HW3 ECE 332 Feedback Control

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0.1 Problem 1

1. Consider the system with transfer function

$$G(s) = \frac{Ks+1}{s^3+s^2+(K+k)s+1}.$$

(a) For $K=1$, find the sensitivity S_k^G of this transfer function with respect to k assuming nominal value $k=2$. Then plot is magnitude as a function of frequency.

(b) Repeat (a) with $K=100$ and compare the effect of a large loop gain on the sensitivity.

SOLUTION:

$$G(s) = \frac{Ks+1}{s^3+s^2+(K+k)s+1}$$

0.1.1 Part (a)

For $K=1$ the above becomes

$$G(s) = \frac{s+1}{s^3+s^2+(1+k)s+1}$$

Hence

$$\begin{aligned}
S_k^G &= \frac{dG}{dk}\frac{k}{G} \\
&= \frac{d}{dk}\frac{s+1}{s^3+s^2+(1+k)s+1}\left(\frac{k}{\frac{s+1}{s^3+s^2+(1+k)s+1}}\right) \\
&= \frac{-s(s+1)}{\left(s+ks+s^2+s^3+1\right)^2}\frac{k\left(s^3+s^2+(1+k)s+1\right)}{s+1} \\
&= \frac{-ks}{1+(1+k)s+s^2+s^3}
\end{aligned}$$

At nominal $k=2$ the above becomes

$$S_k^G\Big|_{k=2} = \frac{-2s}{3s+s^2+s^3+1}$$

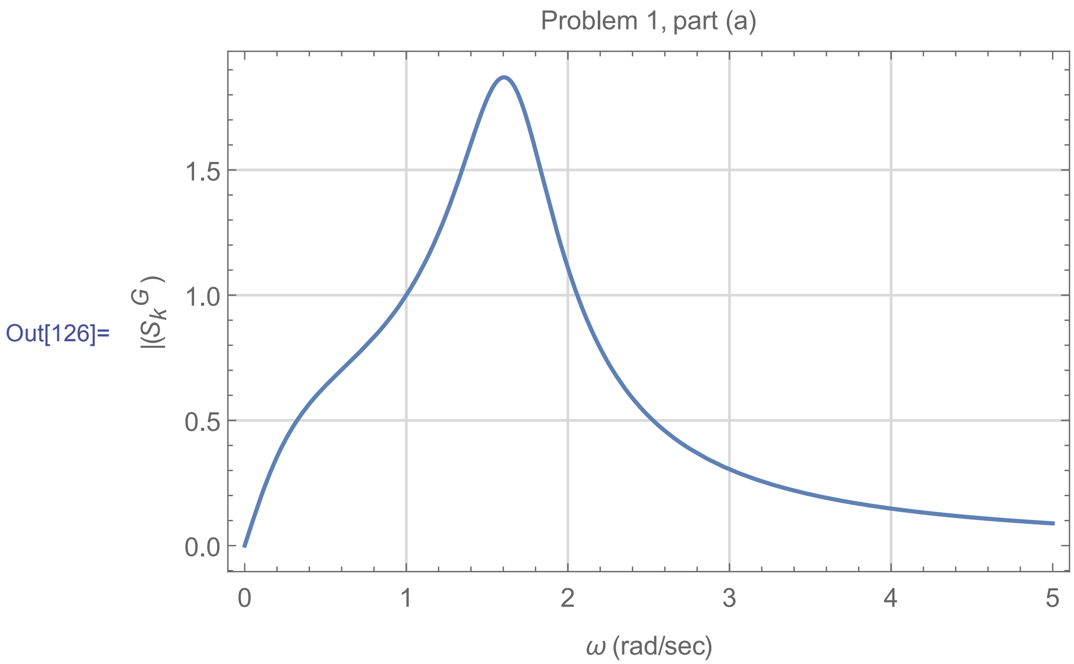
Let $s=j\omega$ then

$$S_k^G = \frac{-2(j\omega)}{3j\omega+(j\omega)^2+(j\omega)^3+1} = \frac{-2j\omega}{j\left(3\omega-\omega^3\right)+\left(1-\omega^2\right)}$$

