0.1 Problem 1

1. (15 points)
Consider the case where a fixed force center scatters a particle of mass m according to an inverse-cube force law $F(r) = k/r^3$. If the initial velocity of m is v, show that the differential cross section is

$$\sigma(\theta) = \frac{k\,\pi^2\,(\pi-\theta)}{m\,v^2\,\theta^2\,(2\pi-\theta)^2\,\sin\theta} \quad .$$

SOLUTION:

Starting from

$$\theta_0(b) = \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{2Em}{l^2} - \frac{2mU}{l^2} - \frac{1}{r^2}}} \tag{1}$$

But

$$\begin{aligned} l &= b\sqrt{2mE} \\ l^2 &= b^2(2mE) \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} \theta_0(b) &= \int_{r_{\min}}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{U}{b^2E} - \frac{1}{r^2}}} \\ &= \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{U}{E} - \frac{b^2}{r^2}}}dr \end{aligned} \tag{1A}$$

In this problem, since $F(r) = \frac{k}{r^3}$, therefore since $F(r) = -\nabla U$

$$\begin{aligned} U(r) &= -\int \frac{k}{r^3}dr \\ &= \frac{k}{2r^2} \end{aligned}$$

Then (1A) becomes

$$\theta_0(b) = \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{k}{2r^2E} - \frac{b^2}{r^2}}}dr \tag{1B}$$

Let $z = \frac{1}{r}$ then $\frac{dr}{dz} = -\frac{1}{z^2}$. When $r = \infty$ then $z = 0$ and when $r = r_{\min}$ then $z = \frac{1}{r_{\min}}$. Now we need to find $r_{\min}$. We know that when $E = U_{effective}$ then $r = r_{\min}$. But

$$\begin{aligned} U_{effective} &= \frac{l^2}{2mr^2} + U(r) \\ &= \frac{l^2}{2mr^2} + \frac{k}{2r^2} \end{aligned}$$

Hence

$$
\begin{aligned}
E &= U_{effective} \\
&= \frac{l^2}{2mr_{\min}^2} + \frac{k}{2r_{\min}^2} \\
&= \frac{l^2 + mk}{2mr_{\min}^2}
\end{aligned}
$$

Solving for $r_{\min}$

$$
\begin{aligned}
r_{\min}^2 &= \frac{l^2 + mk}{2mE} \\
&= \frac{l^2}{2mE} + \frac{k}{2E} \qquad (2)
\end{aligned}
$$

But $l^2 = b^2(2mE)$ then (2) becomes

$$
\begin{aligned}
r_{\min}^2 &= \frac{b^2(2mE)}{2mE} + \frac{k}{2E} \\
&= b^2 + \frac{k}{2E}
\end{aligned}
$$

Therefore

$$
r_{\min} = \sqrt{b^2 + \frac{k}{2E}} \qquad (3)
$$

Now we can finish the limits of integration in (1B). When $r = r_{\min}$ then $z = \frac{1}{r_{\min}} = \frac{1}{\sqrt{b^2 - \frac{k}{2E}}}$, now (1B) becomes (where we now replace r^2 by $\frac{1}{z^2}$)

$$
\begin{aligned}
\theta_0(b) &= \int_{r_{\min}}^{\infty} \frac{b}{r^2\sqrt{1 - \frac{k}{2r^2E} - \frac{b^2}{r^2}}} dr \\
&= \int_{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}}^{0} \frac{z^2 b}{\sqrt{1 - \frac{kz^2}{2E} - b^2 z^2}} \left(-\frac{1}{z^2} dz\right) \\
&= b \int_0^{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}} \frac{1}{\sqrt{1 - \frac{kz^2}{2E} - b^2 z^2}} dz \\
&= b \int_0^{\frac{1}{\sqrt{b^2 + \frac{k}{2E}}}} \frac{dz}{\sqrt{1 - z^2 \left(\frac{k}{2E} + b^2\right)}}
\end{aligned}
$$

