}{180}\right)\right)\left(\frac{3}{2}\right)(9.8)(0.05)}}{\cos\left(45\left(\frac{\pi}{180}\right)\right)(0.1)^2}\\ &= 49.983 \pm 48.398 \text{ rad/sec}\end{aligned}$$

Or

$$\boxed{\dot{\phi} = 939.47 \text{ or } 15.13 \text{ rpm}}$$

0.3.3 Part(3)

From (2) above, repeated below

$$\dot{\phi} = \frac{I_z S \pm \sqrt{I_z^2 S^2 - 4I_y \cos\theta M g l}}{2\cos\theta I_y}$$

Since $\dot{\phi}$ must be real, then $I_z^2 S^2 - 4I_y \cos\theta M g l$ must be either positive or zero.

$$S^2 - 4I_y \cos\theta M g l \geq 0$$

$$S^2 \geq \frac{4I_y \cos\theta M g l}{I_z^2}$$

For $\theta = 0$ the above becomes

$$S^2 \geq \frac{4I_y M g l}{I_z^2}$$

The above is the condition on spin speed S for keeping $\theta = 0$. Hence

$$\begin{aligned} S^2 &\geq \frac{4\left(\frac{2}{3}a^2 m\right)\left(\frac{3}{2}m\right)(9.8)\, l}{\left(m\frac{a^2}{2}\right)^2} \\ &\geq \frac{156.8}{a^2} l \\ &\geq \frac{156.8}{(0.1)^2}(0.05) \\ &\geq 784 \end{aligned}$$

Therefore

$$\begin{aligned} S &\geq \sqrt{784} \\ &\geq 28 \text{ rad/sec} \end{aligned}$$

Or

$$\boxed{S \geq 267.31 \text{ RPM}}$$

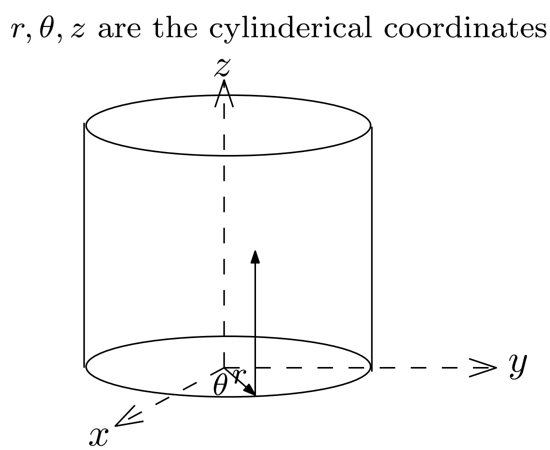
0.4 Problem 4

4. (10 points)
Determine the principal moments of inertia and the corresponding principle axes about the center of mass of a homogeneous circular cone of height h and radius R. (You might find it easier to calculate the moments in a reference frame with the origin at the apex first, and then transform to the center of mass system.)

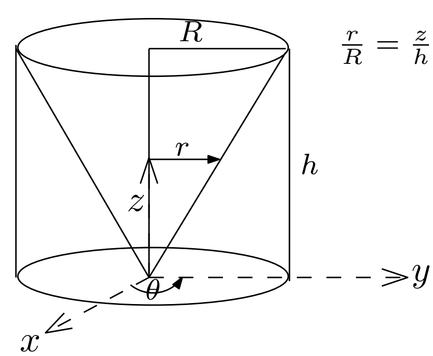
SOLUTION:

0.4.1 Solution using Cylindrical coordinates

Will show the solution using Cylindrical coordinates. Then later will also show the solution using Cartesian coordinates. Using Cylindrical coordinates



The limits of volume integration will be from $z = 0 \cdots h$ and $\theta = 0 \cdots 2\pi$. For r, it depends on z. Since $\frac{r}{R} = \frac{z}{h}$, then $r = \frac{R}{h} z$, therefore the limit for $r = 0 \cdots \frac{R}{h} z$. This is when the tip of the cone at the origin as follows