Taking the magnitude

$$\left|S_k^G\right| = \frac{2\omega}{\sqrt{\left(1-\omega^2\right)^2 + \left(3\omega - \omega^3\right)^2}}$$

Here is a plot of $\left|S_k^G\right|$ as function of ω



0.1.2 Part (b)

For $K = 100$ the transfer function becomes

$$\begin{aligned} G(s) &= \frac{Ks+1}{s^3+s^2+(K+k)s+1} \\ &= \frac{100s+1}{s^3+s^2+(100+k)s+1} \end{aligned}$$

Hence

$$\begin{aligned} S_k^G &= \frac{dG}{dk}\frac{k}{G} \\ &= \frac{d}{dk}\frac{100s+1}{s^3+s^2+(100+k)s+1}\left(\frac{k}{\frac{100s+1}{s^3+s^2+(100+k)s+1}}\right) \\ &= \frac{-s(100s+1)}{\left(100s+ks+s^2+s^3+1\right)^2}\frac{k\left(s^3+s^2+(100+k)s+1\right)}{100s+1} \\ &= \frac{-ks}{100s+ks+s^2+s^3+1} \end{aligned}$$

At nominal $k=2$ the above becomes

$$
\begin{aligned}
S_k^G\Big|_{k=2} &= \frac{-2s}{100s+2s+s^2+s^3+1}\\
&= \frac{-2s}{102s+s^2+s^3+1}
\end{aligned}
$$

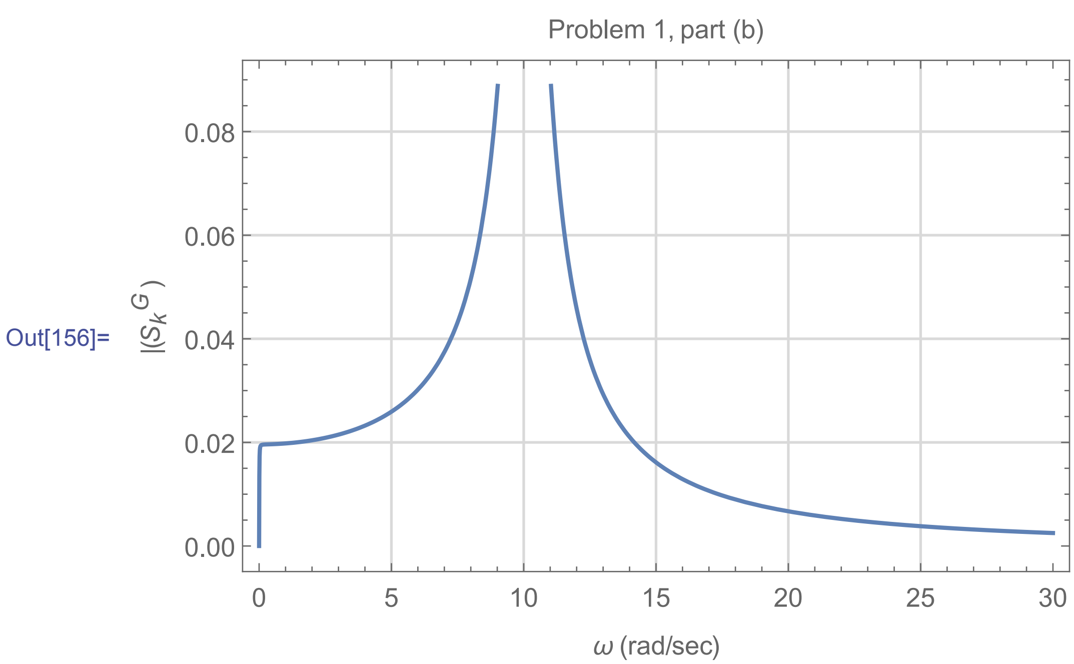
Let $s=j\omega$ then

$$
\begin{aligned}
S_k^G &= \frac{-2\left(j\omega\right)}{102j\omega+\left(j\omega\right)^2+\left(j\omega\right)^3+1}\\
&= \frac{-2j\omega}{102j\omega-\omega^2-j\omega^3+1}\\
&= \frac{-2j\omega}{j\left(102\omega-\omega^3\right)+\left(1-\omega^2\right)}
\end{aligned}
$$

Taking the magnitude

$$
\left|S_k^G\right| = \frac{2\omega}{\sqrt{\left(1-\omega^2\right)^2+\left(102\omega-\omega^3\right)^2}}
$$

Here is a plot of $\left|S_k^G\right|$ as function of ω



We clearly see that as K became much larger, resonance occurs near $\omega=10$. This shows that sensitivity of transfer function to changes in k, depends on the value of K.