Using CAS, it gives $\int \frac{dz}{\sqrt{1 - az^2}} = \frac{1}{\sqrt{a}} \sin^{-1}\left(z\sqrt{a}\right)$. Using this result above, where $a = \left(\frac{k}{2E} + b^2\right)$

gives

$$\theta_0(b) = \frac{b}{\sqrt{\frac{k}{2E}+b^2}}\left(\sin^{-1}\left(z\sqrt{\frac{k}{2E}+b^2}\right)\Bigg|_0^{\frac{1}{\sqrt{b^2+\frac{k}{2E}}}}\right)$$

$$= \frac{b}{\sqrt{\frac{k}{2E}+b^2}}\left[\sin^{-1}\left(\frac{1}{\sqrt{b^2+\frac{k}{2E}}}\sqrt{\frac{k}{2E}+b^2}\right) - \sin^{-1}(0)\right]$$

$$= \frac{b}{\sqrt{\frac{k}{2E}+b^2}}\left[\sin^{-1}(1) - 0\right]$$

$$= \frac{b}{\sqrt{\frac{k}{2E}+b^2}}\frac{\pi}{2}$$

Now we solve for b. Squaring both sides

$$\theta_0^2 = \frac{b^2}{\frac{k}{2E}+b^2}\frac{\pi^2}{4}$$

Using $E = \frac{1}{2}mv^2$ then

$$\begin{aligned}
\theta_0^2 &= \frac{b^2}{\left(\frac{k}{mv^2}+b^2\right)}\frac{\pi^2}{4} \\
4\theta_0^2\left(\frac{k}{mv^2}+b^2\right) &= b^2\pi^2 \\
\frac{k4\theta_0^2}{mv^2} + 4\theta_0^2 b^2 - b^2\pi^2 &= 0 \\
b^2\left(4\theta_0^2 - \pi^2\right) &= -\frac{k4\theta_0^2}{mv^2} \\
b^2\left(\pi^2 - 4\theta_0^2\right) &= \frac{k4\theta_0^2}{mv^2} \\
b^2 &= \frac{k4\theta_0^2}{mv^2\left(\pi^2 - 4\theta_0^2\right)} \\
b &= \frac{2\theta_0}{v}\sqrt{\frac{k}{m\left(\pi^2 - 4\theta_0^2\right)}} \qquad (4)
\end{aligned}$$

But $\theta_0(b) = \frac{\pi}{2} - \frac{\theta_s}{2}$, where θ_s is the scattering angle. Therefore the above becomes

$$\begin{aligned}
b &= \frac{2\left(\frac{\pi}{2} - \frac{\theta_s}{2}\right)}{v}\sqrt{\frac{k}{m\left(\pi^2 - 4\left(\frac{\pi}{2} - \frac{\theta_s}{2}\right)^2\right)}} \\
b &= \frac{\pi - \theta_s}{v}\sqrt{\frac{k}{m\left(\pi^2 - \left(\theta_s^2 - 2\pi\theta_s + \pi^2\right)\right)}} \\
b &= \frac{\pi - \theta_s}{v}\sqrt{\frac{k}{m\left(2\pi\theta_s - \theta_s^2\right)}} \qquad (5)
\end{aligned}$$

Now we are ready to find $\sigma(\theta_s)$

$$\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|$$

From (5)

$$\frac{db}{d\theta_s} = -\frac{\pi^2\sqrt{\frac{k}{m(2\pi\theta_s-\theta_s^2)}}}{v\left(2\pi\theta_s-\theta_s^2\right)}$$

Therefore

$$\begin{aligned}
\sigma(\theta_s) &= \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right| \\
&= \frac{\frac{\pi-\theta_s}{v}\sqrt{\frac{k}{m(2\pi\theta_s-\theta_s^2)}}}{\sin\theta_s}\frac{\pi^2\sqrt{\frac{k}{m(2\pi\theta_s-\theta_s^2)}}}{v\left(2\pi\theta_s-\theta_s^2\right)} \\
&= \frac{\frac{\pi-\theta_s}{v}\frac{k}{m(2\pi\theta_s-\theta_s^2)}}{\sin\theta_s}\frac{\pi^2}{v\left(2\pi\theta_s-\theta_s^2\right)} \\
&= \frac{(\pi-\theta_s)\,k}{mv\sin\theta_s}\frac{\pi^2}{v\left(2\pi\theta_s-\theta_s^2\right)^2} \\
&= \frac{k\pi^2(\pi-\theta_s)}{mv^2\sin\theta_s\left(2\pi\theta_s-\theta_s^2\right)^2}
\end{aligned}$$

Or

$$\sigma(\theta_s) = \frac{k\pi^2(\pi-\theta_s)}{mv^2\theta_s^2\left(2\pi-\theta_s\right)^2\sin\theta_s}$$

Hard problem. Time taken to solve: 6 hrs.

