HW 1
Physics 321 (Mechanics)
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University of Wisconsin, Madison

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Contents

1 Problem 1

1. (5 points)
   A particle is projected with an initial velocity v_0 up a slope that makes an angle α with the horizontal. Assume frictionless motion and calculate the time required for the particle to return to its starting point. Find the time for $v_0 = 2.4\,\text{m/s}$ and $\alpha = 26°$.

SOLUTION

The vertical component of motion is only considered since that is the component that changes due to the action of gravity.

The equation of motion in the vertical y direction is given by $F = ma$. Hence

$$\begin{aligned} my'' &= -mg \\ y'' &= -g \end{aligned}$$

Integrating once gives

$$y' - y'(0) = -gt$$

Where $y'(0) = v_0 \sin(\alpha)$. The time for the particle to reach a final velocity of zero in the vertical direction is now find by solving the above for t

$$y'_{\text{final}} = y'(0) - gt$$

Where $y'_{\text{final}} = 0$. Solving the above for the time t gives

$$\begin{aligned} 0 &= v_0 \sin\alpha - gt \\ t &= \frac{v_0 \sin\alpha}{g} \end{aligned}$$

Hence the total time to reach back to its starting point is twice the above time, which is

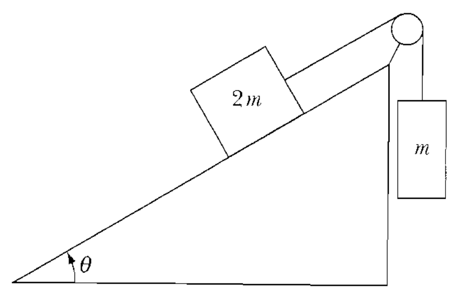
$$\boxed{\text{total time} = 2\left(\frac{v_0 \sin\alpha}{g}\right)}$$

For $\alpha = 20$ degree and $v_0 = 2.4\frac{m}{s}$ and $g = 9.81\frac{m}{s^2}$, the total time is found from

$$\begin{aligned} \text{total time} &= 2\left(\frac{2.4\sin 20^o}{9.81}\right) \\ &= \boxed{0.167\,\text{second}} \end{aligned}$$

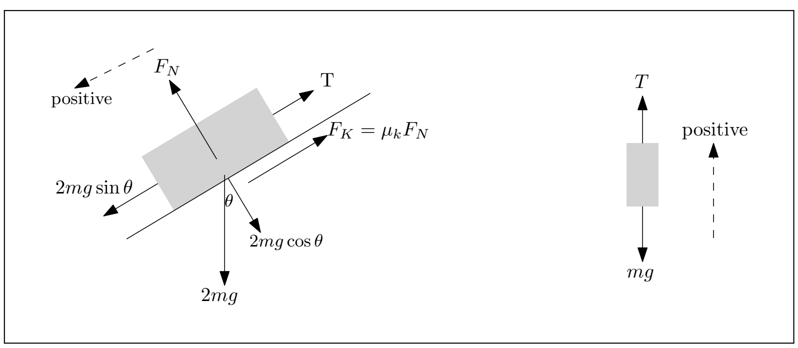
2 Problem 2

2. (10 points)
Two blocks of unequal mass are connected by a string over an ideal pulley (whose mass is negligible and that rotates with negligible friction). If the coefficient of kinetic friction is μ_k, what angle θ allows the mass to move at a constant speed?



SOLUTION

The free body diagram is shown below for each mass.



The acceleration of each body is the same. Let this acceleration be a. From the above free body diagram of the $2m$ body the equation of motion is now derived (using positive direction as show)

$$\begin{aligned} \sum F &= 2ma \\ 2mg\sin\theta - \mu_k F_N - T &= 2ma \end{aligned}$$

Using $F_N = 2mg\cos\theta$ the above becomes

$$2mg\sin\theta - \mu_k 2mg\cos\theta - T = 2ma \tag{1}$$

The tension T in the string is found from the free body diagram of the smaller hanging mass since the tension T is same. From the free body diagram of the small mass the equation of motion is

$$\sum F = ma$$
$$-mg + T = ma$$

Hence

$$T = m\left(a + g\right)$$

Substituting T in (1) gives

$$2mg\sin\theta - \mu_k 2mg\cos\theta - m\left(a + g\right) = (2m)\,a$$
$$2g\sin\theta - \mu_k 2g\cos\theta - g = 3a$$

