HW4 Physics 311 Mechanics

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Physics department
University of Wisconsin, Madison

Instructor: Professor Stefan Westerhoff

By

Nasser M. Abbasi

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Contents

0.1 Problem 1

1. (5 points)
The damping factor λ of a spring suspension system is one-tenth the critical value. Let ω_0 be the undamped frequency. Find (i) the resonant frequency, (ii) the quality factor Q, (iii) the phase angle Φ when the system is driven at frequency $\omega = \omega_0/2$, and (iv) the steady-state amplitude at this frequency.

SOLUTION:

Note that $\lambda_{critical} = \omega_0$. We are told that $\lambda = 0.1\omega_0$ in this problem.

0.1.1 part(1)

The resonant frequency (for this case of under-damped) occurs when the steady state amplitude is maximum

$$b = \frac{\frac{f}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}}$$

This happens when the denominator is *minimum*. Taking derivative of the denominator w.r.t. ω and setting the result to zero gives

$$\begin{aligned}
\frac{d}{d\omega}\left(\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2\right) &= 0 \\
2\left(\omega_0^2 - \omega^2\right)(-2\omega) + 8\lambda^2\omega &= 0 \\
8\lambda^2\omega + 4\omega^3 - 4\omega\omega_0^2 &= 0 \\
2\lambda^2 + \omega^2 - \omega_0^2 &= 0 \\
\omega^2 &= \omega_0^2 - 2\lambda^2
\end{aligned}$$

Taking the positive root (since ω must be positive) gives

$$\omega = \sqrt{\omega_0^2 - 2\lambda^2}$$

When $\lambda = 0.1\omega_0$ the above becomes

$$\begin{aligned}
\omega &= \sqrt{\omega_0^2 - 2\left(\frac{1}{10}\omega_0\right)^2} \\
&= \sqrt{\frac{98}{100}\omega_0^2} \\
&= 0.98995\omega_0 \text{ rad/sec}
\end{aligned}$$

0.1.2 part(2)

Quality factor Q is defined as

$$\begin{aligned} Q &= \frac{\omega_d}{2\lambda} \\ &= \frac{\sqrt{\omega_0^2 - \lambda^2}}{2\lambda} \\ &= \frac{\sqrt{\omega_0^2 - (0.1\omega_0)^2}}{2\,(0.1\omega_0)} \\ &= \frac{\omega_0\sqrt{1-0.1^2}}{0.2\omega_0} \\ &= \frac{\sqrt{1-0.1^2}}{0.2} \end{aligned}$$

Therefore

$$\boxed{Q = 4.975}$$

0.1.3 Part(3)

Given

$$x''(t) + 2\lambda x' + \omega_0^2 x = \frac{f}{m} e^{i\omega t} \tag{1}$$

Assuming the particular solution is $x_p(t) = Be^{i\omega t}$ where $B = be^{i\phi}$ is the complex amplitude and b is the amplitude and ϕ is the phase of B. We want to find the phase. Plugging $x_p(t)$ into (1) and simplifying gives

$$B = \frac{\frac{f}{m}}{\omega_0^2 - \omega^2 + 2\lambda i\omega}$$

Hence

$$\begin{aligned} \phi &= 0 - \tan^{-1}\left(\frac{2\lambda\omega}{\omega_0^2 - \omega^2}\right) \\ &= \tan^{-1}\left(\frac{-2\lambda\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}$$

Since $\lambda = 0.1\omega_0$ and $\omega = \frac{\omega_0}{2}$ the above becomes

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{-2(0.1\omega_0)\,\frac{\omega_0}{2}}{\omega_0^2 - \left(\frac{\omega_0}{2}\right)^2}\right) \\ &= \tan^{-1}(-0.13333) \\ &= -0.13255 \text{ rad} \end{aligned}$$

0.1.4 Part(4)

The steady state amplitude is b from above, which is found as follows

$$b^2 = BB^*$$

Where B^* is the complex conjugate of $B = \frac{\frac{f}{m}}{\omega_0^2 - \omega^2 + 2\lambda i\omega}$. Therefore

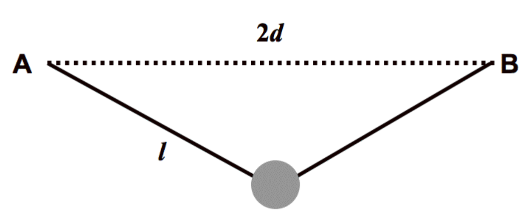
$$\begin{aligned} b &= \frac{\frac{f}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\lambda^2\omega^2}} \\ &= \frac{f}{m}\frac{1}{\sqrt{\left(\omega_0^2 - \left(\frac{\omega_0}{2}\right)^2\right)^2 + 4\left(0.1\omega_0\right)^2\left(\frac{\omega_0}{2}\right)^2}} \\ &= \frac{f}{m}\frac{1}{\sqrt{0.572\,5\omega_0^4}} \\ &= 1.3216\frac{f}{m\omega_0^2} \end{aligned}$$

But $m\omega_0^2 = k$, the stiffness, hence the above is

$$\boxed{b = 1.3216\frac{f}{k}}$$

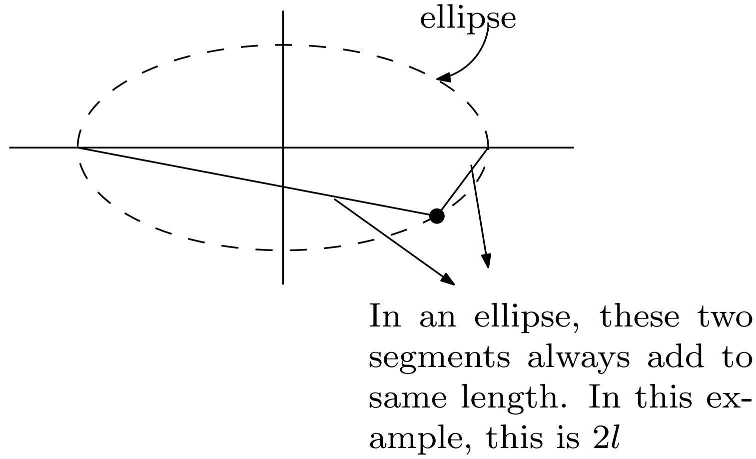
0.2 Problem 2

2. (10 points)
A string of length $2l$ is suspended at points A and B located on a horizontal line. The distance between A and B is $2d$, with $d < l$. A small, heavy bead can slide on the string without friction. Find the period of the small-amplitude oscillations of the bead in the vertical plane containing the suspension points.
Hint: The trajectory of the bead is a section of an ellipse (why?). Move the origin to the equilibrium point and use a Taylor expansion to get an approximate expression for the trajectory around the equilibrium point. Apply Lagrange.

