# HW5 ECE 332 Feedback Control

Fall 2015
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University of Wisconsin, Madison

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By

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## 0.1 Problem 1

**Problem 1:** Performances specifications for a second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with $0 < \zeta < 1$ require that the maximum overshoot to a step input not exceed 30% and that the settling time be less than 0.25 seconds. Using the settling time approximation

$$t_s \approx \frac{3.2}{\zeta\omega_n},$$

find and sketch the region in the complex plane where the two poles need to be located.

SOLUTION:

There are two inequalities to satisfy. The first is given by the settling time requirement

$$t_s = \frac{3.2}{\zeta\omega_n} < 0.25 \tag{1}$$

The second is given by the overshoot requirement

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} < 0.3 \tag{2}$$

From (2), taking the log of both sides gives

$$-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} < \ln(0.3)$$

Multiplying both sides by −1 changes the inequality from < to >

$$\frac{\pi\zeta}{\sqrt{1-\zeta^2}} > -\ln(0.3)$$

Simplifying gives

$$\begin{aligned}\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &> \ln\left(\frac{1}{0.3}\right)\\ &> 1.204\end{aligned}$$

Squaring both sides and solving for $\zeta$

$$\begin{aligned}\frac{\zeta^2}{1-\zeta^2} &> \left(\frac{1.204}{\pi}\right)^2\\ \zeta^2 &> 0.14688\left(1-\zeta^2\right)\\ 1.146\,88\zeta^2 &> 0.14688\\ \zeta^2 &> 0.12807\end{aligned}$$

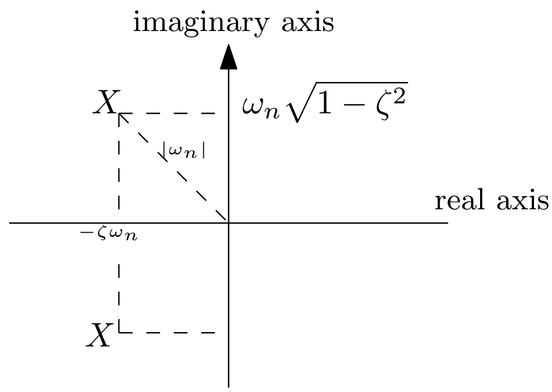
Since $\zeta$ has to be positive then the positive root is used giving

$$\boxed{\zeta > 0.35787}$$

Back to the (1) specifications, which says

$$\frac{3.2}{\zeta\omega_n} < 0.25$$
$$\zeta\omega_n > 12.8$$

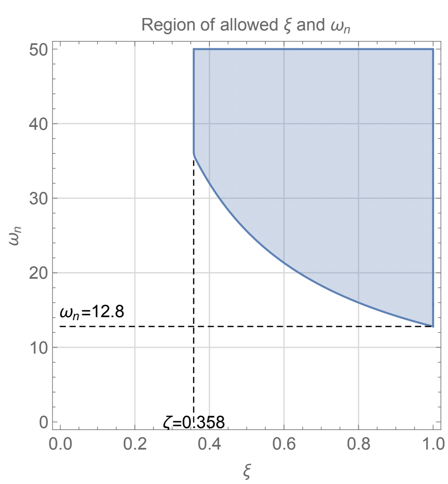
For each $\zeta_i > 0.3578$, we solve for $\omega_n$ from $\zeta\omega_n > 12.8$. This will give full description of where the poles are located.




