# HW5 Physics 311 Mechanics

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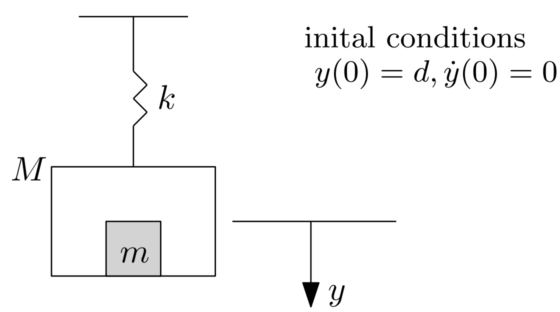
# Contents

## 0.1 Problem 1

1. (5 points)
A spring of spring constant $k$ supports a box of mass $M$, which contains a block of mass $m$. If the system is pulled downward a distance $d$ from the equilibrium position and then released, it starts to oscillate. For what value of $d$ does the block just begin to leave the bottom of the box at the top of the vertical oscillations?

SOLUTION:




The block of mass $m$ will leave the floor of the box when the vertical acceleration is large enough to match the gravity acceleration $g$. The equation of motion of the overall system is given by

$$y'' + \omega_0^2 y = 0 \tag{1}$$

Where $\omega_0$ is the undamped natural frequency

$$\omega_0 = \sqrt{\frac{k}{M+m}}$$

The solution to (1) is

$$y = A\cos\omega_0 t + B\sin\omega_0 t \tag{2}$$

Initial conditions are used to find $A, B$. Since at $t = 0$, $y(0) = d$, then from (2) we find

$$A = d$$

Taking derivative of (2) gives

$$y' = -A\omega_0 \sin\omega_0 t + B\omega_0 \cos\omega_0 t \tag{3}$$

At $t = 0$, $y'(0) = 0$, this gives $B = 0$. Therefore the full solution (2) becomes

$$y = d\cos\omega_0 t$$

The acceleration is now found as

$$y' = -\omega_0 d \sin\omega_0 t$$
$$y'' = -\omega_0^2 d \cos\omega_0 t$$

The period is $T_p = \frac{2\pi}{\omega_0}$. After one $T_p$ from release the box will be the top. Therefore, the acceleration at that moment is

$$\begin{aligned} y''\left(T_p\right) &= -\omega_0^2 d \cos \omega_0 T_p \\ &= -\omega_0^2 d \cos 2\pi \\ &= \omega_0^2 d \end{aligned}$$

The condition for $m$ to just leave the floor of the box is when the above acceleration is the same as $g$.

$$\begin{aligned} \omega_0^2 d &= g \\ d &= \frac{g}{\omega_0^2} \end{aligned}$$

Therefore

$$\boxed{d = \frac{g}{k}\left(M+m\right)}$$

## 0.2 Problem 2

2. (15 points)
(1) Show that the Fourier series of a periodic square wave is

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + ...\right] \;,$$

where

$$\begin{aligned} f(t) &= +1 \quad \text{for} \quad 0 < \omega \mathrm{t} < \pi, \quad 2\pi < \omega \mathrm{t} < 3\pi, ... \\ f(t) &= -1 \quad \text{for} \quad \pi < \omega \mathrm{t} < 2\pi, \quad 3\pi < \omega \mathrm{t} < 4\pi, ... \; . \end{aligned}$$

(2) Use the result from above to find the steady-state motion of a damped harmonic oscillator that is driven by a periodic square-wave force of amplitude $F_0$. In particular, find the relative amplitudes of the first three terms, $A_1$, $A_3$, and $A_5$, of the response function $x(t)$ in the case that the third harmonic $3\omega$ of the driving frequency coincides with the frequency $\omega_0$ of the undamped oscillator. Assume a quality factor of $Q = 100$.

