HW7 ECE 332 Feedback Control

Fall 2015
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By

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0.1 Problem 1

1. For each of the open loop transfer functions below, assume a classical unity feedback connection and sketch the Nyquist plot. Find the values of the gain $K > 0$ for which closed loop stability is guaranteed.

(a) $G(s)H(s) = \frac{K}{s(s^2+s+4)}$

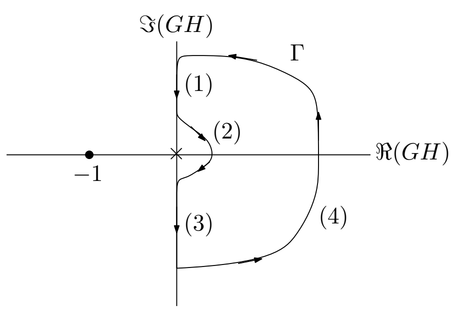
(b) $G(s)H(s) = \frac{K(s+1)}{s^2(s+2)}$

SOLUTION:

0.1.1 Part(a)

$$GH = \frac{K}{s\left(s^2+s+4\right)} \tag{1}$$

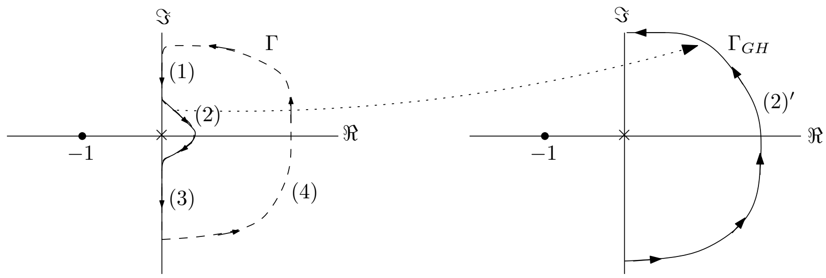
The poles of the open loop are at $s = 0$ and $s = -0.5 \pm j1.94$. So we draw Γ which encloses all the poles in the RHS making sure we avoid the pole at $s = 0$ by making small circle. Here is the result. (we do not care about open loop poles in the LHS).



We now start by mapping each segment from Γ to Γ_{GH}. Starting with *segment 2*. We see that

$$\lim_{s\to 0}\frac{K}{s\left(s^2+s+4\right)} = \lim_{s\to 0}\frac{K}{s} = \infty e^{-j\theta}$$

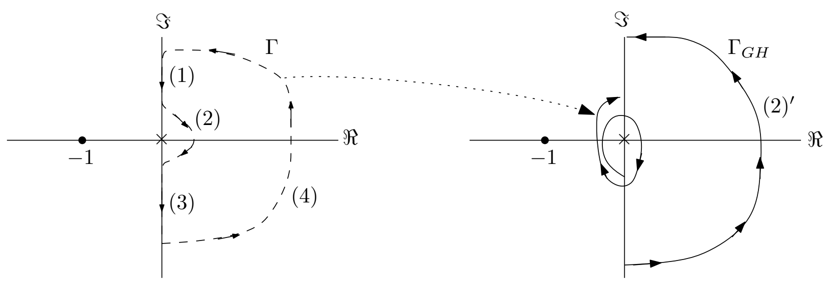
Where θ goes from $+90^0$ to -90^0 in Γ. This means on Γ_{GH} segment (2) will map to a very large circle which goes from -90^0 to $+90^0$ (anti-clockwise). We update the plot after making each segment so we see the progress.



We now go to *segment 4* in Γ.

$$\lim_{s\to\infty}\frac{K}{s\left(s^2+s+4\right)}=\lim_{s\to\infty}\frac{K}{s^3}=0e^{-j3\theta}$$

Where θ goes from -90^0 to $+90^0$ on Γ. This means on Γ_{GH} segment (4) will map to a very small circle which goes from $+270^0$ to -270^0 on Γ_{GH}. This is 1.5 circle rotation in clockwise that goes from around 540^0. updating the plot gives



We now do *segment 1*. For this we need to find the real and imaginary part of GH since we need the axis crossings. From (1)

$$GH=\frac{K}{s^3+s^2+4s}=\frac{K}{\left(j\omega\right)^3+\left(j\omega\right)^2+4\left(j\omega\right)}=\frac{K}{-j\omega^3-\omega^2+4j\omega}=\frac{K}{j\left(4\omega-\omega^3\right)-\omega^2}$$

Multiply numerator and denominator by complex conjugate denominator gives

$$\begin{aligned}GH&=\frac{K}{j\left(4\omega-\omega^3\right)-\omega^2}\frac{\left(-j\left(4\omega-\omega^3\right)-\omega^2\right)}{\left(-j\left(4\omega-\omega^3\right)-\omega^2\right)}\\&=\frac{K\left(-j\left(4\omega-\omega^3\right)-\omega^2\right)}{\left(j\left(4\omega-\omega^3\right)-\omega^2\right)\left(-j\left(4\omega-\omega^3\right)-\omega^2\right)}\\&=\frac{-jK\left(4\omega-\omega^3\right)-K\omega^2}{\omega^6-7\omega^4+16\omega^2}\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{Re}(GH) &= \frac{-K}{\omega^4 - 7\omega^2 + 16}\\ \operatorname{Im}(GH) &= \frac{-K\left(4\omega - \omega^3\right)}{\omega^6 - 7\omega^4 + 16\omega^2} = \frac{K\left(\omega^2 - 4\right)}{\omega\left(\omega^4 - 7\omega^2 + 16\right)}\end{aligned}$$

To find the crossing. Γ_{GH} will cross the real axis when $\operatorname{Im}(GH) = 0$, hence

$$\begin{aligned}\frac{K\left(\omega^2 - 4\right)}{\omega\left(\omega^4 - 7\omega^2 + 16\right)} &= 0\\ K\left(\omega^2 - 4\right) &= 0\\ \omega^2 - 4 &= 0\\ \omega^2 &= 4\\ \omega &= \pm 2 \text{ rad/sec}\end{aligned}$$