Therefore

$$a = \frac{2}{3}\left(g\sin\theta - \mu_k g\cos\theta - \frac{1}{2}g\right)$$

For constant speed, $a = 0$ at some angle θ_c. The above reduces to

$$\frac{2}{3}\left(g\sin\theta_c - \mu_k g\cos\theta_c - \frac{1}{2}g\right) = 0$$
$$\sin\theta_c - \mu_k\cos\theta_c - \frac{1}{2} = 0$$
$$\sin\theta_c - \mu_k\cos\theta_c = \frac{1}{2} \tag{2}$$

To solve this, the following identity is used

$$R\sin\left(\theta_c + \alpha\right) = R\left(\sin\theta_c\cos\alpha + \cos\theta_c\sin\alpha\right) \tag{3}$$

Comparing the RHS of (3) with the LHS of (2) gives

$$R\cos\alpha = 1 \tag{4}$$
$$R\sin\alpha = -\mu_k \tag{5}$$

Dividing (5) by (4) gives $\tan\alpha = -\mu_k$ or

$$\alpha = \tan^{-1}\left(-\mu_k\right) = -\tan^{-1}\left(\mu_k\right)$$

Squaring (4) and (5) and adding gives

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1 + \mu_k^2$$
$$R = \sqrt{1 + \mu_k^2}$$

Therefore the equation $R\sin(\theta+\alpha)=\frac{1}{2}$ becomes

$$\sqrt{1+\mu_k^2}\sin(\theta_c+\alpha)=\frac{1}{2}$$
$$\sin\left(\theta_c-\tan^{-1}\left(\mu_k\right)\right)=\frac{1}{2\sqrt{1+\mu_k^2}}$$
$$\theta_c-\tan^{-1}\left(\mu_k\right)=\sin^{-1}\left(\frac{1}{2\sqrt{1+\mu_k^2}}\right)$$

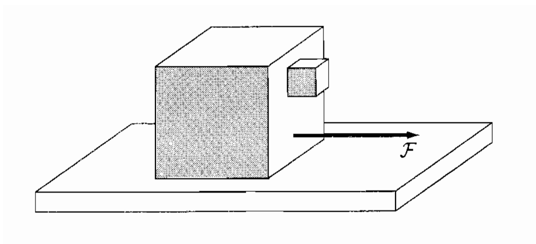
Therefore

$$\theta_c=\sin^{-1}\left(\frac{1}{2\sqrt{1+\mu_k^2}}\right)+\tan^{-1}\left(\mu_k\right)$$

For the case of no friction, where $\mu_k=0$ the above gives $\theta_c=\sin^{-1}\left(\frac{1}{2}\right)=30^0$. As μ_k increases, the angle θ_c will increase. (in the limit, as $\mu_k\rightarrow\infty,\theta_c\rightarrow 90^0$). This is a plot showing how the angle changes as μ_k increases.

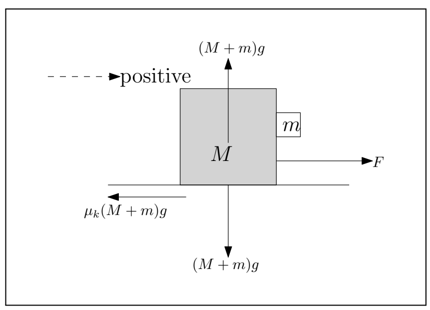
3 Problem 3

3. (10 points)
A small box of mass m is in contact with a large box of mass M as shown in the picture. A force $\vec{F}$ pushes on the large box. Because of friction, the small box will not fall if $\vec{F}$ is large enough. How large does $\vec{F}$ need to be? Take into account *all* frictional forces and assume that the coefficients of friction at all surfaces are μ_s for static and μ_k for kinetic friction.