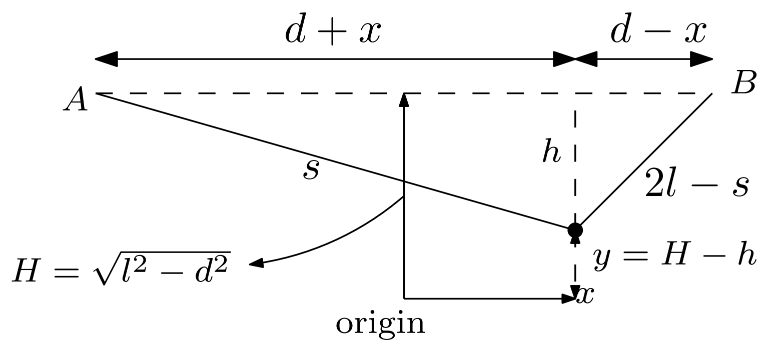


SOLUTION:

The locus the bead describes is an ellipse, since in an ellipse the total distance from any point on it to the points A, B is always the same



To obtain the potential energy, we move the bead a little from the origin and find how much the bead moved above the origin, as shown in the following diagram



$$
\begin{aligned}
s^2 &= h^2 + (d+x)^2 \\
(2l-s)^2 &= h^2 + (d-x)^2
\end{aligned}
$$

From the above, we see that, by applying pythagoras triangle theorem to the left and to the right triangles, we obtain two equations which we solve for h in order to obtain the potential energy

$$
\begin{aligned}
s^2 &= h^2 + (d+x)^2 \\
(2l-s)^2 &= h^2 + (d-x)^2
\end{aligned}
$$

Solving for h gives

$$h = \sqrt{1 - \frac{d^2}{l^2}}\sqrt{l^2 - x^2}$$

Therefore

$$
\begin{aligned}
y &= H - h \\
&= H - \sqrt{1 - \frac{d^2}{l^2}}\sqrt{l^2 - x^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
U &= mgy \\
&= mg\left(H - \sqrt{1-\frac{d^2}{l^2}}\sqrt{l^2-x^2}\right)
\end{aligned}
$$

The kinetic energy is

$$
T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right)
$$

Therefore the Lagrangian is

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mg\left(H - \sqrt{1-\frac{d^2}{l^2}}\sqrt{l^2-x^2}\right)
\end{aligned}
$$

The equation of motion in the x coordinate is now found. From

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= \frac{1}{2}mg\sqrt{1-\frac{d^2}{l^2}}\frac{(-2x)}{\sqrt{l^2-x^2}} \\
&= -mg\sqrt{1-\frac{d^2}{l^2}}\frac{x}{\sqrt{l^2-x^2}}
\end{aligned}
$$

And

$$
\frac{d}{dt}\frac{\partial L}{\partial x} = m\ddot{x}
$$

Applying Euler-Lagrangian equation gives

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial x} - \frac{\partial L}{\partial x} &= 0 \\
\ddot{x} + g\sqrt{1-\frac{d^2}{l^2}}\frac{x}{\sqrt{l^2-x^2}} &= 0
\end{aligned}
$$

For very small x, we drop the x^2 term and the above reduces to

$$
\ddot{x} + g\sqrt{1-\frac{d^2}{l^2}}\frac{x}{l} = 0
$$

Hence the undamped natural frequency is

$$
\omega_0^2 = \frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}
$$

or

$$
\omega_0 = \sqrt{\frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}}
$$

The period of small oscillation is therefore

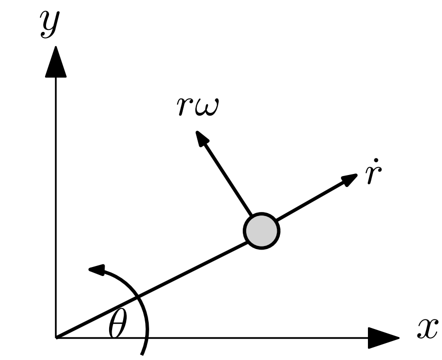
$$
\begin{aligned}
T &= \frac{2\pi}{\omega_0} \\
&= 2\pi\frac{1}{\sqrt{\frac{g}{l}\sqrt{1-\frac{d^2}{l^2}}}}
\end{aligned}
$$

0.3 Problem 3

3. (10 points)
A rod of length L rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially at the stationary end of the rod. It is given a slight push so that its initial speed along the rod is ωL. Find the time it takes the bead to reach the other end of the rod.

0.3.1 SOLUTION method one

The velocity of the particle is as shown in the following diagram



$$\begin{aligned}
\omega &= \dot{\theta} \\
V_x &= \dot{r}\cos\theta - r\omega\sin\theta \\
V_y &= \dot{r}\sin\theta + r\omega\cos\theta
\end{aligned}$$

velocity diagram

There is no potential energy, and the Lagrangian only comes from kinetic energy.

$$\begin{aligned}
v^2 &= V_x^2 + V_y^2 \\
&= (\dot{r}\cos\theta - r\omega\sin\theta)^2 + (\dot{r}\sin\theta + r\omega\cos\theta)^2
\end{aligned}$$

Exapnding and simplifying gives

$$v^2 = \dot{r}^2 + r^2\omega^2$$

Hence

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right)$$

And the equation of motion in the radial r direction is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= 0 \\
\frac{d}{dt}m\dot{r} - mr\omega^2 &= 0
\end{aligned}$$

Hence the equation of motion is

$$\boxed{\ddot{r} - r\omega^2 = 0} \tag{1}$$

The roots of the characteristic equation are $\pm\omega$, hence the solution is

$$r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

At $t = 0$, $r(0) = 0$ and $\dot{r}(t) = L\omega$. Using these we can find c_1, c_2.

$$0 = c_1 + c_2 \tag{2}$$

But $\dot{r}(t) = \omega c_1 e^{\omega t} - \omega c_2 e^{-\omega t}$ and at $t = 0$ this becomes

$$L\omega = \omega c_1 - \omega c_2 \tag{3}$$

From (2,3) we solve for c_1, c_2. From (2), $c_1 = -c_2$ and (3) becomes

$$\begin{aligned} L\omega &= -\omega c_2 - \omega c_2 \\ c_2 &= \frac{L\omega}{-2\omega} = \frac{-1}{2} L \end{aligned}$$

Hence $c_1 = \frac{1}{2} L$ and the solution is

$$\begin{aligned} r(t) &= c_1 e^{\omega t} + c_2 e^{-\omega t} \\ &= \frac{1}{2} L e^{\omega t} - \frac{1}{2} L e^{-\omega t} \\ &= L \left(\frac{e^{\omega t} - e^{-\omega t}}{2} \right) \end{aligned}$$

Or

$$\boxed{r(t) = L(\sinh \omega t)}$$

To find the time it takes to reach end of rod, we solve for t_p from

$$\begin{aligned} L &= L(\sinh \omega t_p) \\ 1 &= \sinh \omega t_p \end{aligned}$$