0.2 Problem 2

2. In many instances, steady state errors in control system are due to some non-linearities such as dead zones. A 'Class B' amplifier is a typical example of a device having a dead zone where it takes ~1 Volt of input signal to turn on the transistor. Once the device is on, however, it can be assumed to function linearly. The Class B amplifier in Figure 1 can be characterized mathematically (assuming a 1V threshold voltage and $V_{CC} = \infty$) by $N(\cdot)$ which is given by:

$$Y = \begin{cases} 0 & \text{if } -1 \leq U \leq 1, \\ U-1 & \text{if } U > 1, \\ U+1 & \text{if } U < -1. \end{cases}$$

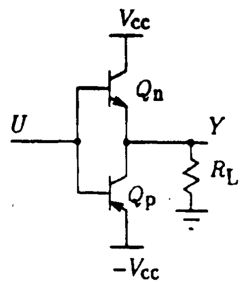


Figure 1: Class B Amplifier Circuit

(a) Develop a plot of Y versus R with $K = 1$ in Figure 2.

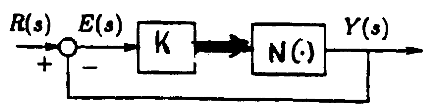


Figure 2: Feedback System

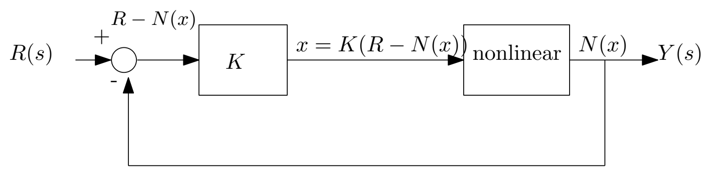
(b) If $r(t) = 5\sin(t)$, sketch $y(t)$.

(c) Now let $K = 10$ and develop a plot of $Y(s)$ versus $R(s)$. Comment on the difference in the output due to the sinusoid - consider amplitude, distortion, etc.

(d) Comment on the change to $Y(s)$ versus $R(s)$ if the gain block K were placed in the feedback loop instead of the forward path.

SOLUTION:

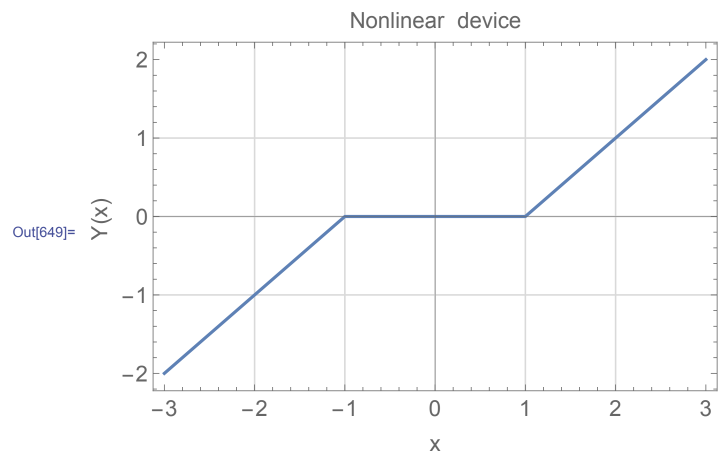
0.2.1 Part (a)



From the above we see that

$$Y = N(x) \tag{1}$$

The plot of $Y(x)$ is given below based on the definition given in the problem



At the output of the controller we have

$$\begin{aligned} x &= k(R - N(x)) \\ x &= k(R - Y) \\ x &= kR - kY \\ Y &= R - \frac{x}{k} \end{aligned} \tag{2}$$