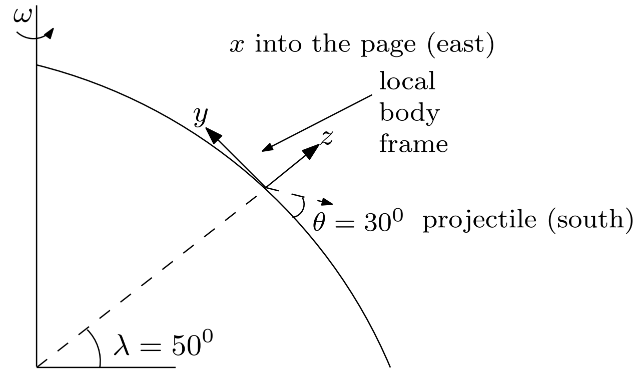
0.2 Problem 2

2. (10 points)
(1) A warship fires a projectile due South at a southern latitude of 50°. The shells are fired at 37° elevation with a speed of $800\,\mathrm{ms}^{-1}$. Neglecting air resistance, calculate by how much the shells will miss their target and in what direction.

(2) A batter hits a baseball a distance of 200 ft in a roughly flat trajectory. Should he take the Coriolis force into account? Neglect air resistance, assume the elevation angle is 15°, and the location is Yankee Stadium (or Wrigley Field, if you prefer).

SOLUTION:

0.2.1 part (1)



Using

$$\begin{aligned} x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 (\dot{z}_0 \cos\lambda - \dot{y}_0 \sin\lambda) + \dot{x}_0 t + x_0 \\ y &= \dot{y}_0 t - \omega t^2 \dot{x}_0 \sin\lambda + y_0 \\ z &= \dot{z}_0 t - \frac{1}{2} g t^2 + \omega t^2 \dot{x}_0 \cos\lambda + z_0 \end{aligned} \tag{1}$$

Where $\{\dot{x}_0, \dot{y}_0, \dot{z}_0\}$ are the initial speeds in each of the body frame directions and $\{x_0, y_0, z_0\}$ are the initial position of the projectile at $t = 0$. Let $v_0 = 800\ m/s^2$ and $\theta = 37^0$. We are given that

$$\begin{aligned} \dot{y}_0 &= -v_0 \cos\theta \\ \dot{z}_0 &= v_0 \sin\theta \\ \dot{x}_0 &= 0 \end{aligned}$$

The minus sign for $\dot{y}_0$ above was added since the direction is south, which is negative y direction for the local frame. And we are given that $x_0 = y_0 = z_0 = 0$. Substituting these in (1) gives (where $\lambda = 50^0$)

$$\begin{aligned} x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 (v_0 \sin\theta \cos\lambda + v_0 \cos\theta \sin\lambda) \\ y &= -(v_0 \cos\theta)\, t \\ z &= (v_0 \sin\theta)\, t - \frac{1}{2} g t^2 \end{aligned} \tag{2}$$

The drift due to the Coriolis force is found from the x component. The projectile will drift west (to the right direction of its motion) since it is moving south. We can now calculate this x drift. We know that $\omega = 7.3 \times 10^{-5}$ rad/sec (rotation speed of earth), so we just need to find time of flight t. From

$$\begin{aligned} \dot{z} &= \dot{z}_0 - g t \\ &= v_0 \sin\theta - g t \end{aligned}$$

The projectile time up (when $\dot{z}$ first becomes zero) is $t = \frac{v_0 \sin\theta}{g} = \frac{800\sin(37(\frac{\pi}{180}))}{9.81} \approx 50$ sec. Hence total time of flight is twice this which is $t_f = 100$ sec. Now we use this time in the x equation in (2) above

$$\begin{aligned} x &= \frac{1}{3}\left(7.3\times 10^{-5}\right)(9.81)(100)^3\cos\left(50^0\right) - \left(7.3\times 10^{-5}\right)(100)^2\left(800\sin 37^0\cos 50^0 + 800\cos 37^0\sin 50^0\right) \\ &= -532 \end{aligned}$$

So it will drift by about 532 meter to the west (since negative sign). In the above $g = 9.81$ was used. This does not include all the terms such as the centrifugal acceleration. But $9.81\frac{m}{s^2}$ is good approximation for this problem.