The density is $\rho = \frac{3M}{\pi R^2 h}$. The center of mass is $\frac{h}{4}$ distance away from the base or $\frac{3}{4}h$ from the tip. The moment of inertia is found at the origin (which is the tip of the cone also), then moved to the center of mass using parallel axes theorem. We know from Cartesian coordinates that the inertia matrix is found using

$$J = \rho \int\int\int \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dzdydx$$

Therefore, in cylindrical coordinates this becomes, after using the mapping $x = r\cos\theta, y = r\sin\theta, z = z$

$$J = \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}z} \begin{pmatrix} r^2\sin^2\theta + z^2 & -r^2\cos\theta\sin\theta & -r\cos\theta z \\ -r\cos\theta z & r^2\cos^2\theta + z^2 & -r\sin\theta z \\ -r\cos\theta z & -r\sin\theta z & r^2 \end{pmatrix} rdrd\theta dz$$

Due to symmetry, the off diagonal elements will be zero. So we only have to perform the following integration

$$J = \rho \int_0^h \int_0^{2\pi} \int_0^{\frac{R}{h}z} \begin{pmatrix} r^2\sin^2\theta + z^2 & 0 & 0 \\ 0 & r^2\cos^2\theta + z^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} rdrd\theta dz$$

For J_{11} we find

$$\begin{aligned}
J_{11} &= \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z}\left(r^2\sin^2\theta+z^2\right)rdrd\theta dz\\
&= \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z}\left(r^2\sin^2\theta\right)rdrd\theta dz+\rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z}z^2rdrd\theta dz\\
&= \rho\int_0^h dz\int_0^{2\pi}d\theta\left(\int_0^{\frac{R}{h}z}\left(r^3\sin^2\theta\right)dr\right)+\rho\int_0^h z^2dz\int_0^{2\pi}d\theta\left(\int_0^{\frac{R}{h}z}rdr\right)\\
&= \rho\int_0^h dz\int_0^{2\pi}\sin^2\theta d\theta\left[\frac{r^4}{4}\right]_0^{\frac{R}{h}z}+\rho\int_0^h z^2dz\int_0^{2\pi}d\theta\left[\frac{r^2}{2}\right]_0^{\frac{R}{h}z}\\
&= \frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz\int_0^{2\pi}\sin^2\theta d\theta+\frac{\rho}{2}\frac{R^2}{h^2}\int_0^h z^4dz\int_0^{2\pi}d\theta\\
&= \frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz\left[\frac{\theta}{2}-\frac{1}{4}\sin\left(2\theta\right)\right]_0^{2\pi}+\frac{\rho}{2}\frac{R^2}{h^2}2\pi\int_0^h z^4dz\\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4dz+\frac{\rho}{2}\frac{R^2}{h^2}2\pi\left[\frac{z^5}{5}\right]_0^h\\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\left[\frac{z^5}{5}\right]_0^h+\rho\frac{R^2}{h^2}\pi\frac{h^5}{5}\\
&= \pi\frac{\rho}{4}\frac{R^4}{h^4}\frac{h^5}{5}+\rho R^2\pi\frac{h^3}{5}\\
&= \pi\frac{\rho}{20}R^4h+\rho R^2\pi\frac{h^3}{5}
\end{aligned}$$

Using $\rho=\frac{3M}{\pi R^2h}$ the above becomes

$$\begin{aligned}
J_{11} &= \frac{3M}{\pi R^2h}\pi\frac{1}{20}R^4h+\frac{3M}{\pi R^2h}R^2\pi\frac{h^3}{5}\\
&= \frac{3M}{20}R^2+\frac{3M}{5}h^2
\end{aligned}$$

For J_{22} it will be the same as the above, since the only difference is $\cos^2\theta$ instead of $\sin^2\theta$ in the integrand. Therefore

$$J_{22}=\frac{3M}{20}R^2+\frac{3M}{5}h^2$$

For the final entry (the easy one) we have

$$\begin{aligned}
J_{33} &= \rho\int_0^h\int_0^{2\pi}\int_0^{\frac{R}{h}z} r^2 r dr d\theta dz \\
&= \rho\int_0^h\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^{\frac{R}{h}z} d\theta dz \\
&= \frac{\rho}{4}\frac{R^4}{h^4}\int_0^h z^4 dz\int_0^{2\pi} d\theta \\
&= \frac{\rho}{4}\frac{R^4}{h^4}2\pi\int_0^h z^4 dz \\
&= \frac{\rho}{4}\frac{R^4}{h^4}2\pi\left[\frac{z^5}{5}\right]_0^h \\
&= \frac{\rho}{20}\frac{R^4}{h^4}2\pi h^5
\end{aligned}$$