Hence

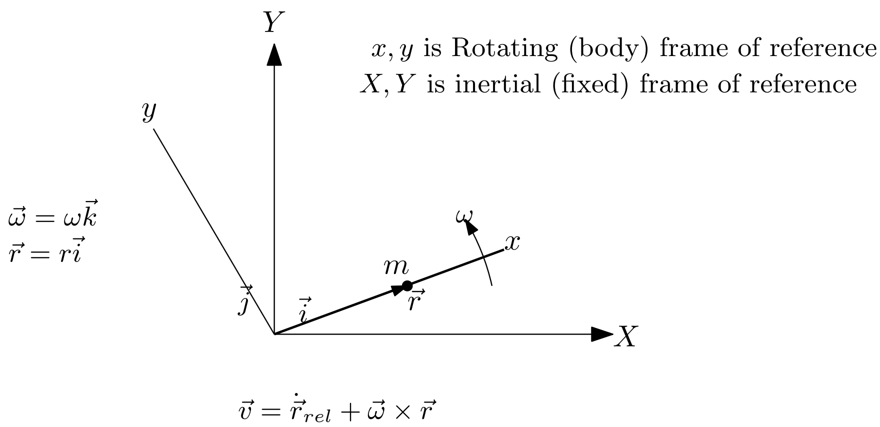
$$\begin{aligned} \omega t_p &= \sinh^{-1}(1) \\ &= 0.88137 \end{aligned}$$

Therefore

$$\boxed{t_p = \frac{0.88137}{\omega} \text{ sec}}$$

0.3.2 another solution

Let the local coordinate frame rotate with the bar, where the bar is oriented along the x axis of the local body coordinate frame as shown below.



The position vector of the particle is $\mathbf{r} = \mathbf{i}r$ where $\mathbf{i}$ is unit vector along the x axis. Taking time derivative, and using the rotating vector time derivative rule which says that $\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_{relative} + \boldsymbol{\omega} \times \mathbf{A}$ where $\boldsymbol{\omega}$ is the angular velocity of the rotating frame then

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \mathbf{r} \tag{1}$$

To find the acceleration of the particle, we take time derivative one more time

$$\frac{d}{dt}\dot{\mathbf{r}} = \frac{d}{dt}(\dot{\mathbf{r}}_{rel}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$$

But $\frac{d}{dt}(\dot{\mathbf{r}}_{rel}) = \ddot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel}$ by applying the rule of time derivative of rotating vector again. Therefore the above equation becomes

$$\frac{d}{dt}\dot{\mathbf{r}} = \ddot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}$$

Replacing $\dot{\mathbf{r}}$ in the above from its value in (1) gives

$$\begin{aligned}
\ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\dot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \mathbf{r}) \\
&= \ddot{\mathbf{r}}_{rel} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel} + \omega \times (\omega \times \mathbf{r}) \\
&= \ddot{\mathbf{r}}_{rel} + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})
\end{aligned}$$

But $\boldsymbol{\omega}$ is constant (bar rotate with constant angular speed), hence the term $\dot{\boldsymbol{\omega}}$ above is zero, and the above reduces to

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_{rel} + 2(\boldsymbol{\omega} \times \dot{\mathbf{r}}_{rel}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{2}$$

The above is the acceleration of the particle as seen in the inertial frame. Now we calculate this acceleration by preforming the vector operations above, noting that $\mathbf{r} = \mathbf{i}r, \boldsymbol{\omega} = \mathbf{k}\omega$, hence (2) becomes

$$
\begin{aligned}
\ddot{\mathbf{r}} &= \mathbf{i}\ddot{r}_{rel} + 2(\mathbf{k}\omega \times \mathbf{i}\dot{r}_{rel}) + \mathbf{k}\omega \times (\mathbf{k}\omega \times \mathbf{i}r) \\
&= \mathbf{i}\ddot{r}_{rel} + 2(\mathbf{j}\omega\dot{r}_{rel}) + \mathbf{k}\omega \times (\mathbf{j}\omega r) \\
&= \mathbf{i}\ddot{r}_{rel} + 2(\mathbf{j}\omega\dot{r}_{rel}) - \mathbf{i}\omega^2 r \\
&= \mathbf{i}\left(\ddot{r}_{rel} - \omega^2 r\right) + \mathbf{j}(2\omega\dot{r}_{rel})
\end{aligned}
$$

The particle has an acceleration along x axis and an acceleration along y axis. We are interested in the acceleration along x since this is where the rod is oriented along. The scalar version of the acceleration in the x direction is

$$a_x = \ddot{r}_{rel} - \omega^2 r$$

Using $F_x = ma_x$ and since $F_x = 0$ (there is no force on the particle) then the equation of motion along the bar (x axis) is

$$\ddot{r}_{rel} - \omega^2 r = 0$$

The roots of the characteristic equation is $\pm\omega$, hence the solution is

$$r(t) = c_1 e^{\omega t} + c_2 e^{-\omega t}$$

At $t = 0$, $r(0) = 0$ and $\dot{r}(t) = L\omega$. Using these we can find c_1, c_2.

$$0 = c_1 + c_2 \tag{3}$$

But $\dot{r}(t) = \omega c_1 e^{\omega t} - \omega c_2 e^{-\omega t}$ and at $t = 0$ this becomes

$$L\omega = \omega c_1 - \omega c_2 \tag{4}$$

From (3,4) we solve for c_1, c_2. From (3), $c_1 = -c_2$ and (4) becomes

$$
\begin{aligned}
L\omega &= -\omega c_2 - \omega c_2 \\
c_2 &= \frac{L\omega}{-2\omega} = \frac{-1}{2} L
\end{aligned}
$$

Hence $c_1 = \frac{1}{2} L$ and the solution is

$$
\begin{aligned}
r(t) &= c_1 e^{\omega t} + c_2 e^{-\omega t} \\
&= \frac{1}{2} L e^{\omega t} - \frac{1}{2} L e^{-\omega t} \\
&= L\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) \\
&= L(\sinh \omega t)
\end{aligned}
$$

To find the time it takes to reach end of rod, we solve for t_p from

$$
\begin{aligned}
L &= L(\sinh \omega t_p) \\
1 &= \sinh \omega t_p
\end{aligned}
$$

Hence

$$\begin{aligned}\omega t_p &= \sinh^{-1}(1)\\ &= 0.88137\end{aligned}$$

Therefore

$$\boxed{t_p = \frac{0.88137}{\omega}\ \text{sec}}$$

0.4 Problem 4

4. (10 points)
Consider a harmonic oscillator with $\omega_0 = 0.5\,\mathrm{s}^{-1}$. Let $x_0 = 1.0\,\mathrm{m}$ be the initial amplitude at $t = 0$ and assume that the oscillator is released with zero initial velocity. Use a computer to plot the phase-space plot ($\dot{x}$ versus x) for the following damping coefficients λ.
(1) $\lambda = 0.05\,\mathrm{s}^{-1}$ (weak damping)
(2) $\lambda = 0.25\,\mathrm{s}^{-1}$ (strong damping)
(3) $\lambda = \omega_0$ (critical damping).