Then

$$\begin{aligned}GH\left(j2\right) &= \frac{K}{\left(j2\right)^3 + \left(j2\right)^2 + 4\left(j2\right)}\\ &= \frac{K}{\left(-j8\right) + (-4) + \left(j8\right)}\\ &= \frac{-1}{4}K\end{aligned}$$

So $\boxed{\Gamma_{GH} \text{ will cross the real axis at } -0.25K.}$

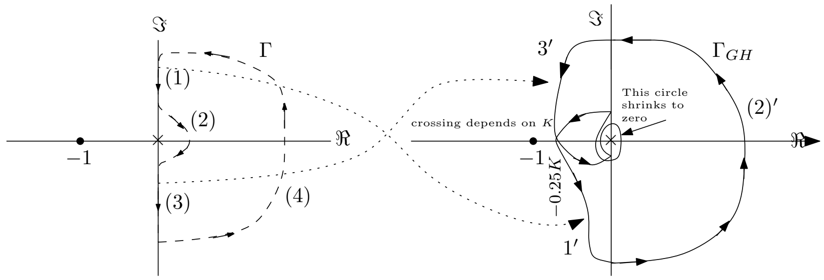
Since $K > 0$, this will be somewhere on the negative real axis. To find where Γ_{GH} crosses the imaginary axis we set $\operatorname{Re}(GH) = 0$, hence

$$\frac{-K}{\omega^4 - 7\omega^2 + 16} = 0$$

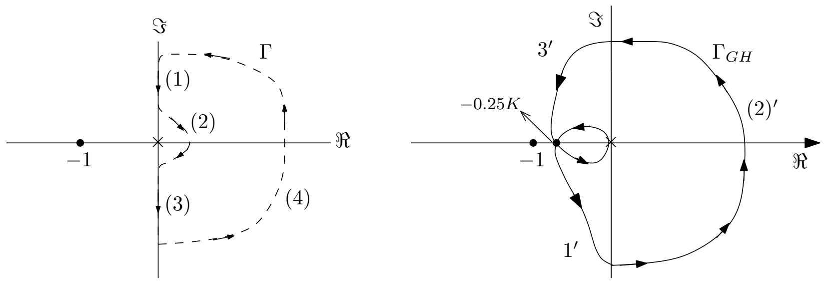
Which gives $\omega = \pm\infty$. When $\omega = \pm\infty$ then $GH(\pm\infty) = 0^{\pm}$. This means the crossing at origin. This makes sense, as the small circle around the origin shrinks to zero size.

To find where segment (1) maps to, we see that for $\omega = +\infty$, $\operatorname{Re}(GH) < 0$ and $\operatorname{Im}(GH) > 0$, so it starts in first quadrant. $\omega = 0^+, \operatorname{Re}(GH) < 0$ and $\operatorname{Im}(GH) < 0$, which means segment (1) starts in the first quadrant but ends in the third quadrant.

Now for segment (3). we see that $\omega = 0^-, \operatorname{Re}(GH) < 0$ and $\operatorname{Im}(GH) > 0$, which means segment (3) starts in the first quadrant. $\omega = -\infty$ then $\operatorname{Re}(GH) < 0$ and $\operatorname{Im}(GH) < 0$ which means segment (3) in Γ_{GH} starts in quadrant 1 and ends up in quadrant 3. There will be crossing at the real axis at $-0.25K$. Therefore the plot now looks like this



As the circle around $s=0$ shrinks to zero, Γ_{GH} becomes as follows



We are now ready to answer the final question about stability and K. Since open loop has zero poles in RHS, then we need to have zero net clock wise encirclements around -1 for the closed loop to be stable. Only condition that will meet that, is to keep the crossing point $-0.25K$ to the right of -1. This means we need $0.25K<1$. This insures zero clockwise encirclements. This means

$$K<4$$

To verify, Routh table is used to determined K using the closed loop transfer function. The closed loop is given by

$$\begin{aligned} T &= \frac{GH}{1+GH} \\ &= \frac{K}{s\left(s^2+s+4\right)+K} \\ &= \frac{K}{s^3+s^2+4s+K} \end{aligned}$$

Hence the Routh table is

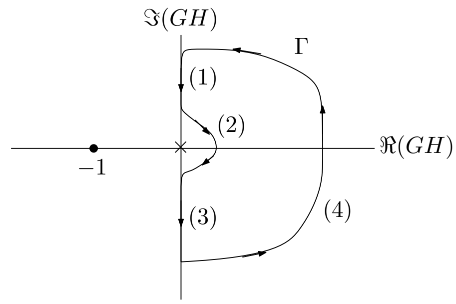
s^3	1	4
s^2	1	K
s^1	$4-K$	
s^0	K	

For no sign change in first column, we need $K>0$ and $4-K>0$. Which means $K<4$ as was found above. Verified OK.

0.1.2 Part(b)

$$GH = \frac{K(s+1)}{s^2(s+2)} = \frac{K(s+1)}{s^3+2s^2} \tag{1}$$

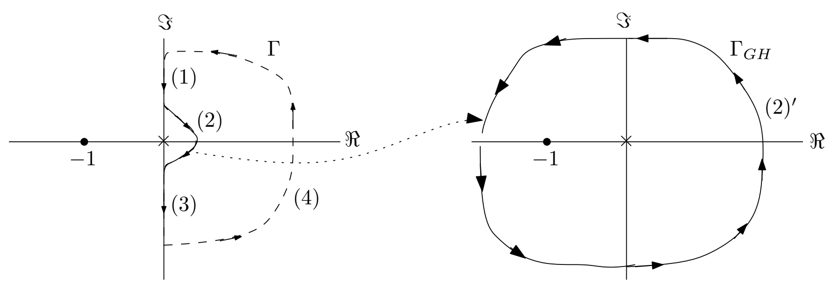
The poles of the open loop are at $s = 0$ (two poles) and $s = -2$. So we draw Γ which encloses all the poles in the RHS making sure we avoid the poles at $s = 0$ by making small circle. Here is the result. (we do not care about open loop pole in the LHS and about the zeros of the open loop).