Here is a plot of the $(\zeta, \omega_n)$ space showing allowed values of $\zeta, \omega_n$.

```
In[69]:= RegionPlot[1 > z > 0.35787 && z w > 12.8, {z, 0, 1}, {w, 0, 50}, GridLines → Automatic, GridLinesStyle → LightGray, Frame → True,
   FrameLabel → {{"ωn", None}, {"ξ", "Region of allowed ξ and ωn"}}, BaseStyle → 14, Epilog → {
    {Dashed, Line[{{0.3578, 0}, {0.3578, 35}}], Line[{{0, 12.8}, {1, 12.8}}]},
    Text["ξ=0.358", {0.358, 0}],
    Text["ωn=12.8", {0.084, 14.9}]
   }]
```

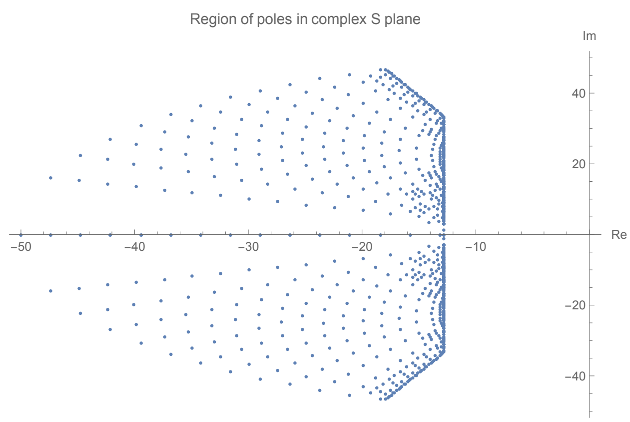
Out[69]=




By taking each point $(\zeta, \omega_n)$ from the above plot, then the pole location with coordinates $-\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$ is generated. The following shows the final result, showing the region where the poles have to be located in order to meet the performance requirements.

```
In[456]:= p = Normal@RegionPlot[1 > z > 0.35787 && z w > 12.8, {z, 0, 1}, {w, 0, 50}];
pts = DeleteDuplicates@Flatten[Cases[p, Polygon[x_] :> x, Infinity], 1];
data = {-First@# Last@#, Last@# Sqrt[1 - (First@#) ^2]} & /@ pts;
data2 = {-First@# Last@#, -Last@# Sqrt[1 - (First@#) ^2]} & /@ pts;
ListPlot[Union[data, data2], AxesOrigin → {0, 0}, AxesLabel → {"Re", "Im"} (*AxesLabel→{"-ξ ωn","ωn √1-ξ² "}*), PlotLabel → "Region of poles in
 ImageSize → 500]
```

Out[460]=

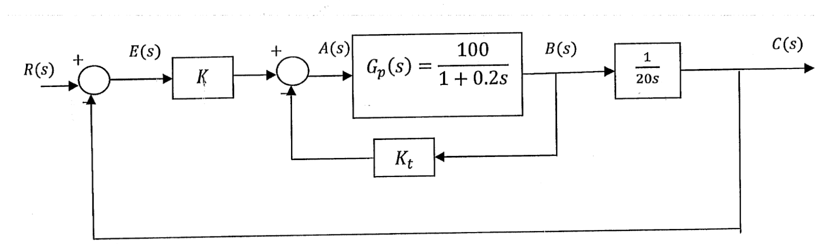



The above diagram shows the location of each pair of poles as a small dot. Complex poles come in pair of conjugates. One pole will be above the real axis and its pair below the real axis.

## 0.2 Problem 2

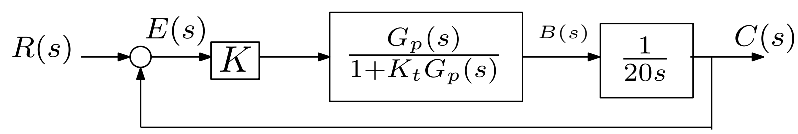
**Problem 2:** For the system shown in the figure below, find gains $K$ and $K_t$ so that the maximum overshoot of the output $c(t)$ to a unit step input $r(t)$ is about 20% and that the rise time is approximately 0.05 seconds. Find the resulting closed loop transfer function $C(s)/R(s)$ and simulate the step response in Matlab. In your solution, use the rise time approximation

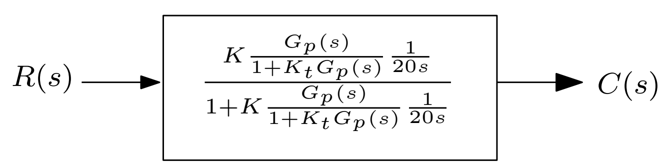
$$t_r \approx \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n}.$$