Using $\rho = \frac{3M}{\pi R^2 h}$ the above becomes

$$\begin{aligned}
J_{33} &= \frac{3M}{\pi R^2 h}\frac{1}{20}\frac{R^4}{h^4}2\pi h^5 \\
&= \frac{6}{20}MR^2
\end{aligned}$$

Therefore

$$J = \begin{pmatrix} \frac{3M}{20}R^2 + \frac{3M}{5}h^2 & 0 & 0 \\ 0 & \frac{3M}{20}R^2 + \frac{3M}{5}h^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

Using $I_{ij} = I_{ij}^{cm} + M\left(a^2\delta_{ij} - a_i a_j\right)$, we now find I. The vector from the origin to the center of mass is $a = \left\{0, 0, \frac{3}{4}h\right\}$, hence

$$\begin{aligned}
I_{11} &= \left(\frac{3M}{20}R^2 + \frac{3M}{5}h^2\right) - M\left(\left(\frac{3}{4}h\right)^2 - \left(0^2\right)\right) \\
&= \frac{3M}{20}R^2 + \frac{3M}{5}h^2 - M\left(\frac{3}{4}h\right)^2 \\
&= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2
\end{aligned}$$

And

$$I_{22} = I_{11}$$

And

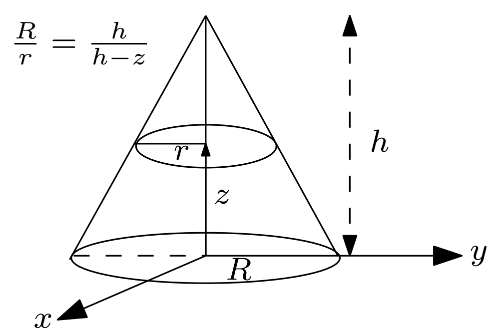
$$
\begin{aligned}
I_{33} &= \frac{3}{10}MR^2 - M\left(\left(\frac{3}{4}h\right)^2 - \left(\frac{3}{4}h\right)^2\right)\\
&= \frac{3}{10}MR^2
\end{aligned}
$$

Therefore the final inertial matrix around the center of the mass of the cone is

$$
I = M\begin{pmatrix} \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 & 0\\ 0 & \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0\\ 0 & 0 & \frac{3}{10}R^2 \end{pmatrix}
$$

0.4.2 Solution using Cartesian coordinates

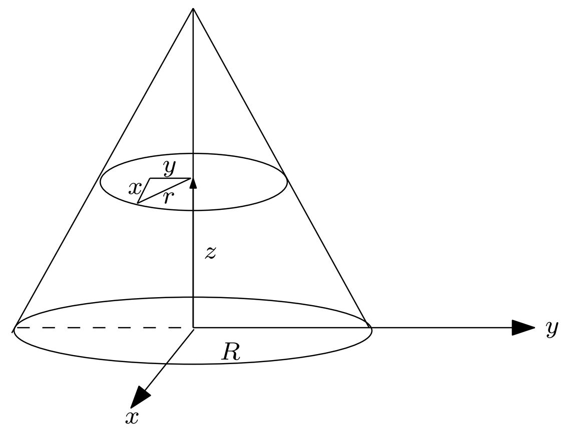
Will find mass moment of inertia tensor at center of base of cone, then use parallel axes to move it to the center of mass of cone.