We now start by mapping each segment from Γ to Γ_{GH}. Starting with *segment 2*. We see that

$$\lim_{s\to 0}\frac{K(s+1)}{s^3+2s^2} = \lim_{s\to 0}\frac{K}{s^2} = \infty e^{-2j\theta}$$

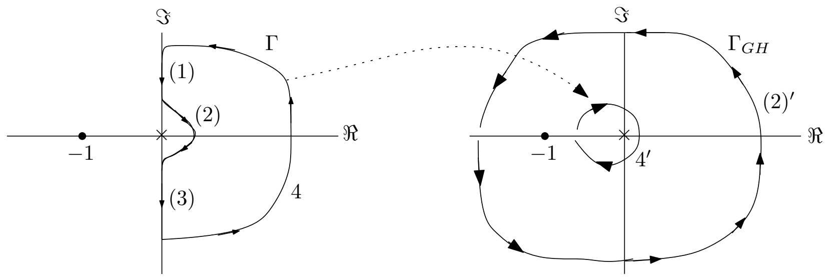
Where θ goes from $+90^0$ to -90^0 in Γ. This means on Γ_{GH} segment (2) will map to a very large circle which goes from -180^0 to $+180^0$. This is a full circle in the anti-clockwise. We update the plot after making each segment so we see the progress.



We now go to *segment 4* in Γ.

$$\lim_{s\to\infty}\frac{K(s+1)}{s^3+2s^2} = \lim_{s\to\infty}\frac{Ks}{s^3} = \lim_{s\to\infty}\frac{K}{s^2} = 0e^{-j2\theta}$$

Where θ goes from -90^0 to $+90^0$ on Γ. This means on Γ_{GH} segment (4) will map to a very small circle which goes from $+180^0$ to -180^0 on Γ_{GH}. This is basically a full circle in clockwise around zero. updating the plot gives



We now do *segment 1*. For this we need to find the real and imaginary part of GH since we need the axis crossings. From (1)

$$GH = \frac{K(s+1)}{s^3+2s^2} = \frac{K\left(j\omega+1\right)}{\left(j\omega\right)^3+2\left(j\omega\right)^2} = \frac{K\left(j\omega+1\right)}{-j\omega^3-2\omega^2}$$

Multiply numerator and denominator by complex conjugate denominator gives

$$\begin{aligned} GH &= \frac{K\left(j\omega+1\right)}{-j\omega^3-2\omega^2}\frac{\left(j\omega^3-2\omega^2\right)}{\left(j\omega^3-2\omega^2\right)} \\ &= \frac{j\left(-K\omega^3\right)-\left(K\omega^4+2K\omega^2\right)}{\omega^4\left(\omega^2+4\right)} \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{Re}(GH) &= \frac{-\left(K\omega^4+2K\omega^2\right)}{\omega^4\left(\omega^2+4\right)} \\ \operatorname{Im}(GH) &= \frac{-K\omega^3}{\omega^4\left(\omega^2+4\right)} = \frac{-K}{\omega\left(\omega^2+4\right)} \end{aligned}$$

To find the crossing. Γ_{GH} will cross the real axis when $\operatorname{Im}(GH)=0$, hence

$$\begin{aligned} \frac{-K}{\omega\left(\omega^2+4\right)} &= 0 \\ \omega &= \pm\infty \text{ rad/sec} \end{aligned}$$

Therefore

$$\begin{aligned} GH\left(j\infty\right) &= \lim_{\omega\to\infty}\frac{K\left(j\omega+1\right)}{-j\omega^3-2\omega^2} \\ &= \lim_{\omega\to\infty}\frac{j\omega}{-j\omega^3} \\ &= \lim_{\omega\to\infty}\frac{j}{-j\omega^2} \\ &= 0^- \end{aligned}$$

So Γ_{GH} will cross the real axis at 0. Which is where the small circle shrinks to zero. To find where Γ_{GH} crosses the imaginary axis we set $\operatorname{Re}(GH) = 0$, hence

$$\frac{-\left(K\omega^4 + 2K\omega^2\right)}{\omega^4\left(\omega^2+4\right)} = 0$$
$$K\omega^4 + 2K\omega^2 = 0$$
$$\omega^2\left(\omega^2+2\right) = 0$$

Hence

$$\omega = 0$$

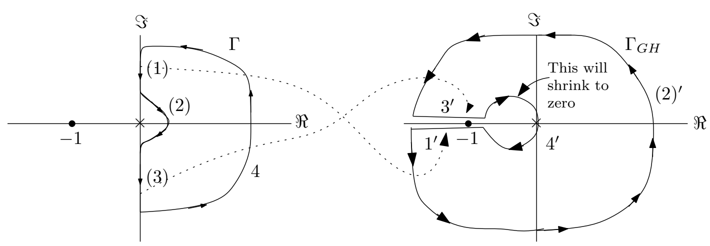
And

$$\omega = \pm i\sqrt{2}$$

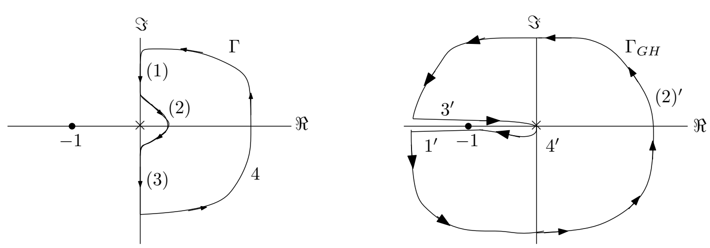
Since ω has to be real. Then $\omega = 0$ is only used. When $\omega = 0$ then $GH\left(j0\right) = \lim_{\omega\to 0} \frac{K(j\omega+1)}{-j\omega^3-2\omega^2} = \lim_{\omega\to 0} \frac{K}{-2\omega^2} = \infty$. Hence Γ_{GH} will cross the imaginary axis at ∞.