SOLUTION:

### 0.2.1 Part (1)

The function $f(t)$ is an odd function, therefore we only need to evaluate $b_n$ terms. To more clearly see the period, the definition of $f(t)$ is written as

$$f(t) = \begin{cases} +1 & 0 < t < \frac{\pi}{\omega}, \cdots \\ -1 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega}, \cdots \end{cases}$$

Therefore the period is

$$T_p = \frac{2\pi}{\omega}$$

Finding $b_n$

$$
\begin{aligned}
b_n &= \frac{1}{\frac{T_p}{2}} \int_0^{T_p} f(t)\sin(n\omega t)\,dt \\
&= \frac{2}{\frac{2\pi}{\omega}} \left( \int_0^{\frac{\pi}{\omega}} (+1)\sin(n\omega t)\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-1)\sin(n\omega t)\,dt \right) \\
&= \frac{\omega}{\pi} \left( \int_0^{\frac{\pi}{\omega}} \sin(n\omega t)\,dt - \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \sin(n\omega t)\,dt \right) \\
&= \frac{\omega}{\pi} \left( \left[ -\frac{\cos(n\omega t)}{n\omega} \right]_0^{\frac{\pi}{\omega}} - \left[ -\frac{\cos(n\omega t)}{n\omega} \right]_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \right) \\
&= \frac{\omega}{\pi} \left( -\frac{1}{n\omega} \left[\cos(n\omega t)\right]_0^{\frac{\pi}{\omega}} + \frac{1}{n\omega} \left[\cos(n\omega t)\right]_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \right) \\
&= \frac{1}{n\pi} \left( -\left[ \cos\left(n\omega\frac{\pi}{\omega}\right) - \cos(0) \right] + \left[ \cos\left(n\omega\frac{2\pi}{\omega}\right) - \cos\left(n\omega\frac{\pi}{\omega}\right) \right] \right) \\
&= \frac{1}{n\pi} \left( -\left[\cos(n\pi) - 1\right] + \left[\cos(2n\pi) - \cos(n\pi)\right] \right) \\
&= \frac{1}{n\pi} \left( -\cos(n\pi) + 1 + \cos(2n\pi) - \cos(n\pi) \right) \\
&= \frac{1}{n\pi} \left( -2\cos(n\pi) + \overbrace{\cos(2n\pi)}^{1} + 1 \right) \\
&= \frac{2}{n\pi} \left(1 - \cos(n\pi)\right)
\end{aligned}
$$

And since $n$ is an integer, then $\cos(n\pi) = (-1)^n$ and the above reduces to

$$
b_n = \frac{2}{n\pi}\left(1 - (-1)^n\right)
$$

Therefore

$$
b_n = \begin{cases} \frac{4}{n\pi} & n = 1, 3, 5, \cdots \\ 0 & \text{otherwise} \end{cases}
$$

Hence

$$
\begin{aligned}
f(t) &= \sum_{n=1,3,5,\cdots}^{\infty} b_n \sin(\omega n t) \\
&= \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi} \sin(\omega n t)
\end{aligned}
$$

Writing down few terms to see the sequence

$$
f(t) = \frac{4}{\pi} \left\{ \sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \frac{1}{7}\sin(7\omega t) + \cdots \right\}
$$

### 0.2.2 Part (2)

When the system is driven by the above periodic square wave of amplitude $F_0$, the steady state response is the sum to the response of each harmonic in the Fourier series expansion of the forcing function. Since the steady state response of a second order system to $F_n \sin(n\omega t)$ is given by

$$y_n(t) = \frac{F_n/m}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda_n^2 (n\omega)^2}} \sin(n\omega t + \delta_n)$$

Where the phase $\delta_n$ is defined as

$$\delta_n = \tan^{-1} \frac{-2\lambda(n\omega)}{\omega_0^2 - (n\omega)^2}$$

Then the steady state response to $f(t) = \sum_{n=1,3,5,\cdots}^{\infty} F_0 \frac{4}{n\pi} \sin(\omega n t)$ is given by

$$\begin{aligned} y_{ss}(t) &= \sum_{n=1,3,5,\cdots}^{\infty} \frac{4}{n\pi} \frac{F_0/m}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda^2 (n\omega)^2}} \sin(n\omega t + \delta_n) \qquad (1) \\ &= \frac{4F_0}{\pi m} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} \frac{\sin(n\omega t + \delta_n)}{\sqrt{\left(\omega_0^2 - (n\omega)^2\right)^2 + 4\lambda^2 (n\omega)^2}} \end{aligned}$$