Equations (1) and (2) must both hold. We now setup a table of R and corresponding Y values, and using $k = 1$ for this part, we obtain

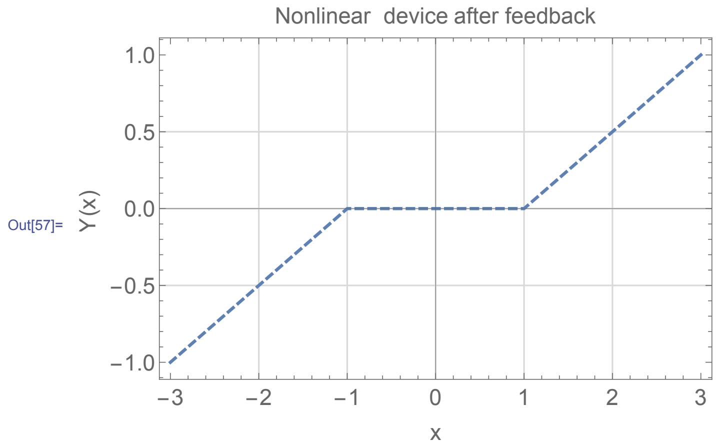
R	$Y = R - x$	solution of $Y = N(x)$	Y at solution
0	$-x$	$x = 0$	0
0.1	$0.1 - x$	$x = 0$	0
0.2	$0.2 - x$	$\vdots$	$\vdots$
see program	$\vdots$	$\vdots$	$\vdots$

Small code was written to finish the above table, using $R = -2 \cdots 2$ range with increments of 0.1. Here is the generated table, followed by the plot

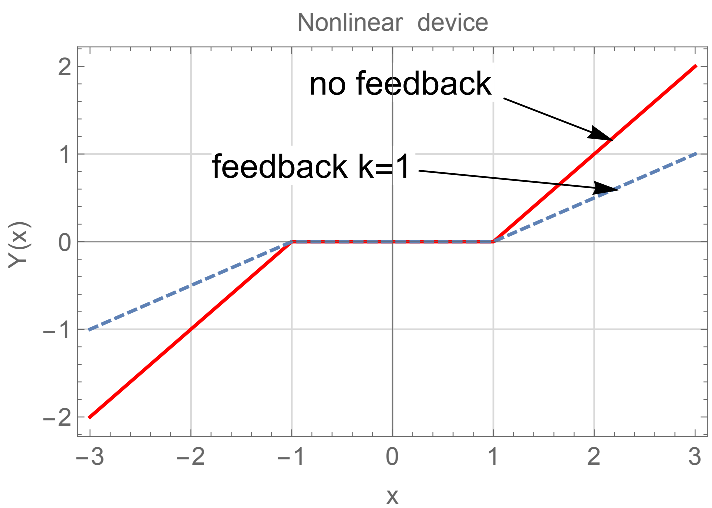
Out[181]=

r	r-x/k	solution of N(x)= r-x/k	Y at solution
-2.	-2.-x	-1.5	-0.5
-1.9	-1.9-x	-1.45	-0.45
-1.8	-1.8-x	-1.4	-0.4
-1.7	-1.7-x	-1.35	-0.35
-1.6	-1.6-x	-1.3	-0.3
-1.5	-1.5-x	-1.25	-0.25
-1.4	-1.4-x	-1.2	-0.2
-1.3	-1.3-x	-1.15	-0.15
-1.2	-1.2-x	-1.1	-0.1
-1.1	-1.1-x	-1.05	-0.05
-1.	-1.-x	-1.	0
-0.9	-0.9-x	-0.9	0
-0.8	-0.8-x	-0.8	0
-0.7	-0.7-x	-0.7	0
-0.6	-0.6-x	-0.6	0
-0.5	-0.5-x	-0.5	0
-0.4	-0.4-x	-0.4	0
-0.3	-0.3-x	-0.3	0
-0.2	-0.2-x	-0.2	0
-0.1	-0.1-x	-0.1	0
0.	0.-x	0	0
0.1	0.1-x	0.1	0
0.2	0.2-x	0.2	0
0.3	0.3-x	0.3	0
0.4	0.4-x	0.4	0
0.5	0.5-x	0.5	0
0.6	0.6-x	0.6	0
0.7	0.7-x	0.7	0
0.8	0.8-x	0.8	0
0.9	0.9-x	0.9	0
1.	1.-x	1.	0
1.1	1.1-x	1.05	0.05
1.2	1.2-x	1.1	0.1
1.3	1.3-x	1.15	0.15
1.4	1.4-x	1.2	0.2
1.5	1.5-x	1.25	0.25
1.6	1.6-x	1.3	0.3
1.7	1.7-x	1.35	0.35
1.8	1.8-x	1.4	0.4
1.9	1.9-x	1.45	0.45
2.	2.-x	1.5	0.5