To find which quadrants segment (1) maps to, we see that for positive ω then $\operatorname{Re}(GH)$ is negative and $\operatorname{Im}(GH)$ is negative (since $K > 0$). Therefore segment (1) maps to third quadrant. And for segment (3), we see that negative ω then $\operatorname{Re}(GH)$ is negative and $\operatorname{Im}(GH)$ is positive (since $K > 0$). Therefore segment (3) maps to first quadrant

Therefore the plot now looks like this



After the small circle shrinks to zero, and since the crossing on the real axis was found at 0 then the final plot looks like



We are now ready to answer the final question about stability and K. Since open loop has zero poles in RHS, then we need to have zero net clockwise encirclements around -1 for the closed loop to be stable. We see that there is no encirclements around -1. No matter what $K > 0$ value is. So the closed loop is stable for all positive K. To verify, Routh table is used to determined K using the closed loop transfer function. The closed loop is given by

$$\begin{aligned} T &= \frac{GH}{1+GH} \\ &= \frac{K(s+1)}{s^3+2s^2+K(s+1)} \end{aligned}$$

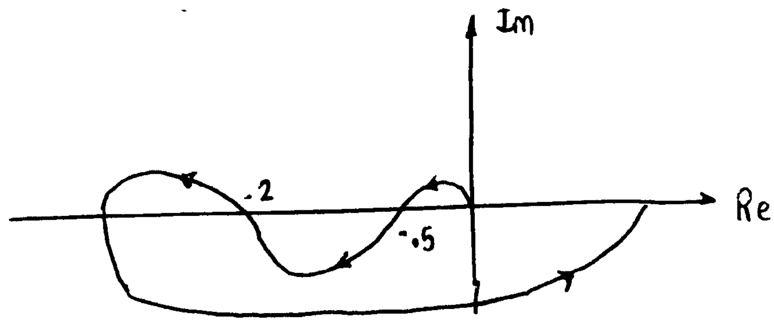
Hence the Routh table is

s^3	1	K
s^2	2	K
s^1	$\frac{2K-K}{2} = \frac{K}{2}$	
s^0	K	

For no sign change in first column, we need $K > 0$ and $\frac{K}{2} > 0$. Which is always true since K is positive. Verified OK.

0.2 Problem 2

2. The Nyquist plot for a transfer function $KG(s)$ is shown below for $K = 1$ and $\omega \geq 0$. Assuming a unity feedback configuration with $G(s)$ having no poles in the closed right half plane, find the ranges for $K > 0$ under which closed loop stability is assured.



SOLUTION:

Since there is zero open loop poles in RHP, we need zero net clockwise encirclements around −1. This means we need to keep point −0.5 to the right of −1 and keep the point −2 to the left of −1. In other words, we need to satisfy

$$0.5K < 1$$
$$2K > 1$$

Or

$$K < 2$$

And

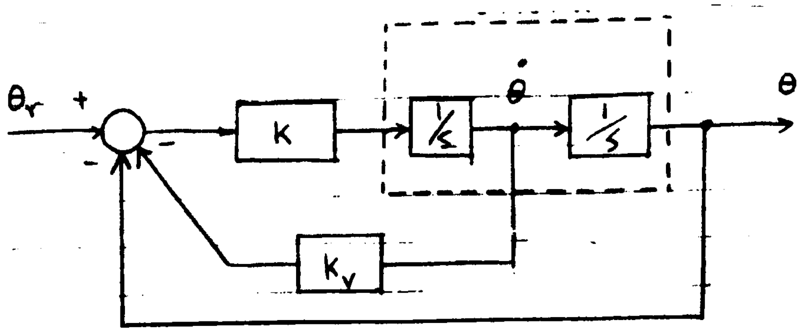
$$K > \frac{1}{2}$$

Hence the range of stable closed loop is for K to meet the following requirement

$$0.5 < K < 2$$

0.3 Problem 3

3. (a) For the attitude control system shown below, assume that the velocity feedback gain K_v is zero and sketch the Nyquist for open loop system $G(s)H(s)$.



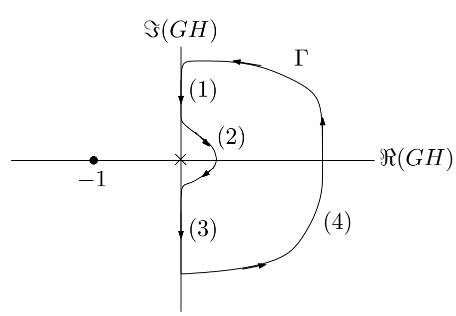
(b) For the closed loop system above, for what range of gains $K > 0$ is stability assured?

(c) Now consider the case with both gains K and K_v non-zero. Using an appropriate Nyquist plot, find the range for these gains under which closed loop stability is assured.

SOLUTION:

0.3.1 Part(a)

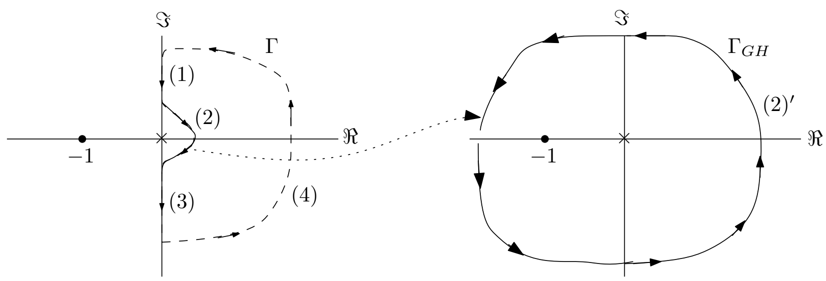
When $K_v = 0$ the open loop is $GH = K\frac{1}{s^2}$. This has no poles in RHP and two poles at zero. Hence Γ is



We start at segment 2.

$$\lim_{s\to 0}\frac{K}{s^2}=\lim_{\varepsilon\to 0}\frac{K}{\left(\varepsilon e^{j\theta}\right)^2}=\infty e^{-2j\theta}$$

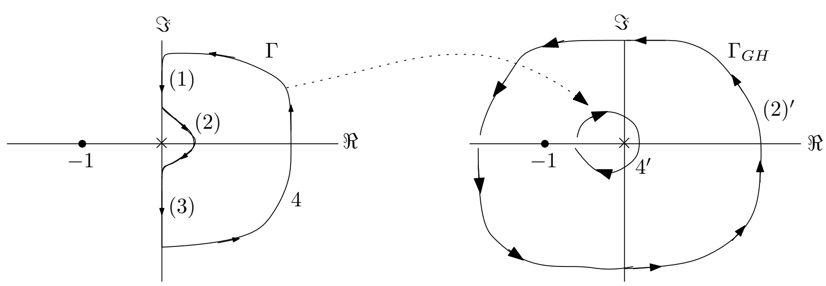
As θ goes from $+90^0$ to -90^0 on Γ segment 2 goes from -180^0 to $+180$ on Γ_{GH} with ∞ radius in anti-clockwise. Hence the plot now looks as follows