0.2.2 part (2)

Taking Latitude as 42^0 (New York). Therefore $\lambda = 42^0$ and $\theta = 15^0$. Initial conditions are

$$\begin{aligned} \dot{y}_0 &= V_0\cos\theta \\ \dot{z}_0 &= V_0\sin\theta \\ \dot{x}_0 &= 0 \end{aligned}$$

Where V_0 is the initial speed the ball was hit with (which we do not know yet), and $x_0 = y_0 = z_0 = 0$. Using

$$\begin{aligned} x &= \frac{1}{3}\omega g t^3\cos\lambda - \omega t^2(\dot{z}_0\cos\lambda - \dot{y}_0\sin\lambda) + \dot{x}_0 t + x_0 \\ y &= \dot{y}_0 t - \omega t^2\dot{x}_0\sin\lambda + y_0 \\ z &= \dot{z}_0 t - \frac{1}{2}g t^2 + \omega t^2\dot{x}_0\cos\lambda + z_0 \end{aligned} \tag{1}$$

Then applying initial conditions the above reduces to

$$\begin{aligned} x &= \frac{1}{3}\omega g t^3\cos\lambda - \omega t^2(V_0\sin\theta\cos\lambda - V_0\cos\theta\sin\lambda) \\ y &= (V_0\cos\theta)\,t \\ z &= (V_0\sin\theta)\,t - \frac{1}{2}g t^2 \end{aligned} \tag{2}$$

From $y(t_f) = (V_0\cos\theta)\,t_f$ then, since we are told that $y(t_f) = 200$ ft,

$$200(0.3048) = (V_0\cos\theta)\,t_f \tag{3}$$

Where t_f is time of flight. But time of flight is also found

$$\begin{aligned} \dot{z} &= \dot{z}_0 - g t \\ &= V_0\sin\theta - g t \end{aligned}$$

And solving for $\dot{z}=0$, which gives $\frac{V_0 \sin\theta}{g}$. So time of flight is twice this or

$$t_f = \frac{2V_0 \sin\theta}{g}$$

Substituting the above into (3) to solve for V_0 gives

$$\begin{aligned}
200(0.3048) &= (V_0 \cos\theta)\frac{2V_0 \sin\theta}{g} \\
60.96 &= \frac{2}{9.81} V_0^2 \left(\cos 15^0\right)\left(\sin 15^0\right) \\
V_0^2 &= \frac{(60.96)(9.81)}{2\cos 15^0 \sin 15^0} \\
&= 1196.0
\end{aligned}$$

Hence

$$V_0 = 34.583 \quad \text{m/s}$$

Now we can go back and solve for time of flight t_f. From

$$\begin{aligned}
200(0.3048) &= (V_0 \cos\theta)\, t_f \\
t_f &= \frac{200(0.3048)}{34.583\ \left(\cos 15^0\right)} \\
&= 1.825 \text{ sec}
\end{aligned}$$

Using (2) we solve for x, the drift due to Coriolis forces.

$$\begin{aligned}
x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 (V_0 \sin\theta \cos\lambda - V_0 \cos\theta \sin\lambda) \\
&= \frac{1}{3}\left(7.3\times 10^{-5}\right)(9.81)(1.825)^3 \cos 42^0 - \left(7.3\times 10^{-5}\right)(1.825)^2 \left(34.58 \sin 15^0 \cos 42^0 + 34.58 \cos 15^0 \sin 42^0 \right. \\
&= 4.897\times 10^{-3} \text{ meter}
\end{aligned}$$

So the ball will drift about $5mm$. This is too small and the ball player can therefore ignore Coriolis forces when hitting the ball.

0.3 Problem 3

3. (5 points)
A bullet is fired straight up with initial speed v_0. Show that the bullet will hit the ground *west* of the initial point of upward motion by an amount $4\,\omega\, v_0^3 \cos\lambda/(3\,g^2)$, where λ is the latitude and ω is the angular velocity of Earth's rotation. Ignore air resistance.

SOLUTION:

Initial conditions are

$$\begin{aligned}\dot{y}_0 &= 0\\ \dot{z}_0 &= v_0\\ \dot{x}_0 &= 0\end{aligned}$$

And $x_0 = y_0 = z_0 = 0$. Using

$$\begin{aligned}x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 (\dot{z}_0 \cos\lambda - \dot{y}_0 \sin\lambda) + \dot{x}_0 t + x_0\\ y &= \dot{y}_0 t - \omega t^2 \dot{x}_0 \sin\lambda + y_0\\ z &= \dot{z}_0 t - \frac{1}{2} g t^2 + \omega t^2 \dot{x}_0 \cos\lambda + z_0\end{aligned} \tag{1}$$

The reduce to (using initial conditions) to

$$\begin{aligned}x &= \frac{1}{3}\omega g t^3 \cos\lambda - \omega t^2 v_0 \cos\lambda\\ y &= 0\\ z &= v_0 t - \frac{1}{2} g t^2\end{aligned} \tag{2}$$