We basically want to perform this integral

$$
J = \rho \int_{z=0}^{z=h} \int_{y=y(z_{\min})}^{y=y(z_{\max})} \int_{x=x(y_{\min})}^{x=x(y_{\max})} \begin{pmatrix} y^2+z^2 & -xy & -xz\\ -xy & x^2+z^2 & -yz\\ -xz & -yz & x^2+y^2 \end{pmatrix} dzdydx
$$

The limit on z is easy. It is from $z=0$ to $z=h$. Now at specific z, we need to know the limit on y. The radius r at some z distance from the origin is $r = \frac{R(h-z)}{h}$ as shown above, which is by proportions. Therefore the limit of integration for y is from $y=-r$ to $+r$. Now we need to find the limit on x. At some specific y distance from origin, we see from the following diagram



We see from the above that $x^2 = r^2 - y^2$ but $r = \frac{R(h-z)}{h}$, hence the limit on x is from $-\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}$ to $+\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}$. Now that we found all the limits, the integration is

$$J = \rho \int_0^h \int_{-\frac{R(h-z)}{h}}^{\frac{R(h-z)}{h}} \int_{-\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}}^{\sqrt{\left(\frac{R(h-z)}{h}\right)^2 - y^2}} \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix} dzdydx$$

Where $\rho = \frac{3M}{\pi R^2 h}$.Using computer algebra software to do the integration (too messy by hand), the above gives

$$J = \begin{pmatrix} \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 & 0 & 0 \\ 0 & \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

Now we use parallel axis to find I at center of mass. The center of mass is at $\vec{a} = \left\{0, 0, \frac{1}{4}h\right\}$, hence

$$\begin{aligned} I_{11} &= J_{11} - M\left(\vec{a}^2 - a_1^2\right) \\ &= \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 - M\left(\frac{1}{4}h\right)^2 \\ &= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2 \end{aligned}$$

And

$$\begin{aligned} I_{12} &= J_{12} - M(-a_1 a_2) \\ &= 0 - M(0) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} I_{13} &= J_{13} - M\left(-a_1 a_3\right) \\ &= -Ma^2 \frac{3}{4} - M\left(-\left(\frac{1}{2}a\right)\frac{3}{2}a\right) \\ &= 0 \end{aligned}$$

And $I_{21} = I_{12}$ And

$$\begin{aligned} I_{22} &= J_{22} - M\left(\vec{a}^2 - a_2^2\right) \\ &= \frac{1}{10}Mh^2 + \frac{3}{20}MR^2 - M\left(\frac{1}{4}h\right)^2 \\ &= \frac{3}{20}MR^2 + \frac{3}{80}Mh^2 \end{aligned}$$

And

$$\begin{aligned} I_{23} &= J_{23} - M\left(-a_2 a_3\right) \\ &= 0 - M\left(0\right) \\ &= 0 \end{aligned}$$

And $I_{31} = I_{31}$ and $I_{32} = I_{23}$ and

$$\begin{aligned} I_{33} &= J_{33} - M\left(\vec{a}^2 - a_3^2\right) \\ &= \frac{3}{10}MR^2 - M\left(\left(\frac{1}{4}h\right)^2 - \left(\frac{1}{4}h\right)^2\right) \\ &= \frac{3}{10}MR^2 \end{aligned}$$

Therefore the moment of inertia tensor around the center of mass

$$I = M\begin{pmatrix} \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 & 0 \\ 0 & \frac{3}{20}R^2 + \frac{3}{80}h^2 & 0 \\ 0 & 0 & \frac{3}{10}R^2 \end{pmatrix}$$

Which is the same as using Cylindrical coordinates (as would be expected).

0.5 Problem 5

5. (15 points)
A homogeneous slab of thickness a is placed on top of a fixed cylinder of radius R whose axis is horizontal (as in the Figure below).

(1) Determine the Lagrangian of the system.

(2) Derive the equations of motion and determine the frequency of *small* oscillations.

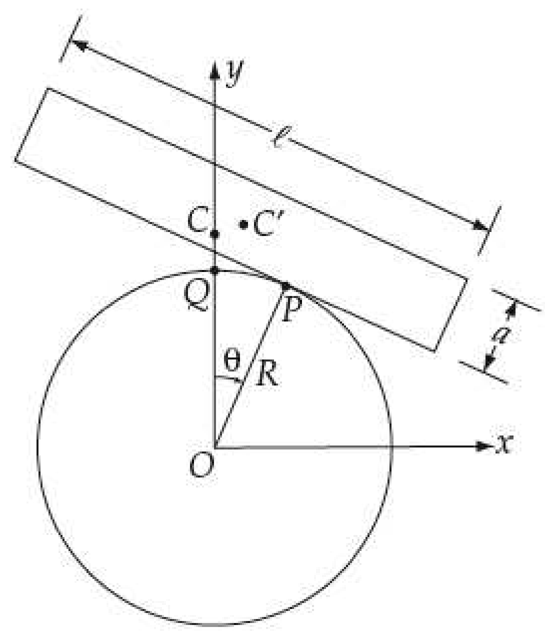
(3) Show that the condition for stable equilibrium of the slab, assuming no slipping, is $R > a/2$.