For segment 4 in Γ. We have

$$\lim_{s\to\infty}\frac{K}{s^2}=\lim_{R\to\infty}\frac{K}{\left(Re^{j\theta}\right)^2}=0e^{-2j\theta}$$

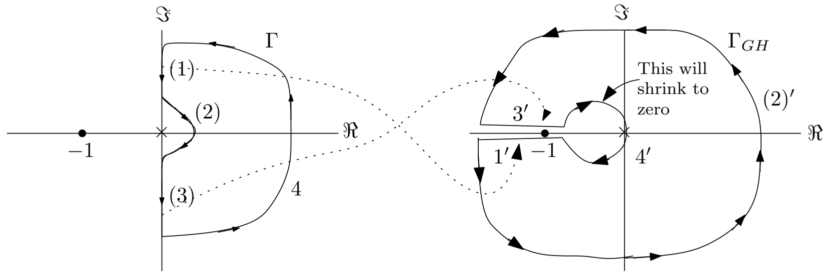
θ goes from -90^0 to $+90^0$ on Γ, segment 4 goes from $+180^0$ to -180 on Γ_{GH} with 0 radius in clockwise. Hence the plot now looks as follows



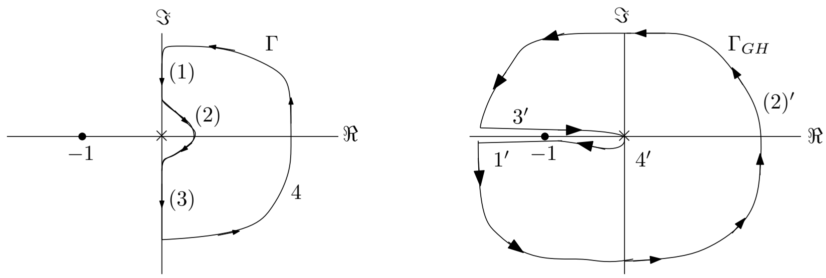
Now

$$\frac{K}{s^2}=\frac{K}{\left(j\omega\right)^2}=\frac{K}{-\omega^2}$$

Hence $\mathrm{Re}\,(GH)=\frac{K}{-\omega^2}$ and there is no imaginary part. Therefore, there is no crossing on the real axis. There is crossing on the imaginary axis at $\omega=\pm\infty$ which occurs at $GH=0$. This is when the small circle shrinks to zero size. It is clear that segment 1 will map to third quadrant and segment 3 will map to second quadrant since there is no crossing. The final plot is



After the small circle shrinks to zero, and since the crossing on the real axis was found at 0 then the final plot looks like



0.3.2 Part(b)

Since open loop has zero poles in RHP, we need Γ_{GH} to have zero net clockwise encirclements. Since Γ_{GH} has no crossing on the real axis that depends on K then Γ_{GH} will remain as shown for any K. So closed loop is stable for all $K > 0$. To verify, we set the Routh table for the closed loop polynomial

The closed loop is given by

$$
\begin{aligned}
T &= \frac{GH}{1+GH} \\
&= \frac{K}{s^2+K}
\end{aligned}
$$

Hence the Routh table is

s^2	1	K
s^1	0	0
s^0	0	

We see that there is no sign change in first column, no matter what K is.

0.3.3 Part (c)

We first find the closed loop transfer function. Let $E(s)$ be the error (just after the summing junction) then

$$\begin{aligned} E &= \theta_r - \theta - K_v s\theta \\ \theta &= EK\frac{1}{s^2} \end{aligned}$$

From the second equation $E = \frac{\theta s^2}{K}$, hence the first equation becomes

$$\begin{aligned} \frac{\theta s^2}{K} &= \theta_r - \theta - K_v s\theta \\ \frac{\theta s^2}{K} + \theta + K_v s\theta &= \theta_r \\ \theta\left(\frac{s^2}{K} + 1 + K_v s\right) &= \theta_r \end{aligned}$$

Therefore the closed loop transfer function $T(s) = \frac{\theta}{\theta_r}$ is

$$\begin{aligned} \frac{\theta}{\theta_r} &= \frac{1}{\frac{s^2}{K} + 1 + K_v s} \\ &= \frac{K}{s^2 + KK_v s + K} \end{aligned}$$

We now find the open loop transfer function with unity feedback using the closed loop transfer function. Since $T(s) = \frac{G}{1+G}$ where $G(s)$ is the closed loop transfer function, then letting $G(s) = \frac{N}{D}$ we have

$$\begin{aligned} \frac{K}{s^2 + KK_v s + K} &= \frac{G}{1+G} \\ &= \frac{\frac{N}{D}}{1 + \frac{N}{D}} \\ &= \frac{N}{N+D} \end{aligned}$$

Therefore $N = K$ and $N + D = s^2 + KK_v s + K$ which means $D(s) = s^2 + KK_v s$. Therefore the open loop transfer function is

$$G(s) = \frac{N}{D} = \frac{K}{s^2 + KK_v s}$$

Hence

$$\boxed{GH = \frac{K}{s(s+KK_v)}}$$

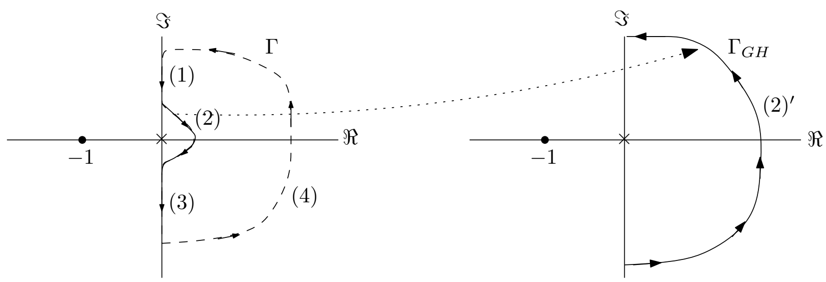
Therefore, the $\boxed{\text{open loop has a pole at } s = 0 \text{ and at } s = -KK_v}$.