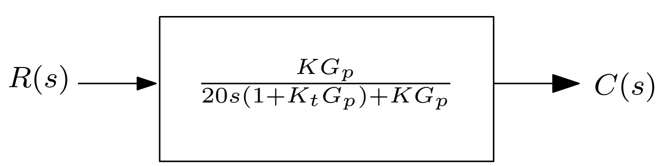



SOLUTION:

The closed loop transfer function, in terms of $K$ and $K_t$ can be found using either Mason rule or simple block reduction. For this problem block reduction seems easier.



Using $G_p = \frac{100}{1+0.2s}$ the above becomes

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{K\left(\frac{100}{1+0.2s}\right)}{20s\left(1+K_t\frac{100}{1+0.2s}\right)+K\frac{100}{1+0.2s}}\\ &= \frac{100K}{20s(1+0.2s+100K_t)+100K}\\ &= \frac{100K}{4s^2+(20+2000K_t)s+100K}\\ &= \frac{25K}{s^2+(5+500K_t)s+25K}\end{aligned}$$

The standard form is $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ therefore by comparing to the above we find

$$\begin{aligned}\omega_n^2 &= 25K\\ \omega_n &= 5\sqrt{K} \qquad (1)\end{aligned}$$

And

$$\begin{aligned}5+500K_t &= 2\zeta\omega_n\\ &= 2\zeta\left(5\sqrt{K}\right)\\ \zeta &= \frac{5+500K_t}{10\sqrt{K}} \qquad (2)\end{aligned}$$

Hence the transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Where $\omega_n = 5\sqrt{K}$ and $\zeta = \frac{5+500K_t}{10\sqrt{K}}$. We now apply the user specifications in order to determine $K$ and $K_t$. From the overshoot requirement, we write

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \leq 0.2 \qquad (3)$$

And from the rise time requirements we have

$$\frac{1-0.4167\zeta+2.917\zeta^2}{\omega_n} = 0.05 \qquad (4)$$

From (3) and (4) we can now solve for $\omega_n$ and $\zeta$ and this allow us to find $K$ and $K_t$ by using

(1,2). From (3), taking logs and solving for $\zeta$ gives

$$
\begin{aligned}
-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\leq \ln(0.2)\\
\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\geq \ln(5)\\
\frac{\pi\zeta}{\sqrt{1-\zeta^2}} &\geq 1.6094\\
\pi^2\zeta^2 &\geq (1.6094)^2\left(1-\zeta^2\right)\\
\pi^2\zeta^2 &\geq 2.5902 - 2.5902\zeta^2\\
\left(\pi^2+2.5902\right)\zeta^2 &\geq 2.5902\\
\zeta &\geq \sqrt{\frac{2.5902}{\left(\pi^2+2.5902\right)}}
\end{aligned}
$$

Hence

$$\zeta \geq 0.456$$

Any $0.456 \leq \zeta < 1$ can be used. In order to find $\omega_n$, let us choose

$$\boxed{\zeta = 0.46}$$

For the rest of the calculations . From (4) we find

$$
\begin{aligned}
\frac{1-0.4167(0.46)+2.917(0.46)^2}{\omega_n} &= 0.05\\
\omega_n &= \frac{1.4256}{0.05}
\end{aligned}
$$

Therefore

$$\boxed{\omega_n = 28.512 \text{ rad/sec}}$$

Now that we found $\zeta$ and $\omega_n$ , we use (1,2) to find the gains. From (1)

$$
\begin{aligned}
\omega_n &= 5\sqrt{K}\\
K &= \frac{\omega_n^2}{25} = \frac{28.512^2}{25}
\end{aligned}
$$