To find time of flight of bullet (going up and then down again), from $\dot{z} = v_0 - gt$, we solve for $\dot{z} = 0$, which gives $t = \frac{v_0}{g}$. So time of flight is twice this amount

$$t_f = \frac{2v_0}{g} \text{ sec}$$

To find the amount x the bullet moves during this time, we use (2) and solve for x

$$\begin{aligned}x(t_f) &= \frac{1}{3}\omega g t_f^3 \cos\lambda - \omega t_f^2 v_0 \cos\lambda\\ &= \frac{1}{3}\omega g \left(\frac{2v_0}{g}\right)^3 \cos\lambda - \omega \left(\frac{2v_0}{g}\right)^2 v_0 \cos\lambda\\ &= \frac{1}{3}\omega \frac{8v_0^3}{g^2} \cos\lambda - \omega \frac{4v_0^3}{g^2} \cos\lambda\\ &= \left(\frac{8}{3} - 4\right)\left(\omega \frac{v_0^3}{g^2} \cos\lambda\right)\\ &= -\frac{4}{3}\omega \frac{v_0^3}{g^2} \cos\lambda\end{aligned}$$

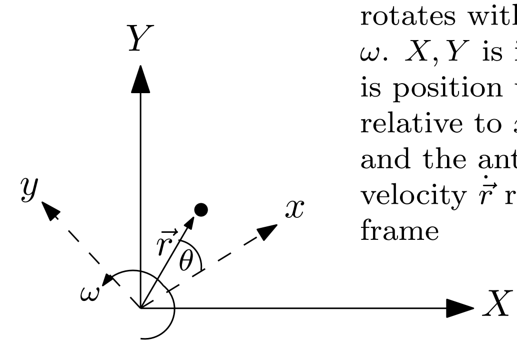
This means when it lands again, the bullet will be $-\frac{4}{3}\omega\frac{v_0^3}{g^2}\cos\lambda$ meters relative to the original point it was fired from (the origin of the local body frame). Since the sign is negative, it means it is west.

0.4 Problem 4

4. (10 points)
A bug crawls with constant speed in a circular path of radius b on a phonograph turntable rotating with constant angular speed ω. The bug's path is concentric with the center of the turntable. If the bug's mass is m and the coefficient of static friction for the bug on the table is μ, how fast (relative to the turntable) can the bug crawl before it starts to slip if it goes (1) in the direction of rotation and (2) opposite to the direction of rotation?

SOLUTION:

0.4.1 Part(1)



x, y is local frame that rotates with angular velocity ω. X, Y is inertial frame. $\vec{r}$ is position vector of ant relative to x, y local frame, and the ant is moving with velocity $\dot{\vec{r}}$ relative to its local frame

When Ant is moving in direction of rotation:

$$\begin{aligned}\vec{r} &= b\cos\theta\vec{i} + b\sin\theta\vec{j} \\ \vec{v} &= \vec{v}_{rel} + \vec{\omega}\times\vec{r}\end{aligned} \tag{1}$$

But

$$\begin{aligned}\vec{v}_{rel} &= \frac{d}{dt}\vec{r} \\ &= -b\dot{\theta}\sin\theta\vec{i} + b\dot{\theta}\cos\theta\vec{j}\end{aligned}$$

And

$$\begin{aligned}\vec{\omega}\times\vec{r} &= \omega\vec{k}\times\left(b\cos\theta\vec{i} + b\sin\theta\vec{j}\right) \\ &= b\omega\cos\theta\vec{j} - b\omega\sin\theta\vec{i}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\vec{v} &= \left(-b\dot{\theta}\sin\theta\vec{i} + b\dot{\theta}\cos\theta\vec{j}\right) + \left(b\omega\cos\theta\vec{j} - b\omega\sin\theta\vec{i}\right) \\ &= \vec{i}\left(-b\dot{\theta}\sin\theta - b\omega\sin\theta\right) + \vec{j}\left(b\dot{\theta}\cos\theta + b\omega\cos\theta\right)\end{aligned}$$

The above is the velocity of the ant, in the inertial frame, using local body unit vector $\vec{\imath}, \vec{\jmath}$. Now we find the ant acceleration, given by

$$\vec{a} = \vec{a}_{rel} + 2\left(\omega\vec{k} \times \vec{v}_{rel}\right) + \left(\dot{\omega}\vec{k} \times \vec{r}\right) + \omega\vec{k} \times (\vec{\omega} \times \vec{r})$$

But $\dot{\omega} = 0$ since disk has constant ω then

$$\vec{a} = \vec{a}_{rel} + 2\left(\omega\vec{k} \times \vec{v}_{rel}\right) + \omega\vec{k} \times (\vec{\omega} \times \vec{r}) \tag{1}$$