Assuming positive gains, $-KK_v$ is in the LHP. Hence the open loop is stable. Which means the Nyquist plot should have zero net clockwise encirclement around -1 for the closed loop to be stable.

We now start by mapping each segment from Γ to Γ_{GH}. Starting with *segment 2.* We see that

$$\lim_{s\to 0}\frac{K}{s^2+sKK_v}=\lim_{s\to 0}\frac{1}{sK_v}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon e^{j\theta}K_v}=\infty e^{-j\theta}$$

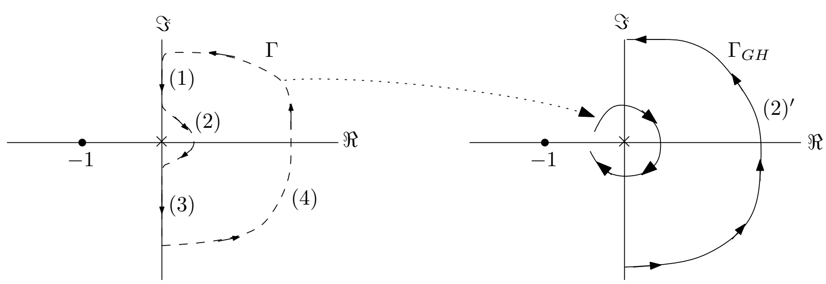
Where θ goes from $+90^0$ to -90^0 in Γ. This means on Γ_{GH} segment (2) will map to a half circle which goes from -90^0 to $+90^0$ in anti-clockwise.



We now go to *segment 4* in Γ.

$$\lim_{s\to\infty}\frac{K}{s^2+sKK_v}=\lim_{s\to\infty}\frac{K}{s^2}=\lim_{R\to\infty}\frac{1}{Re^{2j\theta}}=0e^{-j2\theta}$$

Where θ goes from -90^0 to $+90^0$ on Γ. This means on Γ_{GH} segment (4) will map to a very small circle which goes from $+180^0$ to -180^0 on Γ_{GH}. This is basically a full circle in clockwise around zero. updating the plot gives

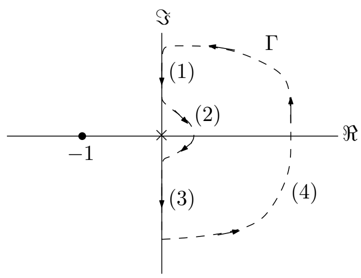


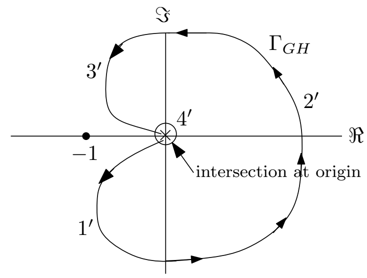
To find the intersections,

$$\begin{aligned}GH&=\frac{K}{s^2+sKK_v}=\frac{K}{-\omega^2+j\omega KK_v}=\frac{K}{\left(-\omega^2+j\omega KK_v\right)}\frac{\left(-\omega^2-j\omega KK_v\right)}{\left(-\omega^2-j\omega KK_v\right)}\\&=\frac{-K\omega^2-j\omega K^2K_v}{K^2\omega^2K_v^2+\omega^4}=\frac{-K}{K^2K_v^2+\omega^2}-j\frac{K^2K_v}{K^2\omega K_v^2+\omega^3}\end{aligned}$$

Hence $\operatorname{Re}(GH)=\frac{-K}{K^2K_v^2+\omega^2}$ and $\operatorname{Im}(GH)=-\frac{K^2K_v}{K^2\omega K_v^2+\omega^3}$. When $\operatorname{Re}(GH)=0$, we get $\omega=\pm\infty$. When means $GH=0$ at this frequency. So Γ_{GH} crosses the imaginary axis at the origin.

When $\text{Im}(GH) = 0$, we also get $\omega = \pm\infty$ which also means Γ_{GH} crosses the real axis at zero. By continuation, and since segment 1 must follow segment 4, then segment 1 maps to third quadrant in Γ_{GH} and segment 3 must map to quadrant 2 in Γ_{GH}. As $\varepsilon \to 0$ the small circle become a point at origin and we get the final plot





Since the intersection is always at the origin, Γ_{GH} will never move to the left passed -1 to make any encirclement around -1. We need at least one net encirclement for the closed loop to be unstable. Hence $\boxed{\text{the closed loop is stable for all positive } K, K_v}$.

To verify, we show Routh table for the closed loop found above, which is $\frac{K}{s^2+KK_vs+K}$. The Routh table is

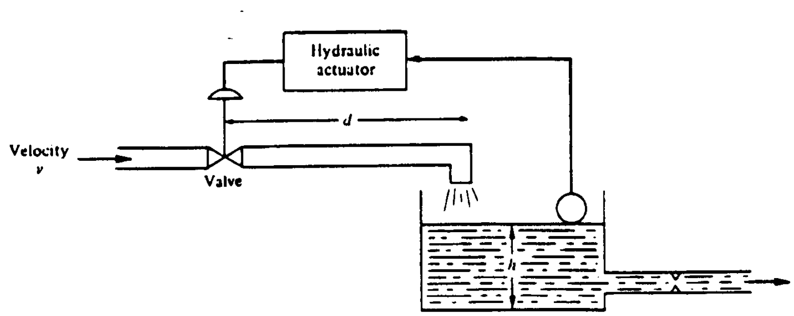
s^2	1	K
s^1	KK_v	0
s^0	K	

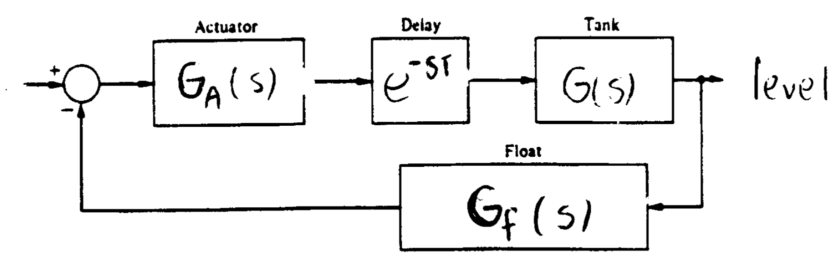
For positive K, K_v, we see that there can not be a sign change. Hence closed loop is stable for all K, K_v. (Note, I assume $K, K_v > 0$. Verified this with instructor via email).