Therefore

$$\boxed{K = 32.517}$$

And from (2)

$$
\begin{aligned}
\zeta &= \frac{5+500K_t}{10\sqrt{K}}\\
0.46 &= \frac{5+500K_t}{10\sqrt{32.517}}
\end{aligned}
$$

Hence

$$\boxed{K_t = 0.04246}$$

The final transfer function is

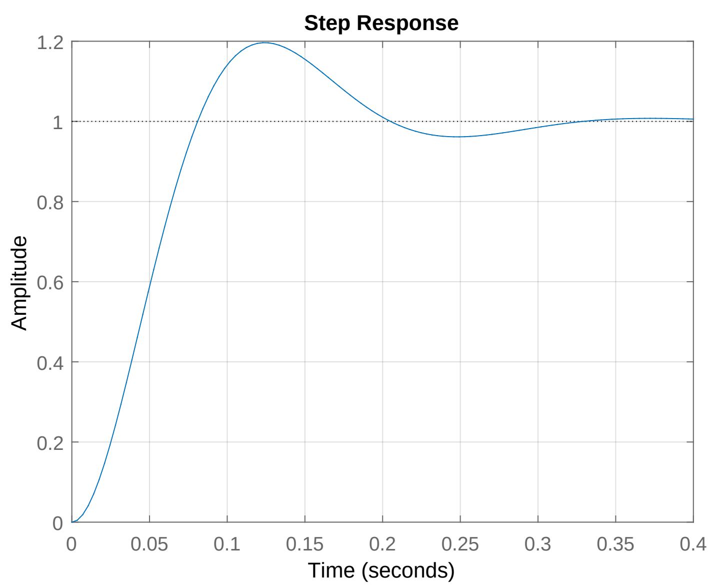
$$\frac{C(s)}{R(s)} = \frac{25K}{s^2 + (5 + 500K_t)\,s + 25K}$$
$$= \frac{25\,(32.517)}{s^2 + (5 + 500\,(0.04246))\,s + 25\,(32.517)}$$

Or

$$\boxed{\frac{C(s)}{R(s)} = \frac{812.93}{s^2+26.23s+812.93}}$$

Matlab is used to simulate the step response, and to also verify the user requirements are met.

```
close all;
clear all;
s   = tf('s');
sys = 812.93/(s^2+26.23*s+812.93);
step(sys)
grid
```




The step information was also verified using the command stepinfo() which showed the specifications was indeed met.

```
stepinfo(sys)

        RiseTime: 0.0549
    SettlingTime: 0.2916
```

```
SettlingMin: 0.9225
SettlingMax: 1.1963
  Overshoot: 19.6310
 Undershoot: 0
       Peak: 1.1963
   PeakTime: 0.1229
```

## 0.3 Problem 3

**Problem 3:** A system with adjustable gain $K$ and open loop transfer function

$$G(s) = \frac{Ks(20s^2+1)}{s^4+5s^2+10s+15}$$

connected in a classical unity feedback configuration. Use the Routh-Hurwitz criterion to find the range of $K$ for which closed loop stability is assured.

SOLUTION:

$$G(s) = \frac{Ks\left(20s^2+1\right)}{s^4+5s^2+10s+15}$$

In classical unity feedback, the closed loop transfer function $T(s)$ is

$$\begin{aligned} T(s) &= \frac{G(s)}{1+G(s)} \\ &= \frac{Ks\left(20s^2+1\right)}{\left(s^4+5s^2+10s+15\right)+Ks\left(20s^2+1\right)} \\ &= \frac{Ks\left(20s^2+1\right)}{s^4+20Ks^3+5s^2+(10+K)s+15} \end{aligned}$$

Applying Routh-Hurwitz to the denominator $D(s) = s^4+20Ks^3+5s^2+(10+K)s+15$ gives

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $5$ | $15$ |
| $s^3$ | $20K$ | $(10+K)$ | $0$ |
| $s^2$ | $\frac{20K(5)-(10+K)}{20K}$ | $15$ | $0$ |
| $s^1$ | $\frac{\frac{(20K(5)-(10+K))}{20K}(10+K)-20K(15)}{\frac{20K(5)-(10+K)}{20K}}$ | $0$ | $0$ |
| $s^0$ | $15$ | | |