But

$$\begin{aligned}\vec{a}_{rel} &= \frac{d}{dt}\vec{v}_{rel} \\ &= \vec{\imath}\left(-b\ddot{\theta}\sin\theta - b\dot{\theta}^2\cos\theta\right) + \vec{\jmath}\left(b\ddot{\theta}\cos\theta - b\dot{\theta}^2\sin\theta\right)\end{aligned}$$

Since Bug moves with constant speed, then $\ddot{\theta} = 0$ and the above becomes

$$\vec{a}_{rel} = \vec{\imath}\left(-b\dot{\theta}^2\cos\theta\right) + \vec{\jmath}\left(-b\dot{\theta}^2\sin\theta\right)$$

Now the Coriolis term $2(\vec{\omega} \times \vec{v}_{rel})$ is found

$$\begin{aligned}2(\vec{\omega} \times \vec{v}_{rel}) &= 2\left(\omega\vec{k} \times \left(-b\dot{\theta}\sin\theta\vec{\imath} + b\dot{\theta}\cos\theta\vec{\jmath}\right)\right) \\ &= 2\left(-\omega b\dot{\theta}\sin\theta\vec{\jmath} - b\omega\dot{\theta}\cos\theta\vec{\imath}\right)\end{aligned}$$

Now the $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is found

$$\begin{aligned}\vec{\omega} \times (\vec{\omega} \times \vec{r}) &= \omega\vec{k} \times \left(b\omega\cos\theta\vec{\jmath} - b\omega\sin\theta\vec{\imath}\right) \\ &= -b\omega^2\cos\theta\vec{\imath} - b\omega^2\sin\theta\vec{\jmath}\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\vec{a} &= \vec{a}_{rel} + 2\left(\omega\vec{k} \times \vec{v}_{rel}\right) + \omega\vec{k} \times (\vec{\omega} \times \vec{r}) \\ &= \vec{\imath}\left(-b\dot{\theta}^2\cos\theta\right) + \vec{\jmath}\left(-b\dot{\theta}^2\sin\theta\right) + 2\left(-\omega b\dot{\theta}\sin\theta\vec{\jmath} - b\omega\dot{\theta}\cos\theta\vec{\imath}\right) - b\omega^2\cos\theta\vec{\imath} - b\omega^2\sin\theta\vec{\jmath} \\ &= \vec{\imath}\left(-b\dot{\theta}^2\cos\theta - 2b\omega\dot{\theta}\cos\theta - b\omega^2\cos\theta\right) + \vec{\jmath}\left(-b\dot{\theta}^2\sin\theta - 2\omega b\dot{\theta}\sin\theta - b\omega^2\sin\theta\right)\end{aligned}$$

Since this is valid for all time, lets take snap shot when $\theta = 0$, which gives

$$\vec{a} = \vec{\imath}\left(-b\dot{\theta}^2 - 2b\omega\dot{\theta} - b\omega^2\right)$$

So when $\theta = 0$, the ant acceleration (as seen in inertial frame) is towards the center of the disk with the above magnitude. If the ant speed is V then $V = b\dot{\theta}$ and the above can be re-written in terms of V as

$$\vec{a} = -\vec{\imath}\left(\frac{V^2}{b} + 2V\omega + b\omega^2\right)$$

The ant will starts to slip, when the force preventing it from sliding radially in the outer direction equals the centrifugal force $m\left(\frac{V^2}{b}+3V\omega+b\omega^2\right)$Hence

$$\mu m g = m\left(\frac{V^2}{b}+2V\omega+b\omega^2\right)$$

$$\begin{aligned}\frac{V^2}{b}+2V\omega+b\omega^2-\mu g&=0\\V^2+2Vb\omega-\left(\mu b g+b^2\omega^2\right)&=0\end{aligned}$$

This is quadratic in V, hence

$$\begin{aligned}V&=\frac{-2b\omega}{2}\pm\frac{1}{2}\sqrt{4b^2\omega^2+4\left(-\mu b g+b^2\omega^2\right)}\\&=-b\omega\pm\sqrt{b^2\omega^2-\mu b g+b^2\omega^2}\\&=-b\omega\pm\sqrt{2b^2\omega^2-\mu b g}\end{aligned}$$

Since $V>0$ then

$$\begin{aligned}V&=-b\omega+b\omega\sqrt{2-\frac{\mu g}{b\omega^2}}\\&=b\omega\left(\sqrt{2-\frac{\mu g}{b\omega^2}}-1\right)\end{aligned}$$