(4) Use a computer to plot the potential energy U as a function of the angular displacement θ for a slab of mass $M = 1\,\text{kg}$ and

(a) $R = 20\,\text{cm}$ and $a = 5\,\text{cm}$, and

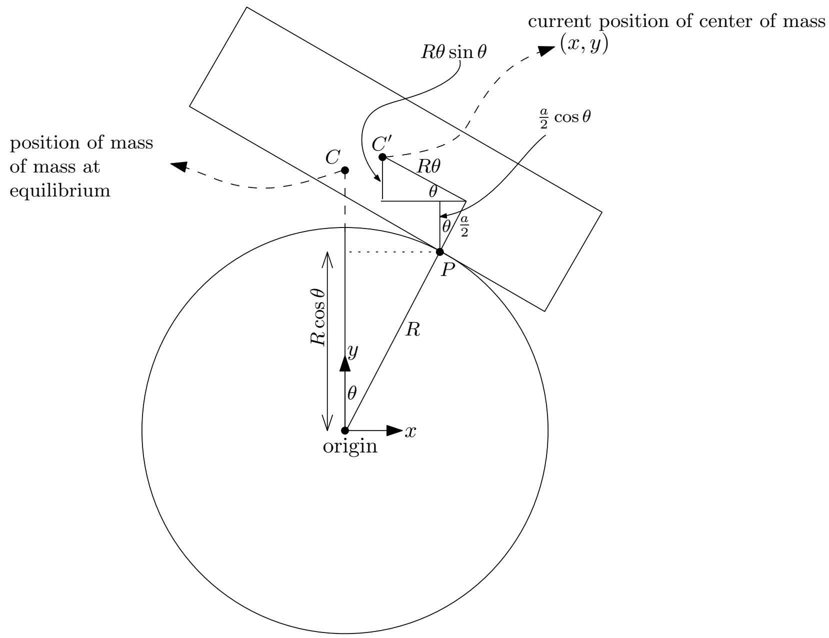
(b) $R = 10\,\text{cm}$ and $a = 30\,\text{cm}$.

(5) Show that the potential energy $U(\theta)$ has a minimum at $\theta = 0$ for $R > a/2$, but not for $R < a/2$.



SOLUTION:

0.5.1 Part (1)



The system has three degrees of freedom (x, y, θ). But they are not independent. Because if we know $\theta(t)$, we can find $x(t)$ and $y(t)$ (for small angle approximation) as shown below in equations (1) and (2).

The cylinder itself does not move or rotate. Only the slab has rotational and translational motion. When the slab center of mass at C it is in equilibrium. When the slab center of mass at point C' the location of the center of mass is (x, y), where from the diagram above we see that (for small angle θ)

$$x = \left(R + \frac{a}{2}\right)\sin\theta - R\theta\cos\theta \tag{1}$$

$$y = \left(R + \frac{a}{2}\right)\cos\theta + R\theta\sin\theta \tag{2}$$

The distance from C' to O which is the zero reference for potential energy is therefore (assuming mass of slab is M)

$$\begin{aligned} U &= Mgy \\ &= Mg\left(R\theta\sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right) \end{aligned}$$

Let the moment of inertial of the slab around the axis of rotation be I therefore

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) \tag{3}$$

Now, we write $\dot{x}^2 + \dot{y}^2$ above in terms of θ using (1) and (2). (Initially I did not know if we should do this or not. So I left the original solution as an appendix in case that was how we are supposed to do it). Using this method below, we find only one equation of motion, not three as in the solution in the appendix.