Simplifying gives

| | | | |
|---|---|---|---|
| $s^4$ | $1$ | $5$ | $15$ |
| $s^3$ | $20K$ | $(10+K)$ | $0$ |
| $s^2$ | $\frac{1}{20K}(99K-10)$ | $15$ | $0$ |
| $s^1$ | $\frac{1}{10-99K}\left(5901K^2-980K+100\right)$ | $0$ | $0$ |
| $s^0$ | $15$ | | |

For stability, we need the first column to be positive. Hence the conditions are

$$20K > 0$$
$$\frac{1}{20K}(99K - 10) > 0$$
$$\frac{1}{10 - 99K}\left(5901K^2 - 980K + 100\right) > 0$$

The first just says that $K > 0$. The second says $99K - 10 > 0$ or $K > \frac{10}{99}$. Now for the third condition

$$\frac{1}{10 - 99K}\left(5901K^2 - 980K + 100\right) > 0$$

Since $K > \frac{10}{99}$ is required, then $\frac{1}{10-99K}$ is negative quantity since $10 - 99K$ is negative for $K > \frac{10}{99}$. This means the above becomes

$$5901K^2 - 980K + 100 < 0$$

Notice the change of inequality from > to < since we multiplied both sides by a negative quantity $(10 - 99K)$ to cancel it out. But $5901K^2 - 980K + 100 < 0$ can not be satisfied with a positive $K > \frac{10}{99}$. For example, using the minimum allowed $K$ which is $\frac{10}{99}$, then the value of $5901K^2 - 980K + 100$ becomes

$$5901\left(\frac{10}{99}\right)^2 - 980\left(\frac{10}{99}\right) + 100 = 61.218$$

But it needs to be negative. So *there does not exist K which makes the closed loop stable.*

## 0.4 Problem 4

**Problem 4:** Consider the system with open loop transfer function

$$G(s) = \frac{K}{s(1+Ts)}$$

is connected in a unity feedback configuration. Given $a > 0$, the specification is that all closed loop poles have real part less than $-a$. Show how the Routh-Hurwitz criterion can modified to address this problem. Subsequently, for $T = 1$, find and sketch the region in the $(a, K)$ plane associated with closed loop stability.

SOLUTION:

Given $G(s) = \frac{K}{s(1+Ts)}$ then the closed loop transfer function is

$$\begin{aligned} G_{closed}(s) &= \frac{G}{1+G} \\ &= \frac{K}{s(1+Ts)+K} \\ &= \frac{K}{Ts^2+s+K} \end{aligned}$$

Therefore $D(s) = Ts^2 + s + K$. For the closed loop poles with real part to be less than $-a$, let $s_1 = s + a$. Then $s = s_1 - a$. We apply Routh-Hurwitz to $D(s)$ but with $s = s_1 - a$. The new denominator polynomial becomes

$$D(s_1) = T(s_1 - a)^2 + (s_1 - a) + K$$

Expanding gives

$$\begin{aligned} D(s_1) &= T\left(s_1^2 + a^2 - 2s_1 a\right) + s_1 - a + K \\ &= Ts_1^2 + s_1(1 - 2Ta) + \left(Ta^2 - a + K\right) \end{aligned}$$

Routh table applied to the above polynomial is

| | | |
|---|---|---|
| $s_1^2$ | $T$ | $Ta^2 - a + K$ |
| $s_1^1$ | $1 - 2Ta$ | $0$ |
| $s_1^0$ | $Ta^2 - a + K$ | |

We need all entries in the first column to be same sign (positive in this case, since $T = 1$) for stability to hold (This is in addition to having the poles be with real part less than $-a$). For $T = 1$ the above becomes

| | | |
|---|---|---|
| $s^2$ | $1$ | $a^2 + a + K$ |
| $s^1$ | $1 - 2a$ | $0$ |
| $s^0$ | $a^2 - a + K$ | |