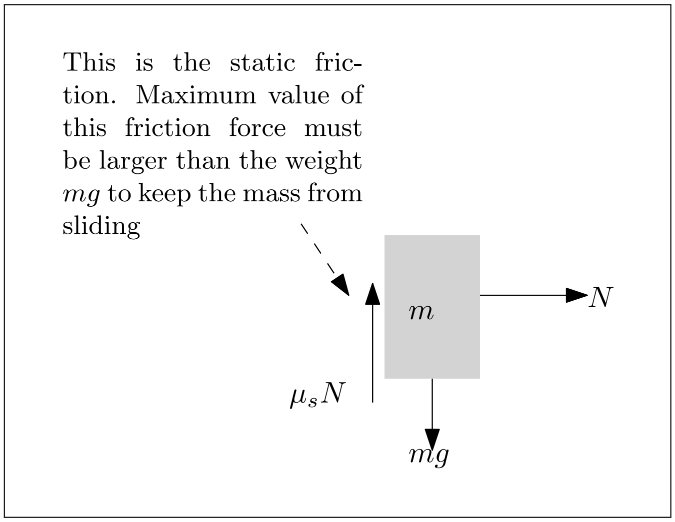
SOLUTION

Looking at the case where the small mass m is not moving (not sliding down the side), and considering both $M + m$ as one body. Let the horizontal acceleration of both bodies be a



$$\begin{aligned}\sum F_x &= (M+m)a \\ F - \mu_k(M+m)g &= (M+m)a \\ a &= \frac{F - \mu_k(M+m)g}{(M+m)}\end{aligned} \tag{1}$$

The small mass m is now considered. The static friction force between m and M has to be larger than the weight mg so that m does not move and fall. This implies $f_{s_{\max}} = \mu_s N$ must be larger than the weight mg



This implies the following condition is required

$$\mu_s N \geq mg \tag{2}$$

Where N is the normal force on m. But

$$ma = N$$

From (1) we find

$$N = m\left(\frac{F - \mu_k(M+m)g}{(M+m)}\right)$$

Therfore (2) becomes

$$\mu_s m\left(\frac{F - \mu_k(M+m)g}{(M+m)}\right) \geq mg$$

Hence

$$\begin{aligned} F - \mu_k(M+m)g &\geq \frac{g}{\mu_s}(M+m) \\ F &\geq \frac{g}{\mu_s}(M+m) + \mu_k(M+m)g \\ &\geq (M+m)g\left(\frac{1}{\mu_s} + \mu_k\right) \end{aligned}$$

Hence

$$\boxed{F \geq (M+m)g\left(\frac{1+\mu_s\mu_k}{\mu_s}\right)}$$

4 Problem 4

4. (5 points)
   Show that the terminal velocity of a falling object is given by

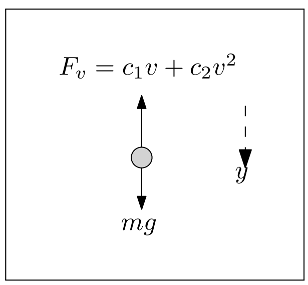
$$v_t = \left[\left(\frac{mg}{c_2} \right) + \left(\frac{c_1}{2\,c_2} \right)^2 \right]^{\frac{1}{2}} - \left(\frac{c_1}{2\,c_2} \right)$$

if the drag force F_v has *both* a linear *and* quadratic term in v:

$$F_v = c_1\,v + c_2\,v^2 \quad .$$

SOLUTION

From the free body diagram



The equation of motion is

$$\sum F_y = my''$$
$$mg - (c_1 y' + c_2 (y'^2)) = my''$$

At the terminal velocity the body is not accelerating. Setting $y'' = 0$ in the above gives an equation to solve for the terminal velocity (where now y' is written as v_t)

$$mg - (c_1 v_t + c_2 v_t^2) = 0$$
$$c_2 v_t^2 + c_1 v_t - mg = 0$$

This is a quadratic equation in v_t, hence the roots are given by

$$\begin{aligned} v_t &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-c_1}{2c_2} \pm \frac{\sqrt{c_1^2 + 4c_2 mg}}{2c_2} \\ &= \frac{-c_1}{2c_2} \pm \sqrt{\left(\frac{c_1}{2c_2} \right)^2 + \frac{mg}{c_2}} \end{aligned}$$

Since the terminal velocity v_t has to be positive as indicated in the diagram above, then the solution is the positive root given by

$$\boxed{v_t = \frac{-c_1}{2c_2} + \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}}}$$

5 Problem 5

5. (10 points)
A projectile is fired with an initial velocity $v_0 = 500\,\text{m/s}$ in a direction making an angle $\alpha = 30^\circ$ with the horizontal. We want to study the effect of air resistance on the range of the projectile. Assume that the drag force has the form $F_v = k\,m\,v$, where m and v are the mass and velocity of the projectile and k is a constant.