$$\dot{x} = \left(R + \frac{a}{2}\right)\dot{\theta}\cos\theta - \left(R\dot{\theta}\cos\theta + R\theta\dot{\theta}\sin\theta\right)$$
$$\dot{y} = -\left(R + \frac{a}{2}\right)\dot{\theta}\sin\theta + \left(R\dot{\theta}\sin\theta + R\theta\dot{\theta}\cos\theta\right)$$

Hence (using CAS for simplification) we find

$$\dot{x}^2 = \frac{1}{4}\dot{\theta}^2\left(a\cos\theta + 2R\theta\sin\theta\right)^2$$

Similarly for $\dot{y}^2$ we find

$$\dot{y}^2 = \frac{1}{4}\dot{\theta}^2\left(a\sin\theta - 2R\theta\cos\theta\right)^2$$

Hence (3) becomes

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{8}M\dot{\theta}^2\left(\left(a\cos\theta + 2R\theta\sin\theta\right)^2 + \left(a\sin\theta - 2R\theta\cos\theta\right)^2\right)$$

And the Lagrangian is

$$\begin{aligned}L &= T - U\\ &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{8}M\dot{\theta}^2\left(\left(a\cos\theta + 2R\theta\sin\theta\right)^2 + \left(a\sin\theta - 2R\theta\cos\theta\right)^2\right) - Mg\left(R\theta\sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right)\end{aligned}$$

0.5.2 Part(2)

$$\frac{\partial L}{\partial\theta} = \frac{1}{2}M\left(ga\sin\theta + 2R\theta\left(-g\cos\theta + R\dot{\theta}^2\right)\right)$$
$$\frac{\partial L}{\partial\dot{\theta}} = \frac{1}{4}\left(4I + a^2M + 4MR^2\theta^2\right)\dot{\theta}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 2MR^2\theta\dot{\theta}^2 + \frac{1}{4}\left(4I + a^2M + 4MR^2\theta^2\right)\ddot{\theta}$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$
$$I\ddot{\theta} + \frac{1}{4}M\left(a^2 + 4R^2\theta^2\right)\ddot{\theta} - \frac{1}{2}agM\sin\theta + MR\theta\left(g\cos\theta + R\dot{\theta}^2\right) = 0$$

For small angles, we use $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, $\dot{\theta}^2 \approx 0$ and $\theta^2 \approx 0$. The above becomes

$$I\ddot{\theta} + \frac{1}{4}Ma^2\ddot{\theta} - \frac{1}{2}agM\theta + MR\theta g = 0$$
$$\ddot{\theta}\left(I + \frac{1}{4}Ma^2\right) + \theta\left(MRg - \frac{1}{2}agM\right) = 0$$
$$\ddot{\theta} + \frac{Mg\left(R - \frac{1}{2}a\right)}{\left(I + \frac{1}{4}a^2M\right)}\theta = 0$$

The above is now in the form $\ddot{\theta} + \omega_0^2 \theta = 0$, therefore the natural frequency is

$$\omega_0 = \sqrt{\frac{Mg\left(R - \frac{1}{2}a\right)}{\left(I + \frac{1}{4}a^2 M\right)}}$$

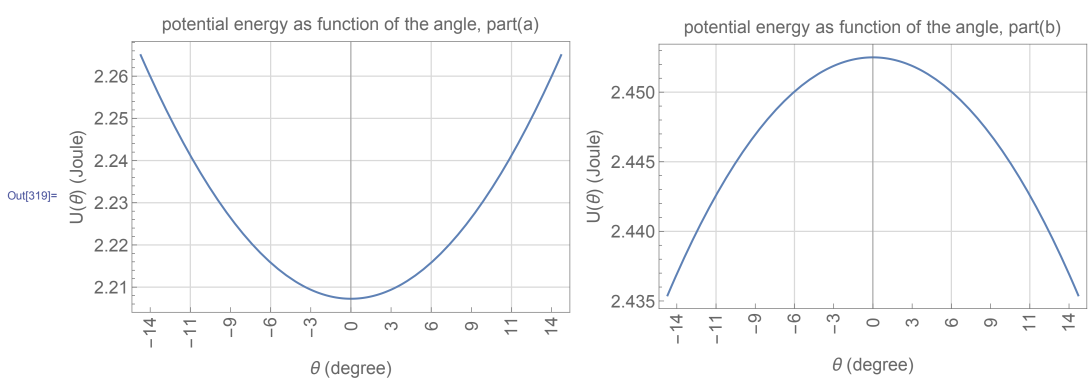
0.5.3 Part(3)