The conditions for stability are

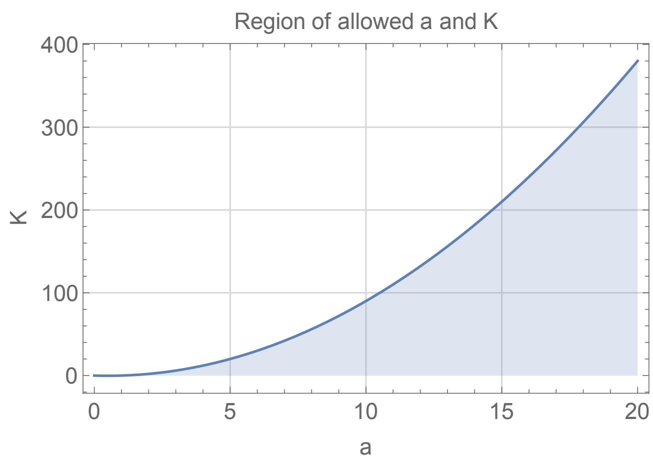
$$1 - 2a > 0$$
$$a(1 - a) + K > 0$$

The first condition gives $a > \frac{1}{2}$. The second condition gives

$$\boxed{K > a^2 - a}$$

Here is plot of the region in the $(a, K)$ plane associated with closed loop stability.

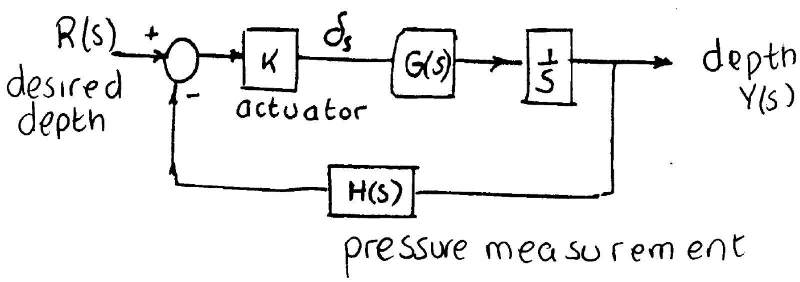
Out[1]=



## 0.5 Problem 5

**Problem 5:** An automatic depth control system for a submarine is depicted in the figure below. The depth is measured by a pressure transducer. For what values of $K$ will the systern be stable? Take $H(s) = 1$ and submarine transfer function

$$G(s) = \frac{(s+2)^2}{s^2 + 0.01}$$




SOLUTION:

The closed loop transfer function is

$$T(s) = \frac{KG(s)\frac{1}{s}}{1 + HKG(s)\frac{1}{s}}$$

Replacing $H(s) = 1$ and $G = \frac{(s+2)^2}{s^2+0.01}$ the above becomes

$$\begin{aligned} T(s) &= \frac{\frac{K(s+2)^2}{s^2+0.01}\frac{1}{s}}{1 + \frac{K(s+2)^2}{s^2+0.01}\frac{1}{s}} \\ &= \frac{K(s+2)^2}{s\left(s^2+0.01\right) + K(s+2)^2} \\ &= \frac{K(s+2)^2}{s^3 + Ks^2 + (0.01+4K)s + 4K} \end{aligned}$$

The Routh table for $D(s) = s^3 + Ks^2 + (0.01+4K)s + 4K$ is

| | | |
|---|---|---|
| $s^3$ | $1$ | $0.01+4K$ |
| $s^2$ | $K$ | $4K$ |
| $s^1$ | $4K-3.99$ | $0$ |
| $s^0$ | $4K$ | |

Therefore for stability we need

$$
\begin{aligned}
K &> 0\\
4K &> 3.99\\
4K &> 0
\end{aligned}
$$

The first and the third conditions give $K > 0$. From the second condition, $K > \frac{3.99}{4} = 0.9975$. Therefore