And plot of Y vs. R is below



This below is the above plot, but with the original device output without feedback, in order to better see the effect of feedback with $k = 1$.



For $k = 1$, the dead zone did not change. But the slope became small after $x = \pm 1$

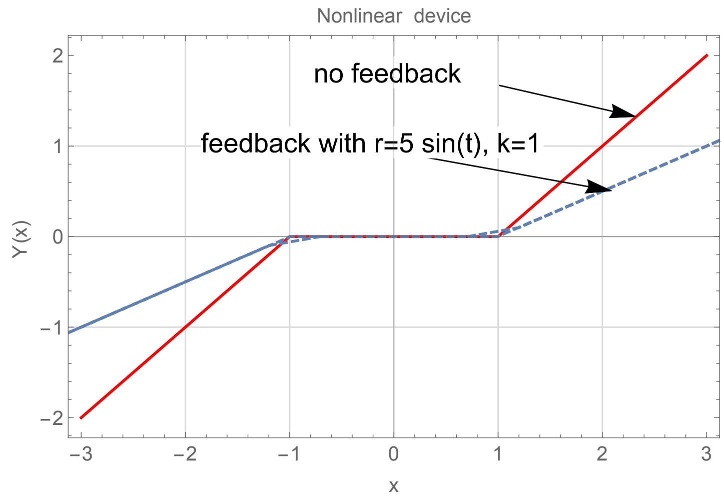
0.2.2 Part (b)

When $r(t) = 5\sin(t)$, the following table shows result for $t = -2 \dots 2$.

Out[188]=

t	r = 5 sin(t)	r-x/k	solution of N(x)=r-x/k	Y at solution
-2.	-4.54649	-4.54649-x	-2.77324	-1.77324
-1.9	-4.7315	-4.7315-x	-2.86575	-1.86575
-1.8	-4.86924	-4.86924-x	-2.93462	-1.93462
-1.7	-4.95832	-4.95832-x	-2.97916	-1.97916
-1.6	-4.99787	-4.99787-x	-2.99893	-1.99893
-1.5	-4.98747	-4.98747-x	-2.99374	-1.99374
-1.4	-4.92725	-4.92725-x	-2.96362	-1.96362
-1.3	-4.81779	-4.81779-x	-2.9089	-1.9089
-1.2	-4.6602	-4.6602-x	-2.8301	-1.8301
-1.1	-4.45604	-4.45604-x	-2.72802	-1.72802
-1.	-4.20735	-4.20735-x	-2.60368	-1.60368
-0.9	-3.91663	-3.91663-x	-2.45832	-1.45832
-0.8	-3.58678	-3.58678-x	-2.29339	-1.29339
-0.7	-3.22109	-3.22109-x	-2.11054	-1.11054
-0.6	-2.82321	-2.82321-x	-1.91161	-0.911606
-0.5	-2.39713	-2.39713-x	-1.69856	-0.698564
-0.4	-1.94709	-1.94709-x	-1.47355	-0.473546
-0.3	-1.4776	-1.4776-x	-1.2388	-0.238801
-0.2	-0.993347	-0.993347-x	-0.993347	0
-0.1	-0.499167	-0.499167-x	-0.499167	0
0.	0.	0.-x	0	0
0.1	0.499167	0.499167-x	0.499167	0
0.2	0.993347	0.993347-x	0.993347	0
0.3	1.4776	1.4776-x	1.2388	0.238801
0.4	1.94709	1.94709-x	1.47355	0.473546
0.5	2.39713	2.39713-x	1.69856	0.698564
0.6	2.82321	2.82321-x	1.91161	0.911606
0.7	3.22109	3.22109-x	2.11054	1.11054
0.8	3.58678	3.58678-x	2.29339	1.29339
0.9	3.91663	3.91663-x	2.45832	1.45832
1.	4.20735	4.20735-x	2.60368	1.60368
1.1	4.45604	4.45604-x	2.72802	1.72802
1.2	4.6602	4.6602-x	2.8301	1.8301
1.3	4.81779	4.81779-x	2.9089	1.9089
1.4	4.92725	4.92725-x	2.96362	1.96362
1.5	4.98747	4.98747-x	2.99374	1.99374
1.6	4.99787	4.99787-x	2.99893	1.99893
1.7	4.95832	4.95832-x	2.97916	1.97916
1.8	4.86924	4.86924-x	2.93462	1.93462
1.9	4.7315	4.7315-x	2.86575	1.86575
2.	4.54649	4.54649-x	2.77324	1.77324