0.4 Problem 4

4. A level control system and its transfer function model is depicted below. Assuming zero delay ($T = 0$), use Matlab to generate the appropriate Nyquist plot which can be used to analyze closed loop stability. Take

$$G_A(s) = \frac{10}{s+1}; \; G(s) = \frac{3.15}{30s+1}; \; G_f(s) = \frac{1}{(s^2/9) + (s/3) + 1}.$$





(b) Using the Nyquist plot from part (a), estimate the gain and phase margins.

(c) With time delay $T > 0$, explain in detail what is meant by the following: "The appropriate Nyquist plot is obtained from the plot in Part (a) by a frequency dependent angular rotation." For the case when $T = 1$, is the closed loop stable? Explain.

SOLUTION:

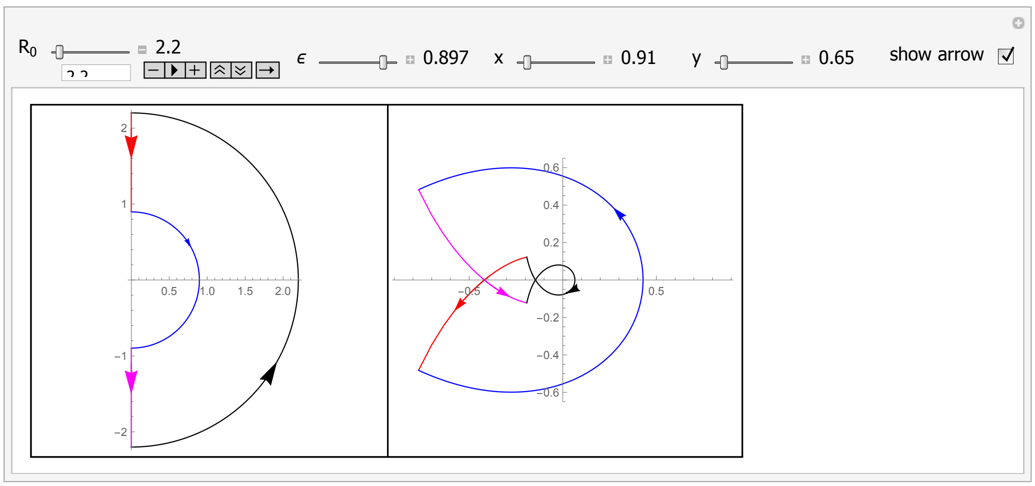
0.4.1 Part (a)

The open loop transfer function $T(s)$ is $G_A(s)\,e^{-sT}G(s)\,G_f(s)$. For $T=0$, we have

$$
\begin{aligned}
T(s) &= G_A(s)\,G(s)\,G_f(s)\\
&= \frac{10}{s+1}\,\frac{3.15}{30s+1}\,\frac{1}{\frac{s^2}{9}+\frac{s}{3}+1}
\end{aligned}
$$

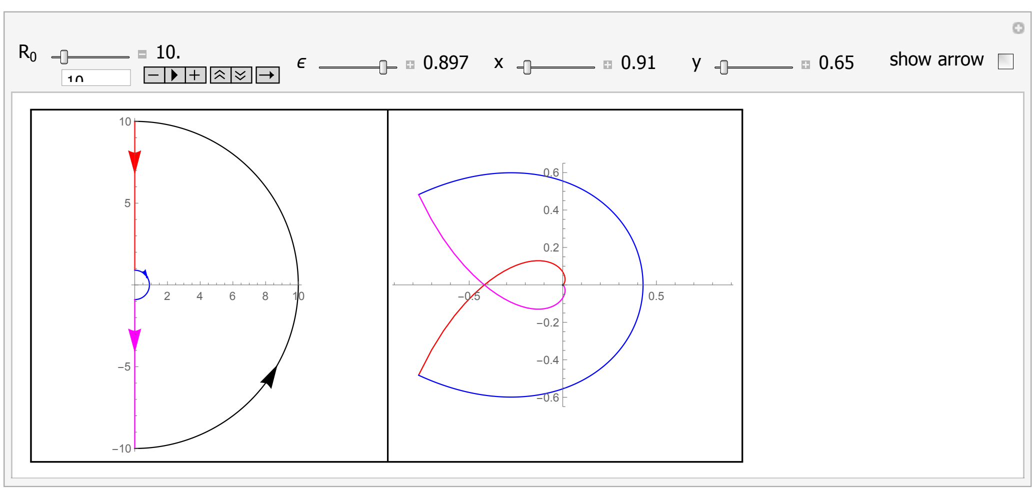
The Nyquist plot is (using the program I wrote which shows Γ and Γ_{GH} side by side)

Out[142]=



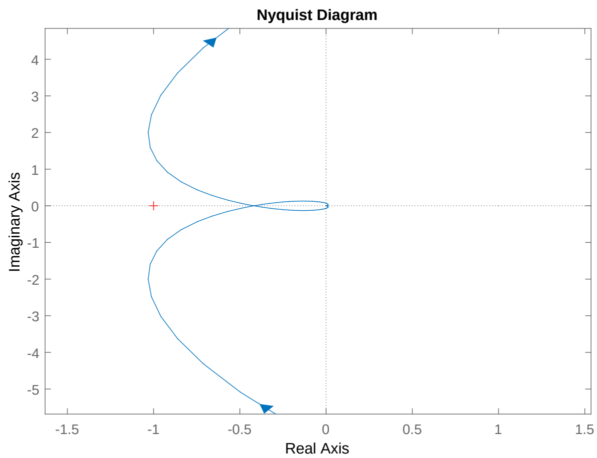
In the limit, as R becomes very large we obtain