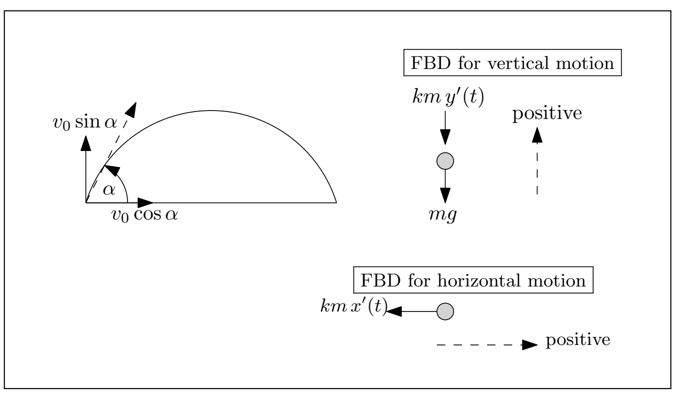
(1) Solve the equations of motions and determine the time T required for the full trajectory.

(2) Use a computer to draw the trajectories of the projectile for $k = 0$ (no air resistance), $k = 0.001$, $k = 0.01$ and $k = 0.1$. From your plots, estimate roughly the range for the different k.

SOLUTION

5.1 Part (1)

The following is the free body diagram used to solve this problem.



In the vertical direction, with positive taken upwards as shown in the diagram, the equation of motion is given by

$$\begin{aligned} \sum F_y &= my'' \\ -mg - kmy' &= my'' \\ y'' + ky' &= -g \end{aligned} \tag{1}$$

In the horizontal direction, the equation of motion is

$$\begin{aligned} \sum F_x &= mx'' \\ -kmx' &= mx'' \end{aligned} \tag{2}$$

The initial conditions for equation of motion in the vertical direction are $y(0) = 0, y'(0) = v_0 \sin\alpha$ and the initial conditions for the equation of motion in the horizontal direction are $x(0) = 0, x'(0) = v_0 \cos\alpha$.

Equation (1) is now solved. The characteristic equation is $\lambda^2 + k\lambda = 0$ or $\lambda(\lambda + k) = 0$, hence the roots are $\lambda = 0, \lambda = -k$, and therefore the homogeneous solution is

$$y_h(t) = A + Be^{-kt}$$

The particular solution is now found. Let $y_p(t) = ct$ where c is some constant. Substituting this into (1) gives

$$\begin{aligned} kc &= -g \\ c &= \frac{-g}{k} \end{aligned}$$

Hence the particular solution is $y_p(t) = -\frac{g}{k}t$, and the complete solution in the y direction is

$$\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \left(A + Be^{-kt}\right) - \frac{g}{k}t \end{aligned}$$

The initial conditions are now applied to determine the constants A, B. (Initial conditions must be used in the complete solution and not the homogeneous solution). When $t = 0$, $y(0) = 0$ and the above gives

$$A = -B$$

Since $y'(t) = -Bke^{-kt} - \frac{g}{k}$ and since $y'(0) = v_0 \sin\alpha$, then at $t = 0$

$$\begin{aligned} v_0 \sin\alpha &= -Bk - \frac{g}{k} \\ B &= -\left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right) \end{aligned}$$

Using values for the constants A, B, the complete solution for equation of motion in the vertical direction becomes

$$\begin{aligned} y(t) &= \left(A + Be^{-kt}\right) - \frac{g}{k}t \\ &= \left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right) - \left(\frac{g}{k^2} + \frac{v_0 \sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t \end{aligned}$$

Hence

$$y(t) = \left(\frac{g + kv_0 \sin\alpha}{k^2}\right)\left(1 - e^{-kt}\right) - \frac{g}{k}t \tag{3}$$

The duration time T is now found by solving for $y = 0$ from (3). Hence

$$0 = \left(\frac{g + kv_0 \sin\alpha}{k^2}\right)\left(1 - e^{-kT}\right) - \frac{g}{k}T$$

$$T = \left(\frac{g + kv_0 \sin\alpha}{gk}\right)\left(1 - e^{-kT}\right) \tag{4}$$