the following is the plot of the output with the feedback for $k = 1$



Matlab code to plot the solution

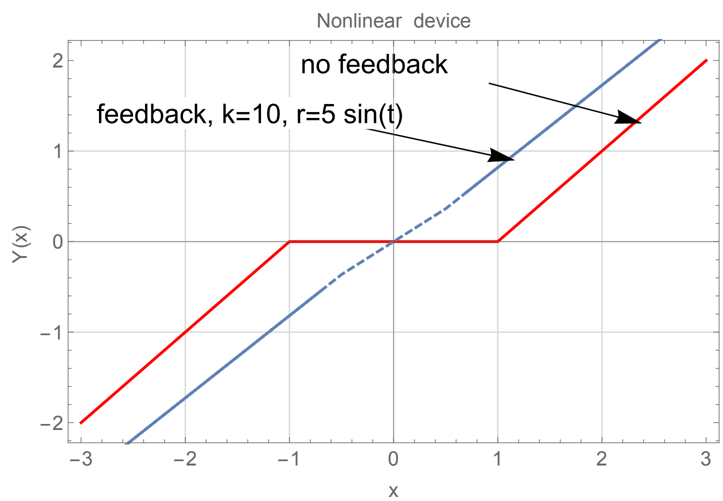
```
%matlab code to generate plot for part(b), HW3, problem 2
%ECE 332
close all; clear all;
figure
t=0:.1:10;
f=@(x) (x+1).*(x<-1)+(x-1).*(x>1)+0; %non-linear device
r = 5 *sin(t); %input
k =1; %change to 10 for second part
x = fsolve(@(x) f(x)-(r-x/k),r);

plot(t,f(x));
grid;
title('output of 5*sin(t), k=1');
```

0.2.3 Part (c)

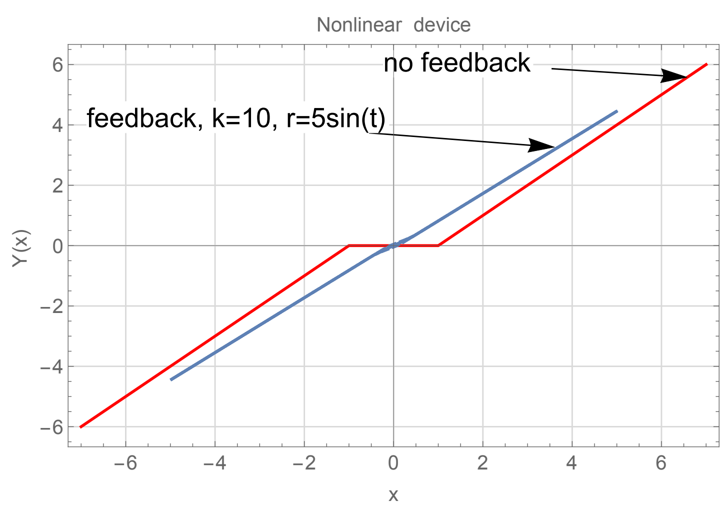
Now $k = 10$, and part(b) was repeated. the following table shows result for $t = -2 \dots 2$.