SOLUTION:

Starting with the equation of motion for damped oscillator

$$x'' + 2\lambda x' + \omega_0^2 x = 0$$

The solution for cases 1,2 (both are underdamped) is

$$x = e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1}$$

Where $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$. While the solution for case (3), the critical damped case is

$$x = (A + tB)\,e^{-\lambda t} \tag{2}$$

For (1) above, at $t = 0$ we obtain

$$1 = A$$

Hence (1) becomes $x = e^{-\lambda t}(\cos\omega_d t + B\sin\omega_d t)$, and taking derivative gives

$$\dot{x} = -\lambda e^{-\lambda t}(\cos\omega_d t + B\sin\omega_d t) + e^{-\lambda t}(-\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t)$$

At $t = 0$ we have

$$\begin{aligned}0 &= -\lambda + B\omega_d\\ B &= \frac{\lambda}{\omega_d}\end{aligned}$$

Hence the complete solution for (1) is

$$x = e^{-\lambda t}\left(\cos\omega_d t + \frac{\lambda}{\omega_d}\sin\omega_d t\right) \tag{3}$$
$$\dot{x} = -\lambda x + e^{-\lambda t}(-\omega_d \sin\omega_d t + \lambda\cos\omega_d t) \tag{4}$$

Now we find the solution for (2), the critical damped case. At $t = 0$

$$1 = A$$

Hence (2) becomes $x = (1 + tB)\, e^{-\lambda t}$, and taking derivative gives

$$\dot{x} = Be^{-\lambda t} - \lambda(1 + tB)\, e^{-\lambda t}$$

At $t = 0$

$$0 = B - \lambda$$
$$B = \lambda$$

Hence the solution to (2) becomes

$$x = (1 + \lambda t)\, e^{-\lambda t} \tag{5}$$
$$\dot{x} = \lambda e^{-\lambda t} - \lambda(1 + \lambda t)\, e^{-\lambda t} \tag{6}$$

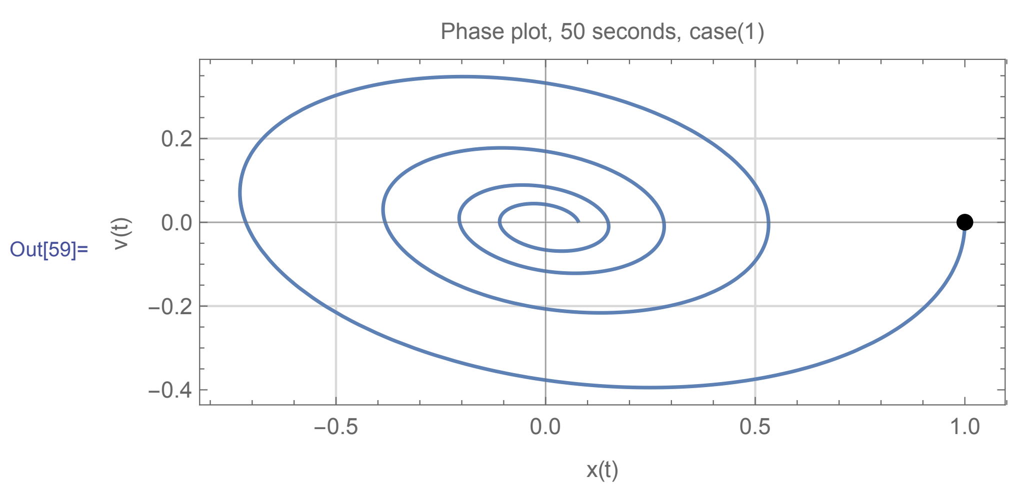
Now that the solutions are found, we plot the phase space using the computer, using parametric plot command

0.4.1 case (1)

For $\lambda = 0.05$, and $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{0.5^2 - 0.05^2} = 0.4975$, then equations (3,4) become

$$x = e^{-0.05t}(\cos 0.4975t + 0.1005\sin 0.4975t) \tag{3A}$$
$$\dot{x} = -0.05x + e^{-0.05t}(-0.4975\sin 0.4975t + 0.05\cos 0.4975t) \tag{4A}$$

Here is the plot generated, showing starting point $(1, 0)$ with the code used



```
am = 0.05;
wn = 0.5;
wd = Sqrt[wn^2 - lam^2];
x = Exp[-lam t] (Cos[wd t] + lam/wd Sin[wd t]);
y = -lam x + Exp[-lam t] (-wd Sin[wd t] + lam Cos[lam t]);
ParametricPlot[{x, y}, {t, 0, 50}, Frame -> True,
 GridLines -> Automatic, GridLinesStyle -> LightGray,
 FrameLabel -> {{"v(t)", None}, {"x(t)",
    "Phase plot, 50 seconds, case(1)"}}, Epilog -> Disk[{1, 0}, .02],
 ImageSize -> 400]
```

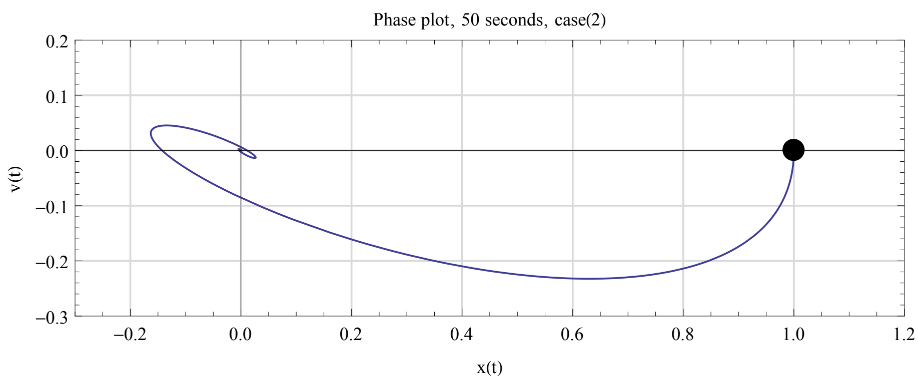
0.4.2 case (2)

For $\lambda = 0.25$, and $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{0.5^2 - 0.25^2} = 0.433$, equations (3,4) become

$$x = e^{-0.25t}(\cos 0.433t + 0.5774 \sin 0.433t) \tag{3A}$$
$$\dot{x} = -0.05x + e^{-0.25t}(-0.433 \sin 0.433t + 0.05 \cos 0.433t) \tag{4A}$$