Looking at the first three responses gives

$$y_{ss}(t) = \frac{4F_0}{\pi m} \left\{ \frac{\sin(\omega t + \delta_1)}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}} + \frac{1}{3} \frac{\sin(3\omega t + \delta_3)}{\sqrt{\left(\omega_0^2 - (3\omega)^2\right)^2 + 4\lambda^2 (3\omega)^2}} + \frac{1}{5} \frac{\sin(5\omega t + \delta_5)}{\sqrt{\left(\omega_0^2 - (5\omega)^2\right)^2 + 4\lambda^2 (5\omega)^2}} + \cdots \right\} \qquad (2)$$

We are told that $3\omega = \omega_0$ or $\omega = \frac{1}{3}\omega_0$ and in addition, using using $Q = \frac{\omega_0}{2\lambda}$ we find

$$\begin{aligned} 100 &= \frac{\omega_0}{2\lambda} \\ \lambda &= \frac{\omega_0}{200} \end{aligned}$$

Using this $\lambda$ and given value of $\omega$ then the phase $\delta_n$ becomes

$$\begin{aligned} \delta_n &= \tan^{-1} \frac{-2\lambda(n\omega)}{\omega_0^2 - (n\omega)^2} \\ &= \tan^{-1} \frac{-2\left(\frac{\omega_0}{200}\right)\left(n\frac{\omega_0}{3}\right)}{\omega_0^2 - \left(n\frac{\omega_0}{3}\right)^2} \\ &= \tan^{-1} \frac{3n}{100n^2 - 900} \end{aligned}$$

Using the above phase in (2) gives[1]

$$y_{ss}\left(t\right)=\frac{4F_0}{\pi m}\left\{\frac{\sin\left(\frac{\omega_0}{3}t+\tan^{-1}\frac{-3}{800}\right)}{\sqrt{\left(\omega_0^2-\left(\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)^2\left(\frac{\omega_0}{3}\right)^2}}+\frac{\frac{1}{3}\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\sqrt{\left(\omega_0^2-\left(3\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)^2\left(3\frac{\omega_0}{3}\right)^2}}+\frac{\frac{1}{5}\sin\left(5\frac{\omega_0}{3}t+\tan^{-1}\frac{3}{32\cdots}\right.}{\sqrt{\left(\omega_0^2-\left(5\frac{\omega_0}{3}\right)^2\right)^2+4\left(\frac{\omega_0}{200}\right)\cdots}}\right.$$

$$=\frac{4}{\pi}\frac{F_0}{m}\left\{\frac{\sin\left(\frac{\omega_0}{3}t-\tan^{-1}\frac{3}{800}\right)}{\sqrt{\frac{640\,009}{810\,000}\omega_0^4}}+\frac{1}{3}\frac{\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\sqrt{\frac{1}{10\,000}\omega_0^4}}+\frac{1}{5}\frac{\sin\left(5\frac{\omega_0}{3}t+\tan^{-1}\frac{3}{320}\right)}{\sqrt{\frac{102\,409}{32\,400}\omega_0^4}}+\cdots\right\}$$

$$=\frac{4}{\pi}\frac{F_0}{m}\left\{1.125\frac{\sin\left(0.333\omega_0 t-\tan^{-1}\frac{3}{800}\right)}{\omega_0^2}+33.333\frac{\sin\left(\omega_0 t+\frac{\pi}{2}\right)}{\omega_0^2}+0.11249\frac{\sin\left(1.6667\omega_0 t+\tan^{-1}\frac{3}{320}\right)}{\omega_0^2}+\cdots\right\}$$

The relative amplitudes of $A_1, A_3, A_5$ are given by

$$\{1.125, 33.333, 0.11249\}$$

We see that the third harmonic ($n=3$) has the largest amplitude, since this is where $3\omega=\omega_0$. In normalized size, dividing all amplitudes by the smallest amplitude gives

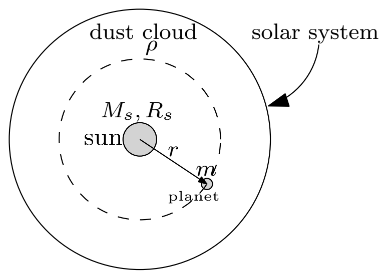
$$\{A_1, A_3, A_5\}_{normalized}=\{10, 296, 1\}$$

[1] The third harmonic $n=3$ has $\frac{\pi}{2}$ phase since $\tan^{-1}(\infty)=\frac{\pi}{2}$

## 0.3 Problem 3

3. (5 points)
   If the solar system were imbedded in a uniform dust cloud of density $\rho$, what would be the force on a planet a distance $r$ from the center of the Sun?