Out[142]=



Here is also Matlab `nyquist` output (zoomed in version)

```
s=tf('s');
sys=10/(s+1)*(3.15)/(30*s+1)*1/(s^2/9+s/3+1);
sys =
850.5
-----------------------------------------
90 s^4 + 363 s^3 + 1092 s^2 + 846 s + 27
nyquist([850.5],[90 363 1092 846 27])
```



0.4.2 Part (b)

For the gain margin, the Γ_{GH} curve crosses the real axis at about -0.41. Therefore we need

$$0.41K_{\max} < 1$$

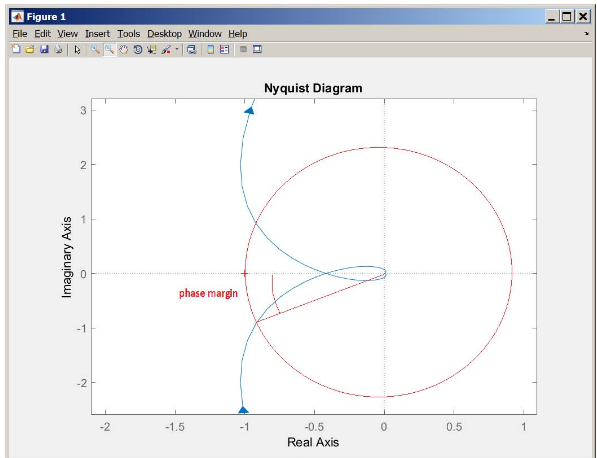
For stability. Hence

$$K_{\max} = \frac{1}{0.41} = 2.439$$

In dB, the above becomes

$$\begin{aligned} gm &= 20\log_{10} 2.439 \\ &= 7.74 \text{ db} \end{aligned}$$

For the phase margin, we draw a unit circle and find the intersection with Γ_{GH} and estimate the angle between the line from origin to the intersection and the -180^0 line. As follows



The angle seems to be approximately between 30^0 and 35^0. This is the phase margin. To get exact values, Matlab margin command can be used as follows

```
s=tf('s');
sys=10/(s+1)*(3.15)/(30*s+1)*1/(s^2/9+s/3+1);
sys =
850.5
----------------------------------------
90 s^4 + 363 s^3 + 1092 s^2 + 846 s + 27
>> [Gm,Pm,~,~] = margin(sys)
Gm =
2.3854
Pm =
35.6025
```

Converting the Gm value given in Matlab to dB, gives the result shown above. Matlab gives 35.6^0 as the exact phase margin.

0.4.3 Part (c)

Let the open loop GH when $T = 0$ (which is what we analyzed in part (b)) be called $GH(s)$ which can be written, in frequency domain as

$$GH(s)|_{s=j\omega} = |GH|\, e^{j\theta}$$

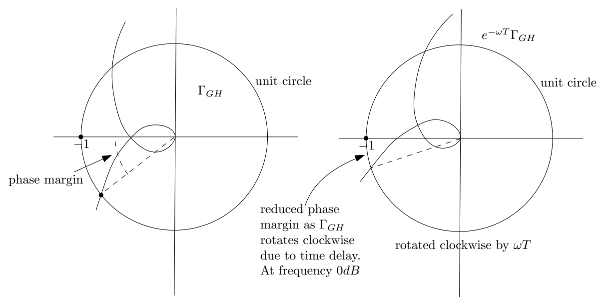
Where both the magnitude $|GH|$ and the phase θ in the above, are functions of the frequency ω. The above is polar representation of the complex quantity $GH\left(j\omega\right)$. When $T > 0$, then the open loop is now $e^{-sT}GH(s)$, which can be written in frequency domain as

$$e^{-j\omega T}GH\left(j\omega\right) = \left|e^{-j\omega T}\right| |GH|\, e^{j(\theta-\omega T)}$$

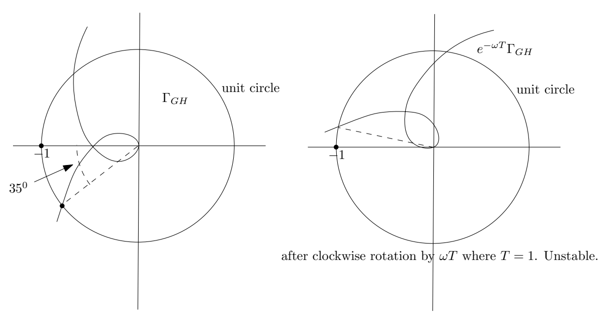
In other words, magnitudes multiply and phases are added. But $\left|e^{-j\omega T}\right| = 1$, so the above is

$$e^{-j\omega T} GH\left(j\omega\right) = |GH|\, e^{j(\theta-\omega T)}$$

We see that the resulting open loop has the same magnitude as before, but its phase has change. We subtract angle ωT from the original phase θ. subtract angle ωT is the same as rotating the complex vector representation clockwise ωT. So this causes the whole Nyquist plot, which is a frequency plot $GH\left(j\omega\right)$, to just rotate by ωT clockwise (since negative angle) to what it was before. This makes Γ_{GH} become closer to -1. This is illustrated in the following diagram



When $T = 1$, the angle is ω radians. Since we found the phase margin to be 35^0 or about 0.61 radians, then the closed loop, which corresponds to the open $e^{-j\omega T} GH\left(j\omega\right)$, will have new phase reduced by ω radians. Since phase margin is measured at $0dB$ angle (or $\omega = 1$ radian, or 57.3^0). This is larger than the phase margin 35^0. Therefore the new system is $\boxed{\text{unstable}}$. Γ_{GH} will rotate and will cross over -1.



What the above shows, is that adding delays e^{-sT} makes the system less stable (closer to becoming unstable). Delays causes the phase margin to reduce. We can find the amount of delay T before the system becomes unstable. We need $\omega T < 35^0$ or $T < \frac{35^0}{57.3^0} < 0.611$ seconds. This is the maximum delay T we can have before the closed loop phase margin is all used up and the system become unstable.