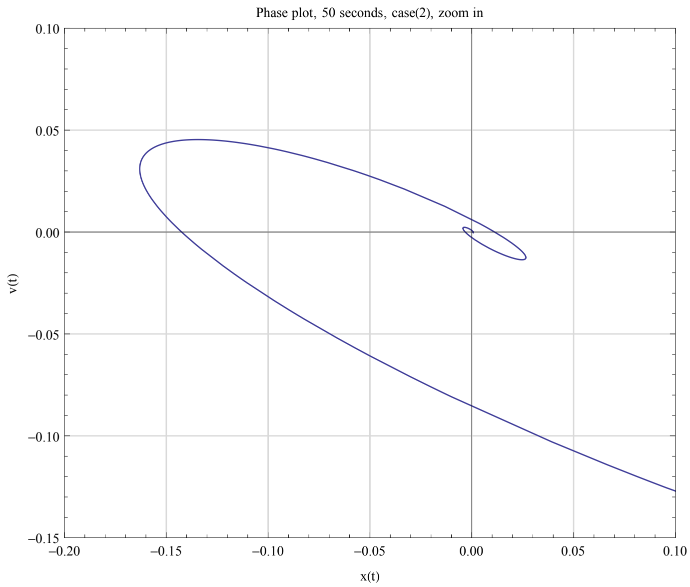
Here is the plot generated where the starting point was $(1, 0)$

Out[150]=



This below is a zoomed in version of the above close to the origin

Out[151]=



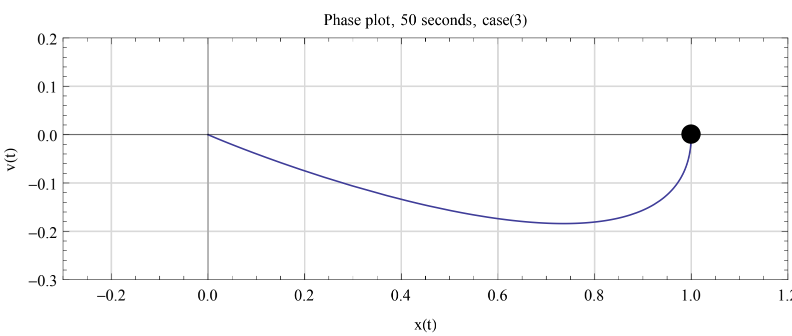
0.4.3 case(3)

For this case, equations (5,6) are used. For $\lambda = 0.5$, equations (5,6) become

$$x = (1 + 0.5t)\, e^{-0.5t} \tag{5A}$$
$$\dot{x} = 0.5e^{-0.5t} - 0.5(1 + 0.5t)\, e^{-0.5t} \tag{6A}$$

Here is the plot generated, showing starting point $(1, 0)$ with the code used

Out[155]=




```
lam = 0.5;
x = (1 + lam*t) Exp[-lam t];
y = lam*Exp[-lam t] - lam*(1 + lam t) Exp[- lam t]
ParametricPlot[{x, y}, {t, 0, 30}, Frame -> True,
 GridLines -> Automatic, GridLinesStyle -> LightGray,
 FrameLabel -> {{"v(t)", None}, {"x(t)",
    "Phase plot, 50 seconds, case(3)"}}, Epilog -> Disk[{1, 0}, .02],
 ImageSize -> 500, PlotRange -> {{-.3, 1.2}, {-.3, .2}},
 PlotTheme -> "Classic"]
```

0.5 Problem 5

5. (15 points)
A damped harmonic oscillator has a period of free oscillation (with no damping) of $T_0 = 1.0\,\mathrm{s}$. The oscillator is initially displaced by an amount $x_0 = 0.1\,\mathrm{m}$ and released with zero initial velocity.
(1) Consider the case that the oscillator is critically damped. Determine the displacement x as a function of time and use a computer program to plot $x(t)$ for $0 \le t \le 2\,\mathrm{s}$.
(2) Now consider the case that the system is overdamped. Determine the displacement as a function of time and use a computer program to plot $x(t)$ for damping coefficients (i) $\lambda = 2.2\,\pi\mathrm{s}^{-1}$, (ii) $\lambda = 4\,\pi\mathrm{s}^{-1}$, and (iii) $\lambda = 10\,\pi\mathrm{s}^{-1}$ for $0 \le t \le 2\,\mathrm{s}$. Compare to the critically damped case.
(3) Now consider the case that the system is underdamped. Determine the displacement as a function of time and use a computer program to plot $x(t)$ for damping coefficients (i) $\lambda = 5.0\,\mathrm{s}^{-1}$, (ii) $\lambda = 1.0\,\mathrm{s}^{-1}$, and (iii) $\lambda = 0.1\,\mathrm{s}^{-1}$ for $0 \le t \le 2\,\mathrm{s}$. Compare to the critically damped case.

SOLUTION:

Since $\omega_0 = \frac{2\pi}{T_0}$, then $\omega_0 = \frac{2\pi}{1} = 2\pi$.

0.5.1 Part (1)

For critical damping $\lambda = \omega_0$ and the solution is

$$x(t) = (A + Bt)\,e^{-\lambda t} \tag{1}$$
$$\dot{x}(t) = Be^{-\lambda t} - \lambda(A + Bt)\,e^{-\lambda t} \tag{2}$$

Initial conditions are now used to find A, B. At $t = 0$, $x(0) = x_0 = 0.1$. From (1) we obtain

$$x_0 = A$$

And since $\dot{x}(0) = 0$, then from (2)

$$\begin{aligned} 0 &= B - \lambda A \\ B &= \lambda A \\ &= \lambda x_0 \end{aligned}$$

Putting values found for A, B, back into (1) gives

$$x(t) = (x_0 + \lambda x_0 t)\,e^{-\lambda t}$$

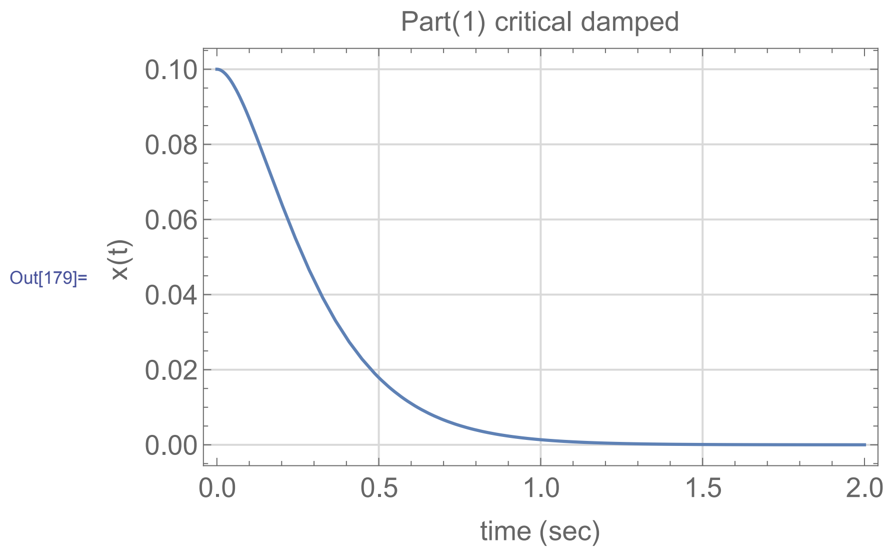
Since this is critical damping, then $\lambda = \omega_0 = 2\pi$, hence

$$x(t) = (x_0 + 2\pi x_0 t)\,e^{-2\pi t}$$

Finally, since $x_0 = 0.1$ meter, then

$$x(t) = \left(\frac{1}{10} + \frac{2\pi}{10}t\right) e^{-2\pi t}$$

A plot of the above for $0 \leq t \leq 2s$ is given below



0.5.2 Part(2)

For overdamped, $\lambda > \omega_0$ the two roots of the characteristic polynomial are real, hence no oscillation occur. The solution is given by

$$x(t) = Ae^{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} + Be^{\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)t} \tag{1}$$