SOLUTION:




The total force on the planet $m$ is due to the mass inside the region centered at the center of the sun. The mass outside can be ignored since its effect cancels out. Let the radius of the sun be $R_{sun}$, then the total mass that pulls the planet toward the center of the solar system is given by

$$M_{total} = M_{sun} + \frac{4}{3}\pi\left(r^3 - R_{sun}^3\right)\rho$$

The force on the planet is therefore

$$\begin{aligned}\bar{F} &= -\frac{GM_{total}m}{r^2}\hat{r} \\ &= -\frac{G\left(M_{sun} + \frac{4}{3}\pi\left(r^3 - R_{sun}^3\right)\rho\right)m}{r^2}\hat{r}\end{aligned}$$

Where $\hat{r}$ is a unit vector pointing from the sun towards the planet $m$ and $G$ is the gravitational constant and $\rho$ is the cloud density.

## 0.4 Problem 4

4. (10 points)
(1) What is the speed (in km/s) for a satellite in a low-lying orbit close to Earth? Assume that the radius of the satellite's orbit is roughly equal to the Earth's radius.
(2) Show that the radius for a circular orbit of a synchronous (24-h) Earth satellite is about 6.6 Earth radii.
(3) The distance to the Moon is about 60.3 Earth radii. From this, calculate the length of the sidereal month (the period of the Moon's orbital revolution).

SOLUTION:

### 0.4.1 Part (1)

The force on the satellite is $mr_e\omega^2$ where $r_e$ is taken as the earth radius since this is low-lying orbit. Therefore

$$\frac{GM_e m}{r_e^2} = mr_e\omega^2$$

But $v = r_e\omega$ where $v$ is the satellite speed we want to find. Hence $\omega^2 = \frac{v^2}{r_e^2}$ and the above becomes

$$\begin{aligned}\frac{GM_e}{r_e^2} &= r_e\frac{v^2}{r_e^2}\\ v &= \sqrt{\frac{GM_e}{r_e}}\\ &= \sqrt{\frac{\left(6.67408\times10^{-11}\right)\left(5.972\times10^{24}\right)}{6.371\times10^{6}}}\\ &= 7909.6 \text{ meter/sec}\\ &= 7.9 \text{ km/sec}\end{aligned}$$

### 0.4.2 Part (2)

Let the radius of the satellite orbit be $r$. Using

$$\begin{aligned}\frac{GM_e m}{r^2} &= mr\omega^2\\ r &= \left(\frac{GM_e}{\omega^2}\right)^{\frac{1}{3}}\end{aligned}$$

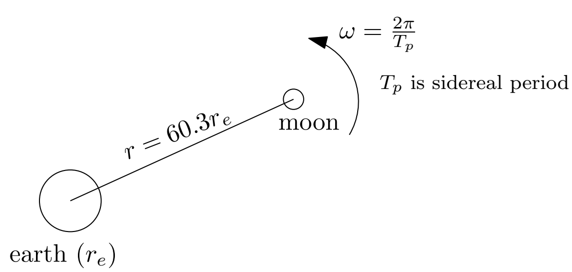
where $\omega = \frac{2\pi}{T_p}$ where $T_p$ is the period of the satellite. But for synchronous satellite, this period is 24 hrs. Hence the above becomes

$$
\begin{aligned}
r &= \left( \frac{GM_e}{\left(\frac{2\pi}{T_p}\right)^2} \right)^{\frac{1}{3}} \\
&= \left( \frac{\left(6.67408 \times 10^{-11}\right)\left(5.972 \times 10^{24}\right)}{\left(\frac{2\pi}{24(60)(60)}\right)^2} \right)^{\frac{1}{3}} \\
&= 4.224 \times 10^7 \text{ meter}
\end{aligned}
$$