0.4.2 Part(2)

When Ant is moving the opposite direction of rotation, then the Coriolis term $2\left(\omega\vec{k}\times\vec{v}_{rel}\right)$ will have the opposite sign from the above. Then means the final answer will be

$$\vec{a}=-\vec{i}\left(\frac{V^2}{b}-2V\omega+b\omega^2\right)$$

Which means

$$\begin{aligned}V&=\frac{2b\omega}{2}\pm\frac{1}{2}\sqrt{4b^2\omega^2+4\left(-\mu b g+b^2\omega^2\right)}\\&=b\omega\pm\sqrt{b^2\omega^2-\mu b g+b^2\omega^2}\\&=b\omega\pm\sqrt{2b^2\omega^2-\mu b g}\end{aligned}$$

Or

$$\begin{aligned}V&=b\omega+b\omega\sqrt{2-\frac{\mu g}{b\omega^2}}\\&=b\omega\left(\sqrt{2-\frac{\mu g}{b\omega^2}}+1\right)\end{aligned}$$

0.5 Problem 5

5. (10 points)
(1) Show that the small angular deviation ϵ of a plumb line from the true vertical (toward the center of the Earth) at a point on Earth's surface is

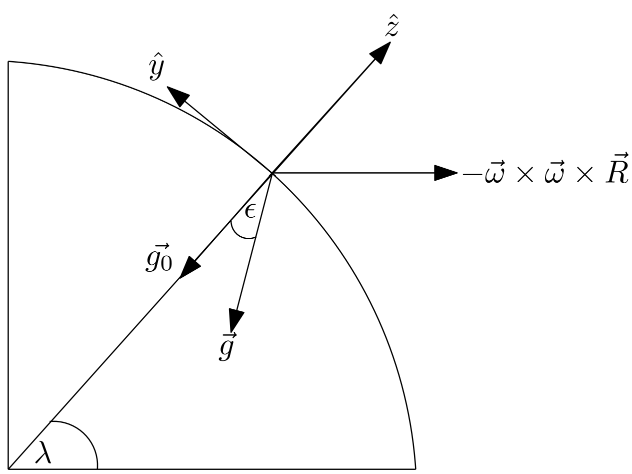
$$\epsilon = \frac{R\omega^2 \sin\lambda \cos\lambda}{g_0 - R\omega^2\cos^2\lambda} \ ,$$

where g_0 is the acceleration due to gravity, λ is the latitude, and R is the radius of the Earth.

(2) Use a computer to plot ϵ as a function of latitude. At what latitude do we observe the largest deviation, and how large is it?

SOLUTION:

0.5.1 Part(1)



$$\vec{g} = \vec{g}_0 - \vec{\omega}\times\vec{\omega}\times\vec{R}$$

Using $a\times(b\times c) = b(a\cdot c) - c(a\cdot b)$ the above becomes

$$\begin{aligned}\vec{g} &= \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - (\vec{\omega}\cdot\vec{\omega})\,\vec{R}\right)\\ &= \vec{g}_0 - \left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right) - \omega^2\vec{R}\right)\end{aligned}$$

Then using

$$\vec{g}\times\vec{g}_0 = g g_0 (\sin\varepsilon)\,\vec{n} \tag{1}$$

Where $\vec{n}$ is perpendicular to plane of $\vec{g},\vec{g}_0$ which is $\hat{x}$ in this case. Then the LHS of the above is

$$\begin{aligned}\vec{g}\times\vec{g}_0 &= \left[\vec{g}_0-\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)\right]\times\vec{g}_0\\ &= \vec{g}_0\times\vec{g}_0-\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)\times\vec{g}_0\right)+\left(\omega^2\vec{R}\times\vec{g}_0\right)\end{aligned}$$

But $\vec{R}\times\vec{g}_0=0$ since they are in same direction, also $\vec{g}_0\times\vec{g}_0=0$ and the above becomes

$$\vec{g}\times\vec{g}_0=-\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)\times\vec{g}_0 \tag{2}$$

But

$$\vec{\omega}\cdot\vec{R}=\omega R\cos\left(\frac{\pi}{2}-\lambda\right)$$

Therefore (2) becomes

$$\vec{g}\times\vec{g}_0=-\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\vec{\omega}\times\vec{g}_0$$

But $\vec{\omega}\times\vec{g}_0=-\omega g_0\sin\left(\frac{\pi}{2}-\lambda\right)\hat{x}$, hence the above becomes

$$\vec{g}\times\vec{g}_0=\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\omega g_0\sin\left(\frac{\pi}{2}-\lambda\right)\hat{x}$$