A, B are found from initial conditions. When $t = 0$ the above becomes

$$x_0 = A + B \tag{2}$$

Taking derivative of (1) gives

$$\dot{x}(t) = A\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) e^{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} + B\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right) e^{\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)t}$$

At $t = 0$ the above becomes

$$0 = \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) A + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right) B \tag{3}$$

We have two equations (2,3) which we solve for A, B. From (2), $A = x_0 - B$, and (3) becomes

$$
\begin{aligned}
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)(x_0 - B) + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right) B \\
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) x_0 - B\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) + \left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right) B \\
0 &= \left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) x_0 - 2B\sqrt{\lambda^2 - \omega_0^2} \\
B &= \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) x_0}{2\sqrt{\lambda^2 - \omega_0^2}} \qquad (4)
\end{aligned}
$$

Using B found in (4) then (3) now gives A as

$$
\begin{aligned}
A &= x_0 - B \\
&= x_0 - \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right) x_0}{2\sqrt{\lambda^2 - \omega_0^2}} \\
&= x_0 \left(1 - \frac{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)}{2\sqrt{\lambda^2 - \omega_0^2}}\right) \\
&= x_0 \left(\frac{\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right)
\end{aligned}
$$

Hence the complete solution from (1) becomes

$$
x(t) = x_0 \left(\frac{\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right) e^{\left(-\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} + x_0 \left(\frac{-\lambda + \sqrt{\lambda^2 - \omega_0^2}}{2\sqrt{\lambda^2 - \omega_0^2}}\right) e^{\left(-\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)t} \qquad (5)
$$

The above is now used for each case below to plot the solution..

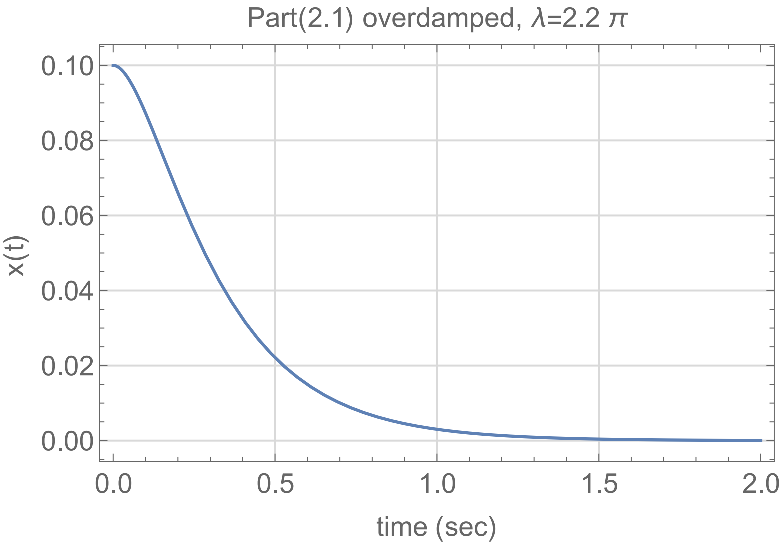
case (i)

$\lambda = 2.2\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

$$
\begin{aligned}
x(t) &= 0.1 \left(\frac{2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}}{2\sqrt{(2.2\pi)^2 - (2\pi)^2}}\right) e^{\left(-2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}\right)t} + 0.1 \left(\frac{-2.2\pi + \sqrt{(2.2\pi)^2 - (2\pi)^2}}{2\sqrt{(2.2\pi)^2 - (2\pi)^2}}\right) e^{\left(-2.2\pi\right.} \\
&= 0.17e^{-4.032\,2t} - 0.07\,e^{-9.791t}
\end{aligned}
$$

A plot of the above for $0 \leq t \leq 2s$ is given below

Out[297]=



case (ii)

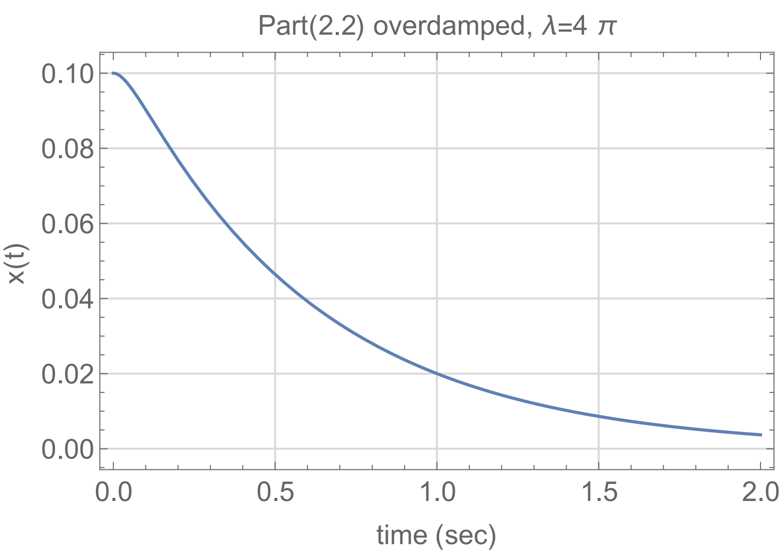
$\lambda = 4\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

$$x(t) = 0.1\left(\frac{4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}}{2\sqrt{(4\pi)^2 - (2\pi)^2}}\right) e^{\left(-4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}\right)t} + 0.1\left(\frac{-4\pi + \sqrt{(4\pi)^2 - (2\pi)^2}}{2\sqrt{(4\pi)^2 - (2\pi)^2}}\right) e^{\left(-4\pi - \sqrt{(4\pi)^2 - (2\pi)\right.}$$

$$= 0.1077e^{-1.6836t} - 0.00774e^{-23.449t}$$

A plot of the above for $0 \leq t \leq 2s$ is given below

Out[310]=



case (iii)