For stable equilibrium, we need $\frac{Mg\left(R-\frac{1}{2}a\right)}{\left(I+\frac{1}{4}a^2M\right)} > 0$ in order to obtain an oscillator (simple harmonic motion), otherwise the solution will contain pure exponential term and it will blow up. Hence we need

$$\begin{aligned} Mg\left(R - \frac{1}{2}a\right) &> 0 \\ R - \frac{1}{2}a &> 0 \\ R &> \frac{1}{2}a \end{aligned}$$

0.5.4 Part(4)

Here is a plot of $Mg\left(R\theta \sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right)$, for small angle, using $M = 1\text{kg}$. For parts (a) and (b)



We see from the above, that in part(b), where $R < \frac{a}{2}$, the potential energy at $\theta = 0$ is not minimum. This implies $\theta = 0$ is not a stable equilibrium. While in part(a) it is stable.

0.5.5 Part(5)

$$U(\theta) = Mg\left(R\theta \sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right)$$

Hence to find where the minimum is

$$U'(\theta) = gR\theta\cos\theta - \frac{1}{2}ga\sin\theta$$

Setting this to zero and for small angle we obtain

$$0 = gR\theta - \frac{1}{2}ga\theta$$
$$0 = \theta g\left(R - \frac{1}{2}a\right)$$

This implies $\theta = 0$ is where the minimum potential energy is. We know this is stable equilibrium. Therefore we expect $U''(\theta = 0)$ to be positive for a local minimum (from calculus). We now check the condition for this.

$$U''(\theta) = -\frac{1}{2}g\left((a - 2R)\cos\theta + 2R\theta\sin\theta\right)$$

At $\theta = 0$ we obtain

$$U''(\theta = 0) = -\frac{1}{2}g\,(a - 2R)$$

For the above to be positive, then

$$a - 2R < 0$$
$$2R > a$$
$$R > \frac{a}{2}$$

The above is the condition for having stable equilibrium at $\theta = 0$. If $R < \frac{a}{2}$, then at $\theta = 0$ the slab will not be stable, which is not we have shown in part(3).

0.5.6 Appendix. Second Solution of problem 5

Part(1)

In this solution, we find three equations of motion.

$$T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right)$$

Hence the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) - Mg\left(R\theta\sin\theta + \left(\frac{a}{2} + R\right)\cos\theta\right) \end{aligned}$$

Part(2)

For θ

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -Mg\left(R\left(\sin\theta + \theta\cos\theta\right) - \left(\frac{a}{2} + R\right)\sin\theta\right) \\ \frac{\partial L}{\partial \dot{\theta}} &= I\dot{\theta} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= I\ddot{\theta} \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ I\ddot{\theta} + Mg\left(R\left(\sin\theta + \theta\cos\theta\right) - \left(\frac{a}{2} + R\right)\sin\theta\right) &= 0 \end{aligned}$$

For small angles, we use $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and the above becomes

$$\begin{aligned} I\ddot{\theta} + Mg\left(2R\theta - \left(\frac{a}{2} + R\right)\theta\right) &= 0 \\ I\ddot{\theta} + Mg\left(R - \frac{1}{2}a\right)\theta &= 0 \\ \ddot{\theta} + \frac{Mg\left(R - \frac{1}{2}a\right)}{I}\theta &= 0 \end{aligned}$$

The above is now in the form $\ddot{\theta} + \omega_0^2\theta = 0$, therefore the natural frequency is

$$\omega_0 = \sqrt{\frac{Mg\left(R - \frac{1}{2}a\right)}{I}}$$

For x, we have

$$\begin{aligned}\frac{\partial L}{\partial x} &= 0\\ \frac{\partial L}{\partial \dot{x}} &= M\dot{x}\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x}\end{aligned}$$

Hence

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0\\ M\ddot{x} &= 0\end{aligned}$$

For y we also obtain

$$M\ddot{y} = 0$$

The rest follows as first solution above and will not be repeated.