An analytical solution based on perturbation method for this is given in the text book at page 67 as

$$\boxed{T \simeq \frac{2v_0 \sin\alpha}{g}\left(1 - \frac{kv_0 \sin\alpha}{3g}\right)}$$

However in this solution equation (4) was solved numerically instead for T for the numerical values given in this problem, and the results are summarized on the following table

k	T (sec)
0.001	50.5427
0.01	47.2597
0.1	34.3395

The equation of motion in the x direction is now solved. This equation is given above in (2) as $x'' + kx' = 0$. The characteristic equation is $\lambda^2 + k\lambda = 0$ or $\lambda(\lambda + k) = 0$, hence the roots are $\lambda = 0, \lambda = -k$, and therefore, the homogeneous solution is

$$x_h(t) = A + Be^{-kt}$$

Since there is no forcing function, the complete solution is the same

$$x(t) = A + Be^{-kt} \tag{5}$$

The constants are found from the initial conditions. At $t = 0$

$$0 = A + B$$
$$A = -B$$

Since $x'(t) = -Bke^{-kt}$, then at $t = 0$

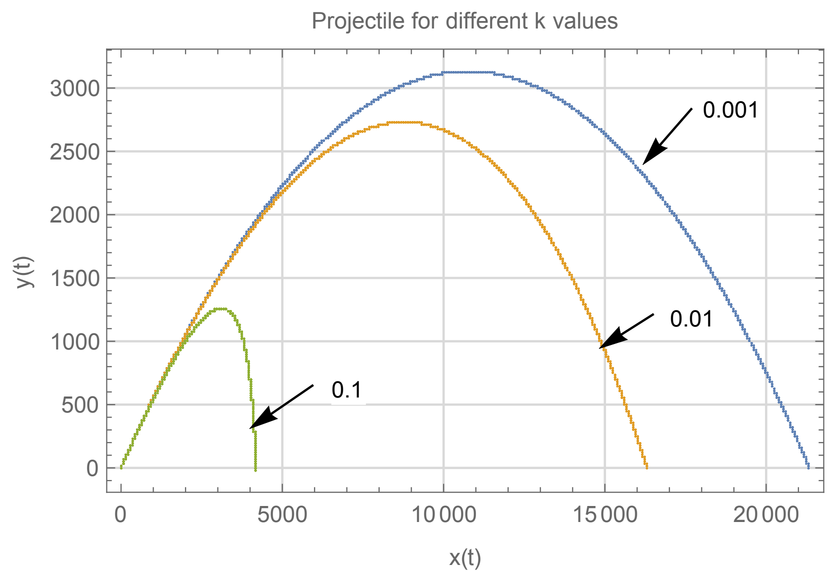
$$v_0 \cos\alpha = -Bk$$
$$B = \frac{-v_0 \cos\alpha}{k}$$

Substituting the above values for A, B into (4) gives the solution for the motion in the horizontal direction

$$x(t) = \frac{v_0 \cos\alpha}{k}\left(1 - e^{-kt}\right) \tag{6}$$

5.2 Part (2)

The following shows the projectile path for each different k value.



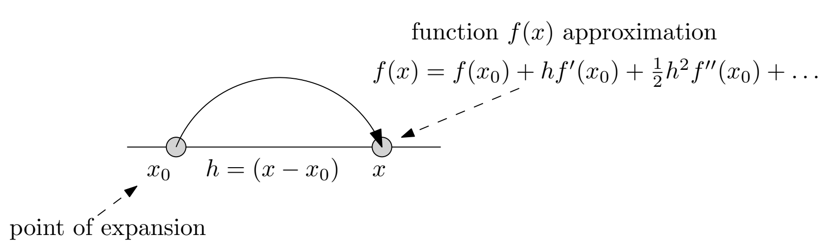
From the above, an estimate of the range for each k is given in the following table

k	range (meters)
0.001	21500
0.01	16500
0.1	4100

6 Problem 6

6. (10 points)
Find the Taylor series expansion of
(1) $f(x) = \cos x$ about $x = 0$,
(2) $f(x) = \cosh x$ about $x = 0$,
(3) $f(x) = \ln x$ about $x = 2$,
(4) $f(x) = \frac{1}{x^2}$ about $x = -1$,
(5) $f(x) = \sqrt{1+x}$ about $x = 0$.
Check out *Appendix A* of Thornton/Marion if you are unfamiliar with Taylor expansions.