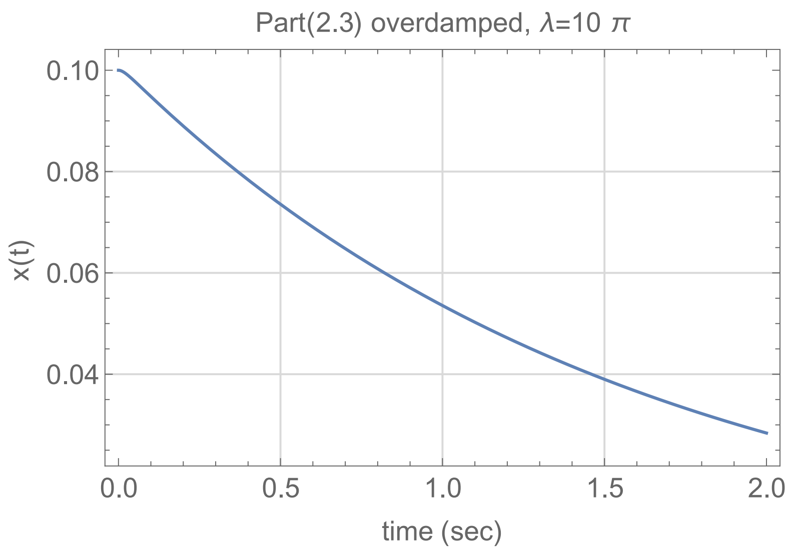
$\lambda = 10\pi, \omega_0 = 2\pi, x_0 = 0.1$, hence (5) becomes

$$x(t) = 0.1\left(\frac{10\pi + \sqrt{(10\pi)^2 - (2\pi)^2}}{2\sqrt{(10\pi)^2 - (2\pi)^2}}\right) e^{\left(-10\pi + \sqrt{(10\pi)^2 - (2\pi)^2}\right)t} + 0.1\left(\frac{-10\pi + \sqrt{(10\pi)^2 - (2\pi)^2}}{2\sqrt{(10\pi)^2 - (2\pi)^2}}\right) e^{\left(-10\pi - \sqrt{(1\right.}}$$

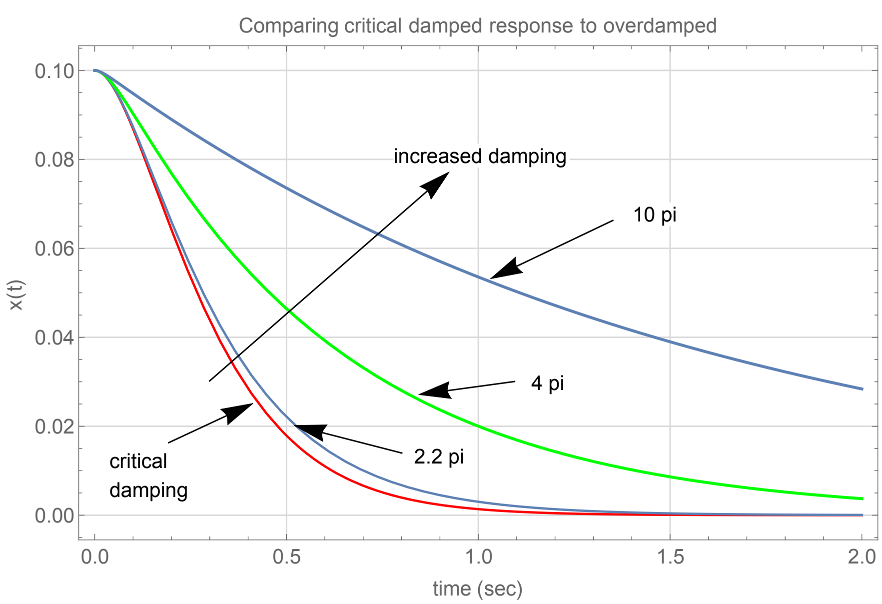
$$= 0.101\, e^{-0.634\,73t} - 0.001034 e^{-62.197t}$$

A plot of the above for $0 \leq t \leq 2s$ is given below

Out[314]=



To compare to the critical damped case, the above three plots are plotted on the same figure against the critical damped case in order to get a better picture and be able to compare the results



From the above we see that critical damping has the fastest decay of the response $x(t)$. As the damping increases, it takes longer for the response to decay.

0.5.3 Part(3)

For the underdamped case, the solution is given by

$$x(t) = e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t) \tag{1}$$

Where $\omega_d = \sqrt{\omega_0^2 - \lambda^2}$ and A, B are constant of integration that can be found from initial conditions. And

$$\dot{x}(t) = -\lambda e^{-\lambda t}(A\cos\omega_d t + B\sin\omega_d t) + e^{-\lambda t}(-A\omega_d\sin\omega_d t + B\omega_d\cos\omega_d t) \tag{2}$$

Applying initial conditions $x(0) = x_0$ then (1) becomes

$$x_0 = A$$

Applying initial conditions $\dot{x}(0) = 0$ then (2) becomes

$$\begin{aligned} 0 &= -\lambda x_0 + B\omega_d \\ B &= \frac{\lambda x_0}{\omega_d} \end{aligned}$$

Replacing A, B back into the solution (1) gives the solution

$$x(t) = e^{-\lambda t}\left(x_0\cos\omega_d t + \frac{\lambda x_0}{\omega_d}\sin\omega_d t\right) \tag{3}$$

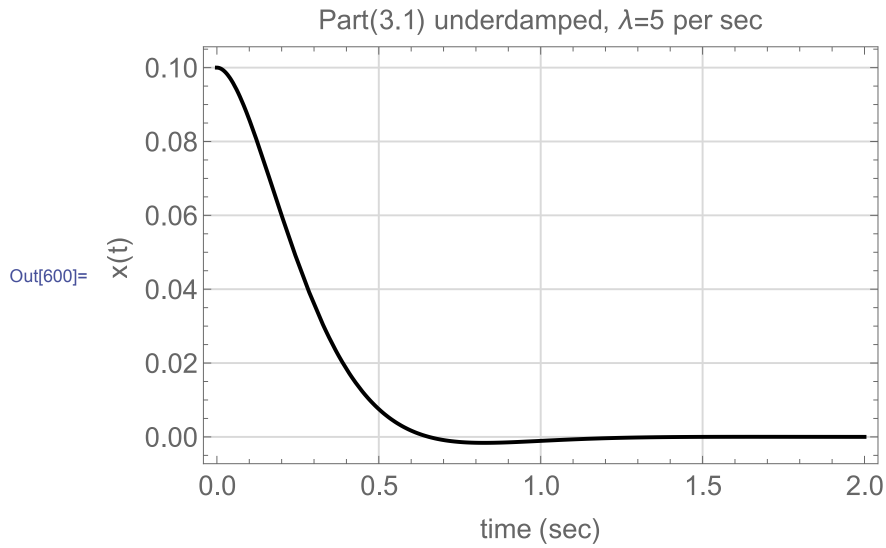
We now use the above solution for the rest of the problem

case(i)