Now we go back to (1) and apply the definition, therefore

$$\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\omega g_0\sin\left(\frac{\pi}{2}-\lambda\right)\hat{x}=gg_0(\sin\varepsilon)\,\hat{x}$$

Or

$$\begin{aligned}\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\omega g_0\sin\left(\frac{\pi}{2}-\lambda\right) &= gg_0(\sin\varepsilon)\\ \sin\varepsilon &= \frac{\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\omega g_0\sin\left(\frac{\pi}{2}-\lambda\right)}{gg_0}\\ &= \frac{R\omega^2\cos\left(\frac{\pi}{2}-\lambda\right)\sin\left(\frac{\pi}{2}-\lambda\right)}{g}\end{aligned}$$

But $\sin\left(\frac{\pi}{2}-\lambda\right)=\cos\lambda$ and $\cos\left(\frac{\pi}{2}-\lambda\right)=\sin\lambda$ hence the above becomes

$$\sin\varepsilon=\frac{R\omega^2\sin\lambda\cos\lambda}{g} \tag{3}$$

To find $g=|\vec{g}|$, since $\vec{g}=\vec{g}_0-\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)$, then taking dot product gives

$$
\begin{aligned}
|\vec{g}| &= \vec{g}\cdot\vec{g}\\
&= \left[\vec{g}_0-\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)\right]\cdot\left[\vec{g}_0-\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)\right]\\
&= g_0^2-2\vec{g}_0\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)+\overbrace{\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)}^{\text{ignore. All }\omega^4\text{ powers. too small}}\\
&\approx g_0^2-2\vec{g}_0\cdot\left(\vec{\omega}\left(\vec{\omega}\cdot\vec{R}\right)-\omega^2\vec{R}\right)\\
&= g_0^2-\left(-2g_0\hat{z}\right)\cdot\left(\left(\omega\cos\lambda\hat{y}+\omega\sin\lambda\hat{z}\right)\left(\omega R\cos\left(\frac{\pi}{2}-\lambda\right)\right)-\omega^2R\hat{z}\right)\\
&= g_0^2-\left(-2g_0\hat{z}\right)\cdot\left(\left(\omega\cos\lambda\hat{y}+\omega\sin\lambda\hat{z}\right)\left(\omega R\sin\lambda\right)-\omega^2R\hat{z}\right)\\
&= g_0^2-\left(-2g_0\hat{z}\right)\cdot\left(\omega^2R\sin\lambda\cos\lambda\hat{y}+\left(\omega^2R\sin^2\lambda-\omega^2R\right)\hat{z}\right)\\
&= g_0^2-\left(-2g_0\left(\omega^2R\sin^2\lambda-\omega^2R\right)\right)\\
&= g_0^2+2g_0\omega^2R\sin^2\lambda-2g_0\omega^2R\\
&= g_0^2+2g_0\omega^2R\left(1-\cos^2\lambda\right)-2g_0\omega^2R\\
&= g_0^2+2g_0\omega^2R-2g_0\omega^2R\cos^2\lambda-2g_0\omega^2R\\
&= g_0^2-2g_0\omega^2R\cos^2\lambda
\end{aligned}
$$

Therefore (3) becomes

$$\sin\varepsilon=\frac{R\omega^2\sin\lambda\cos\lambda}{g_0^2-2g_0\omega^2R\cos^2\lambda}$$

Since ε is small, then $\sin\varepsilon\approx\varepsilon$, therefore

$$\varepsilon\approx\frac{R\omega^2\sin\lambda\cos\lambda}{g_0^2-2g_0\omega^2R\cos^2\lambda}$$

The solutions has an extra g_0 in the denominator. I am not sure why. I will what is given for part(2) to plot it.

0.5.2 Part(2)

This plot shows the maximum ε is at $\lambda=45^0$. Here is the code used and the plot generated

```
R0 = 6371*10^3; (*earth radius*)
omega = 7.27*10^(-5); (*earth rotation*)
g0 = 9.81;
e[lam_] := (R0 omega^2 Sin[lam] Cos[lam])/(g0 - R0 omega^2 Cos[lam]^2)*180/Pi;
newTicks[min_, max_] := Table[{i, Round[i*180/Pi]}, {i, 0, Pi/2, .1}];

Plot[e[lam], {lam, 0, Pi/2}, Frame -> True,
 FrameLabel -> {{"\[CurlyEpsilon] degree", None}, {"\[Lambda] (degree)",
```

```
"Part(2) solution"}}, GridLines -> Automatic,
FrameTicks -> {{Automatic, Automatic}, {newTicks, Automatic}}]
```

Out[80]=