But radius of earth is $r_e = 6.371 \times 10^6$ meters. Hence

$$
\frac{r}{r_e} = \frac{4.224 \times 10^7}{6.371 \times 10^6} = 6.63
$$

### 0.4.3 Part (3)




From

$$
\begin{aligned}
\frac{GM_e m}{r^2} &= mr\omega^2 \\
\frac{GM_e}{r^3} &= \omega^2 \\
\frac{GM_e}{r^3} &= \left(\frac{2\pi}{T_p}\right)^2
\end{aligned}
$$

We solve for $T_p$, hence

$$
\begin{aligned}
\frac{2\pi}{T_p} &= \sqrt{\frac{GM_e}{r^3}} \\
T_p &= \frac{2\pi}{\sqrt{\frac{GM_e}{r^3}}} = \frac{2\pi}{\sqrt{\frac{\left(6.67408\times10^{-11}\right)\left(5.972\times10^{24}\right)}{\left((60.3)\left(6.371\times10^{6}\right)\right)^3}}} \\
&= 2.3698 \times 10^6 \text{ sec}
\end{aligned}
$$

Therefore, in days, the above becomes

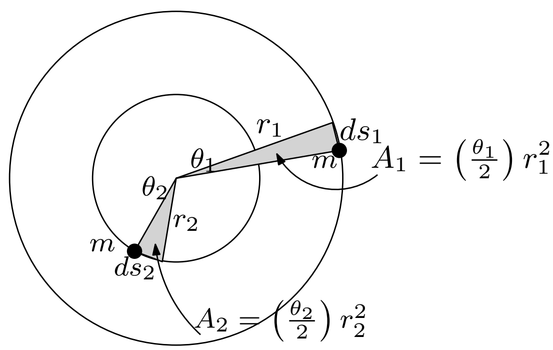
$$T_p = \frac{2.369\,8 \times 10^6}{(24)\,(60)\,(60)} = 27.428 \text{ days}$$

## 0.5 Problem 5

5. (15 points)
(1) A particle is subject to an attractive force $f(r)$, where $r$ is the distance between the particle and the center of the force. Find $f(r)$ if all circular orbits are to have identical areal velocities.
(2) The orbit of a particle moving in a central field is a circle passing through the origin, $r = r_0 \cos(\theta)$. Show that the force law is inverse-fifth power.

SOLUTION:

### 0.5.1 Part(1)




From the above diagram, where we have two particles of same mass $m$ in two circular orbits. The area of each sector is given by

$$A = \frac{\theta}{2} r^2$$

The time rate of each sector area is

$$\frac{dA_1}{dt} = \frac{\dot{\theta}_1}{2} r_1^2 \tag{1}$$

Similarly

$$\frac{dA_2}{dt} = \frac{\dot{\theta}_2}{2} r_2^2 \tag{2}$$

Since we have a central force, then this force attracts each mass with a force given by $f = mr\dot{\theta}^2$. Therefore $f_{r_1} = mr_1\dot{\theta}_1^2$, Similarly $f_{r_2} = mr_2\dot{\theta}_2^2$. Substituting for $\dot{\theta}$ from these expressions back into (1) and (2) gives

$$\frac{dA_1}{dt} = \sqrt{\frac{f_1}{mr_1}} \frac{r_1^2}{2} \tag{1B}$$

Similarly

$$\frac{dA_2}{dt} = \sqrt{\frac{f_1}{mr_1}\frac{r_1^2}{2}} \tag{2B}$$

We are told the areal speeds are the same, therefore equating the above gives

$$\begin{aligned}
\frac{dA_1}{dt} &= \frac{dA_2}{dt} \\
\sqrt{\frac{f_1}{mr_1}\frac{r_1^2}{2}} &= \sqrt{\frac{f_2}{mr_2}\frac{r_1^2}{2}} \\
\frac{f_1}{mr_1}\frac{r_1^4}{4} &= \frac{f_2}{mr_2}\frac{r_2^2}{4} \\
f_1 r_1^3 &= f_2 r_2^3
\end{aligned}$$

Hence

$$\frac{f_{r_1}}{f_{r_2}} = \frac{r_2^3}{r_1^3}$$

This says that, since we using the same mass, that the force $f(r)$ on a mass is inversely proportional to the cube of the mass distance from the center. To see this more clearly, let $r_1 = 1$ then

$$f_{r_2} = \frac{1}{r_2^3} f_{r_1}$$

So if we move the mass from $r_1 = 1$ to say 3 times as far to $r_2 = 3$, then the force on the same mass becomes $\frac{1}{27}$ smaller than it was.