$\lambda = 5s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 5^2} = 3.8051$ and (3) becomes

$$\begin{aligned} x(t) &= e^{-5t}\left(0.1\cos(3.8051t) + \frac{(5)(0.1)}{3.8051}\sin(3.8051t)\right) \\ &= e^{-5t}(0.1\cos(3.8051t) + 0.1314\sin(3.8051t)) \end{aligned}$$

A plot of the above solution $x(t)$ for $0 \le t \le 2s$ is given below

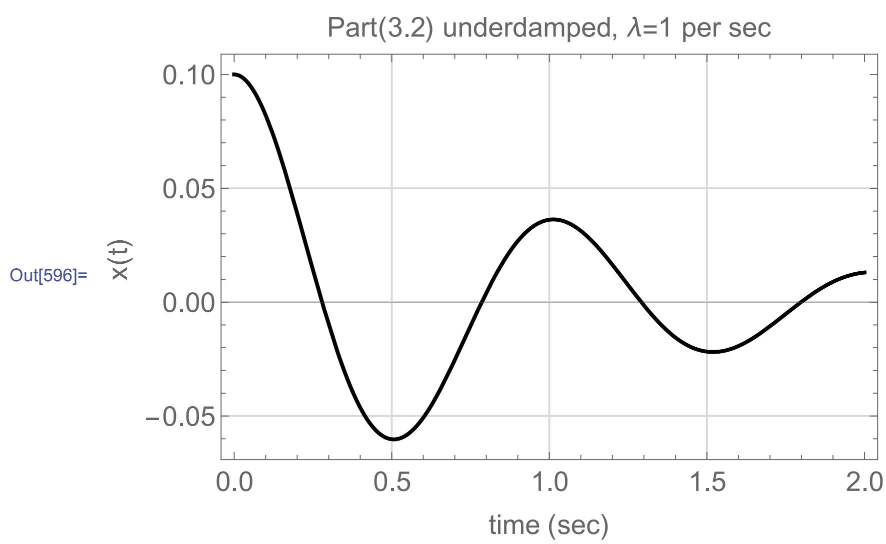


case(ii)

$\lambda = 1s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 1^2} = 6.2031$ and (3) becomes

$$\begin{aligned} x(t) &= e^{-t}\left(0.1\cos(6.2031t) + \frac{(1)(0.1)}{6.2031}\sin(6.2031t)\right) \\ &= e^{-t}(0.1\cos(6.2031t) + 0.016\sin(6.2031t)) \end{aligned}$$

A plot of the above solution $x(t)$ for $0 \leq t \leq 2s$ is given below



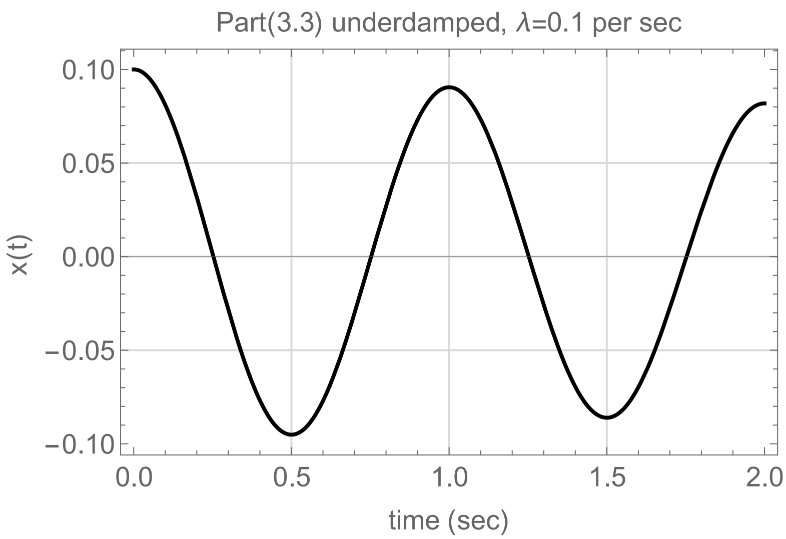
case(iii)

$\lambda = 0.1s^{-1}, \omega_0 = 2\pi, x_0 = 0.1$, hence $\omega_d = \sqrt{\omega_0^2 - \lambda^2} = \sqrt{(2\pi)^2 - 0.1^2} = 6.2824$ and (3) becomes

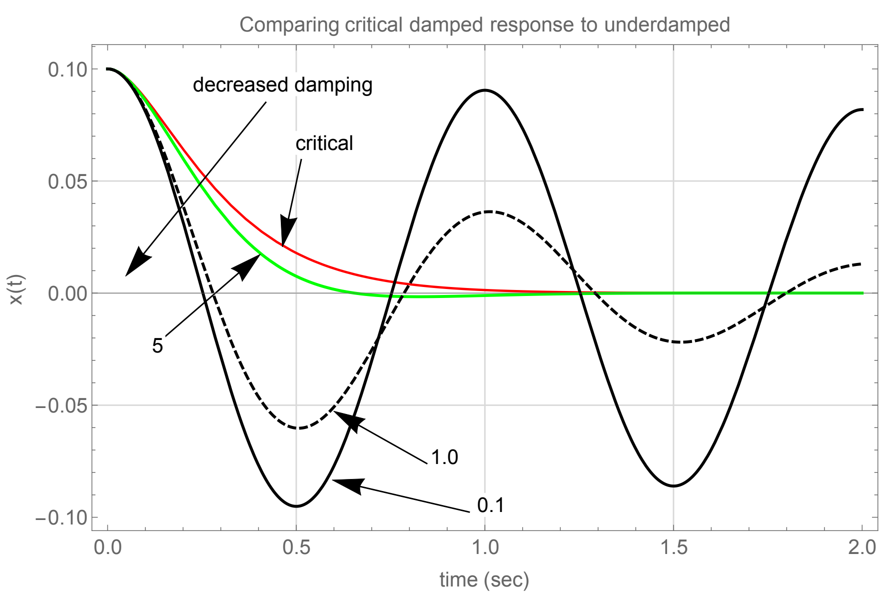
$$\begin{aligned} x(t) &= e^{-0.1t}\left(0.1\cos(6.2824t) + \frac{(0.1)(0.1)}{6.2824}\sin(6.2824t)\right) \\ &= e^{-0.1t}(0.1\cos(6.2824t) + 0.001592\sin(6.2824t)) \end{aligned}$$

A plot of the above solution $x(t)$ for $0 \le t \le 2s$ is given below

Out[591]=



To compare to the critical damped case, the above 3 plots are now plotted on the same figure against the critical damped case in order to get a better picture and be able to compare the results



As the damping becomes smaller, more oscillation occur. The case for $\lambda = 5s^{-1}$ had the smallest oscillation.