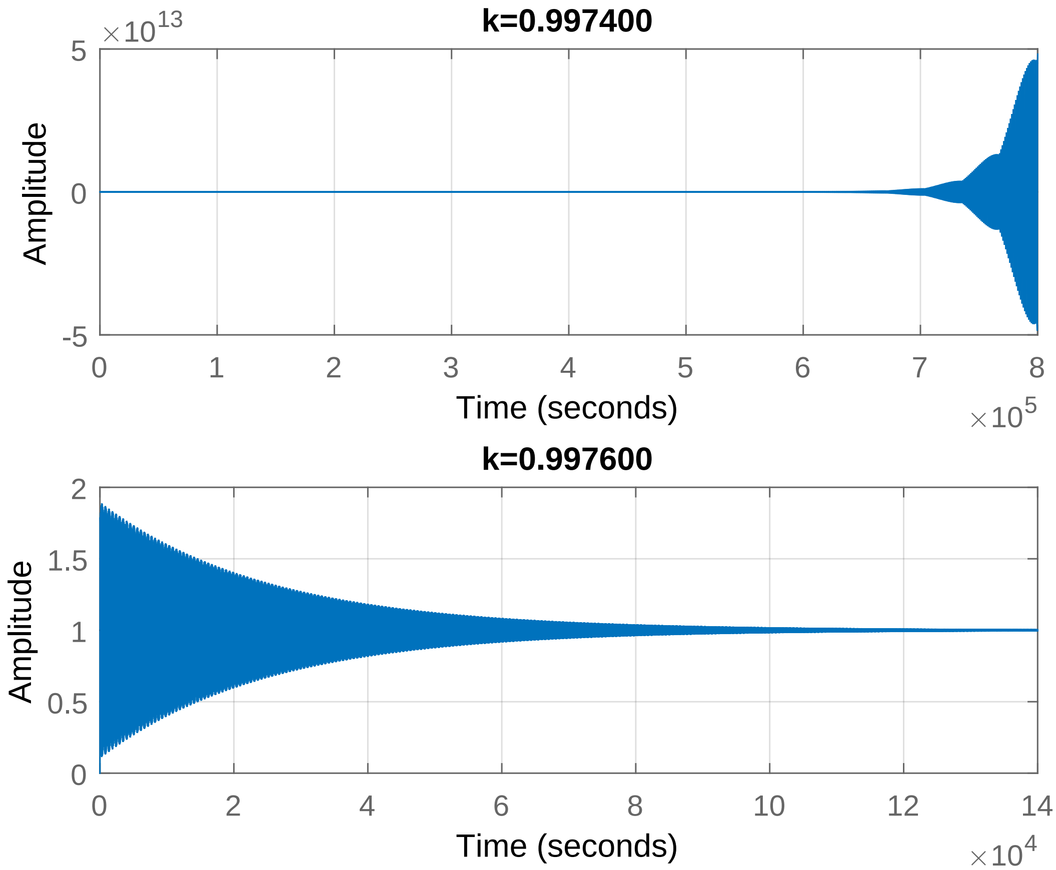
$$\boxed{K > 0.9975}$$

To verify, here is the step response for $k = 0.9974$ and $k = 0.9976$, showing one is unstable and the second is stable.

```
close all; clear all;
s   = tf('s');
G   = (s+2)^2/(s^2+0.01);

k   = .9974;
sys = feedback(k*G*1/s,1);
subplot(2,1,1);
step(sys);
title(sprintf('k=%f',k)); grid

subplot(2,1,2);
k   = .9976;
sys = feedback(k*G*1/s,1);
step(sys);
title(sprintf('k=%f',k)); grid
```


k=0.997400
×10^13
5
0
-5
Amplitude
0
1
2
3
4
5
6
7
8
Time (seconds)
×10^5
k=0.997600
2
1.5
1
0.5
0
Amplitude
0
2
4
6
8
10
12
14
Time (seconds)
×10^4