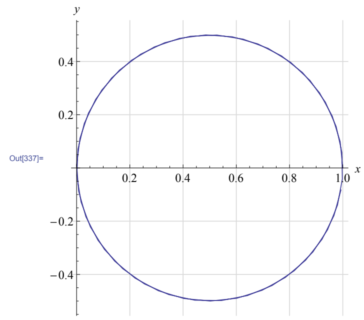
### 0.5.2 Part(2)

The orbit first is plotted as follows

```
Clear[r0, r]
r0 = 1;
r[angle_] := r0 Cos[angle]
xyData = Table[{r[a] Cos[a], r[a] Sin[a]}, {a, 0, 2 Pi, .1}];

ListLinePlot[xyData, GridLines -> Automatic,
 GridLinesStyle -> LightGray, AxesOrigin -> {0, 0},
 AxesLabel -> {x, y}, BaseStyle -> 14, PlotTheme -> "Classic",
 AspectRatio -> Automatic]
```

Which produces the following plot




Using 8.21 in textbook, page 293

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu r^2}{l^2}F(r) \tag{1}$$

Where $\mu$ is the reduces mass, $l$ is the angular momentum and $F(r)$ is the force we are solving for. Since $r=r_0\cos\theta$ then

$$\begin{aligned}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) &= \frac{d}{d\theta}\left(\frac{d}{d\theta}\frac{1}{r}\right)=\frac{d}{d\theta}\left(\frac{d}{d\theta}\frac{1}{r_0\cos\theta}\right)\\ &= \frac{d}{d\theta}\left(\frac{(-1)(-\sin\theta)}{r_0\cos^2\theta}\right)\\ &= \frac{d}{d\theta}\left(\frac{\sin\theta}{r_0\cos^2\theta}\right)\\ &= \left(\frac{\cos\theta}{r_0\cos^2\theta}+\frac{2\sin^2\theta}{r_0\cos^3\theta}\right)\\ &= \left(\frac{1}{r_0\cos\theta}+\frac{2\sin^2\theta}{r_0\cos^3\theta}\right) \qquad (2)\end{aligned}$$

But from $r=r_0\cos\theta$ we see that $\cos\theta=\frac{r}{r_0}$ and $\sin^2\theta=1-\cos^2\theta=1-\left(\frac{r}{r_0}\right)^2$, hence (2) becomes

$$\begin{aligned}
\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) &= \left(\frac{1}{r_0\left(\frac{r}{r_0}\right)} + \frac{2\left(1-\left(\frac{r}{r_0}\right)^2\right)}{r_0\left(\frac{r}{r_0}\right)^3}\right)\\
&= \left(\frac{1}{r} + \frac{2\left(1-\frac{r^2}{r_0^2}\right)}{\frac{r^3}{r_0^2}}\right)\\
&= \left(\frac{1}{r} + \frac{2-\frac{2r^2}{r_0^2}}{\frac{r^3}{r_0^2}}\right)\\
&= \left(\frac{1}{r} + \frac{2r_0^2-2r^2}{r^3}\right)\\
&= \frac{r^2+2r_0^2-2r^2}{r^3}\\
&= \frac{2r_0^2-r^2}{r^3}
\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}
\frac{2r_0^2-r^2}{r^3}+\frac{1}{r} &= -\frac{\mu r^2}{l^2}F(r)\\
\frac{2r_0^2-r^2+r^2}{r^3} &= -\frac{\mu r^2}{l^2}F(r)
\end{aligned}$$

Solving for $F(r)$

$$\begin{aligned}
F(r) &= -\frac{2l^2r_0^2}{\mu r^5}\\
&= -\left(\frac{2l^2r_0^2}{\mu}\right)\frac{1}{r^5}
\end{aligned}$$

The above shows that the force is an inverse fifth power.