SOLUTION



6.1 Part (1)

$f(x) = \cos(x)$ **about** $x = 0$

$$\begin{aligned} f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\ &= \cos(0) + x(-\sin(0)) + \frac{1}{2}x^2(-\cos(0)) + \frac{1}{6}x^3 \sin(0) + \frac{1}{24}x^4 \cos(0) + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots \end{aligned}$$

6.2 Part (2)

$f(x) = \cosh(x)$ **about** $x = 0$

$$\begin{aligned} f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\ &= \cosh(0) + x(\sinh(0)) + \frac{1}{2}x^2(\cosh(0)) + \frac{1}{6}x^3 \sinh(0) + \frac{1}{24}x^4 \cosh(0) + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots \end{aligned}$$

6.3 Part(3)

$f(x) = \ln(x)$ **about** $x = 2$

$$\begin{aligned} f(x) &\simeq f(2) + hf'(2) + \frac{1}{2}h^2 f''(2) + \frac{1}{3!}h^3 f'''(2) + \frac{1}{4!}h^4 f^{(4)}(2) + \cdots \\ &= \ln(2) + (x-2)\left(\frac{1}{x}\right)_{x=2} + \frac{1}{2}(x-2)^2\left(\frac{-1}{x^2}\right)_{x=2} + \frac{1}{6}(x-2)^3\left(\frac{2}{x^3}\right)_{x=2} + \frac{1}{24}(x-2)^4\left(\frac{-6}{x^4}\right)_{x=2} + \cdots \\ &= \ln(2) + \frac{x-2}{2} - \frac{1}{2}\frac{(x-2)^2}{4} + \frac{1}{3}\frac{(x-2)^3}{8} - \frac{1}{4}\frac{(x-2)^4}{16} + \cdots \\ &= \ln(2) + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \frac{(x-2)^4}{64} + \cdots \end{aligned}$$

6.4 Part(4)

$f(x) = \frac{1}{x^2}$ **about** $x = -1$

$$\begin{aligned} f(x) &\simeq f(-1) + hf'(-1) + \frac{1}{2}h^2 f''(-1) + \frac{1}{3!}h^3 f'''(-1) + \frac{1}{4!}h^4 f^{(4)}(-1) + \cdots \\ &= 1 + (x+1)\left(\frac{-2}{x^3}\right)_{x=-1} + \frac{1}{2}(x+1)^2\left(\frac{6}{x^4}\right)_{x=-1} + \frac{1}{6}(x+1)^3\left(\frac{-24}{x^5}\right)_{x=-1} + \frac{1}{24}(x+1)^4\left(\frac{120}{x^6}\right)_{x=-1} + \cdots \\ &= 1 + (x+1)\left(\frac{-2}{-1}\right) + \frac{1}{2}(x+1)^2\left(\frac{6}{1}\right) + \frac{1}{6}(x+1)^3\left(\frac{-24}{-1}\right) + \frac{1}{24}(x+1)^4\left(\frac{120}{1}\right) + \cdots \\ &= 1 + 2(x+1) + 3(x+1)^2 + 4(x+1)^3 + 5(x+1)^4 + \cdots \end{aligned}$$

6.5 Part (5)

$f(x) = \sqrt{1+x}$ about $x = 0$

$$
\begin{aligned}
f(x) &\simeq f(0) + hf'(0) + \frac{1}{2}h^2 f''(0) + \frac{1}{3!}h^3 f'''(0) + \frac{1}{4!}h^4 f^{(4)}(0) + \cdots \\
&= 1 + x\left(\frac{1}{2\sqrt{1+x}}\right)_{x=0} + \frac{1}{2}x^2\left(\frac{-1}{4(1+x)^{\frac{3}{2}}}\right)_{x=0} + \frac{1}{6}x^3\left(\frac{3}{8(1+x)^{\frac{5}{2}}}\right)_{x=0} + \frac{1}{24}x^4\left(\frac{-15}{16(1+x)^{\frac{7}{2}}}\right)_{x=0} + \cdots \\
&= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5}{128}x^4 + \cdots
\end{aligned}
$$