t	r = 5 sin(t)	r-x/10	solution of N(x)=r-x/10	Y at solution
-2.	-4.54649	-4.54649-x	-5.04226	-4.04226
-1.9	-4.7315	-4.7315-x	-5.21045	-4.21045
-1.8	-4.86924	-4.86924-x	-5.33567	-4.33567
-1.7	-4.95832	-4.95832-x	-5.41666	-4.41666
-1.6	-4.99787	-4.99787-x	-5.45261	-4.45261
-1.5	-4.98747	-4.98747-x	-5.44316	-4.44316
-1.4	-4.92725	-4.92725-x	-5.38841	-4.38841
-1.3	-4.81779	-4.81779-x	-5.2889	-4.2889
-1.2	-4.6602	-4.6602-x	-5.14563	-4.14563
-1.1	-4.45604	-4.45604-x	-4.96003	-3.96003
-1.	-4.20735	-4.20735-x	-4.73396	-3.73396
-0.9	-3.91663	-3.91663-x	-4.46967	-3.46967
-0.8	-3.58678	-3.58678-x	-4.1698	-3.1698
-0.7	-3.22109	-3.22109-x	-3.83735	-2.83735
-0.6	-2.82321	-2.82321-x	-3.47565	-2.47565
-0.5	-2.39713	-2.39713-x	-3.0883	-2.0883
-0.4	-1.94709	-1.94709-x	-2.67917	-1.67917
-0.3	-1.4776	-1.4776-x	-2.25236	-1.25236
-0.2	-0.993347	-0.993347-x	-1.81213	-0.812133
-0.1	-0.499167	-0.499167-x	-1.36288	-0.362879
0.	0.	0.-x	0	0
0.1	0.499167	0.499167-x	1.36288	0.362879
0.2	0.993347	0.993347-x	1.81213	0.812133
0.3	1.4776	1.4776-x	2.25236	1.25236
0.4	1.94709	1.94709-x	2.67917	1.67917
0.5	2.39713	2.39713-x	3.0883	2.0883
0.6	2.82321	2.82321-x	3.47565	2.47565
0.7	3.22109	3.22109-x	3.83735	2.83735
0.8	3.58678	3.58678-x	4.1698	3.1698
0.9	3.91663	3.91663-x	4.46967	3.46967
1.	4.20735	4.20735-x	4.73396	3.73396
1.1	4.45604	4.45604-x	4.96003	3.96003
1.2	4.6602	4.6602-x	5.14563	4.14563
1.3	4.81779	4.81779-x	5.2889	4.2889
1.4	4.92725	4.92725-x	5.38841	4.38841
1.5	4.98747	4.98747-x	5.44316	4.44316
1.6	4.99787	4.99787-x	5.45261	4.45261
1.7	4.95832	4.95832-x	5.41666	4.41666
1.8	4.86924	4.86924-x	5.33567	4.33567
1.9	4.7315	4.7315-x	5.21045	4.21045
2.	4.54649	4.54649-x	5.04226	4.04226

And the following is the plot of the result



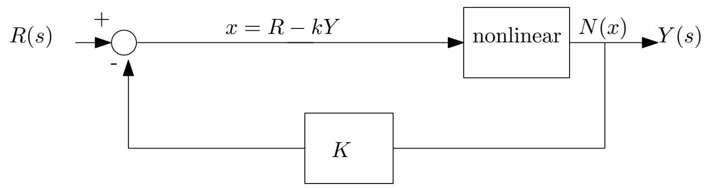
The above plot shows that with $k = 10$, the dead zone has shrunk to almost zero, and the output of the nonlinear device is now linear. This is good. This is another plot, for larger range of input values.



As range of input values become large, the output of the feedback linear device approaches the open loop device output. This is outside the dead zone region as can be seen from the above. Very close to the origin, there is very small non-linearity remains, but it is hard to see.

0.2.4 part (d)

When the gain k is in the feedback loop, as shown in the following diagram



Therefore

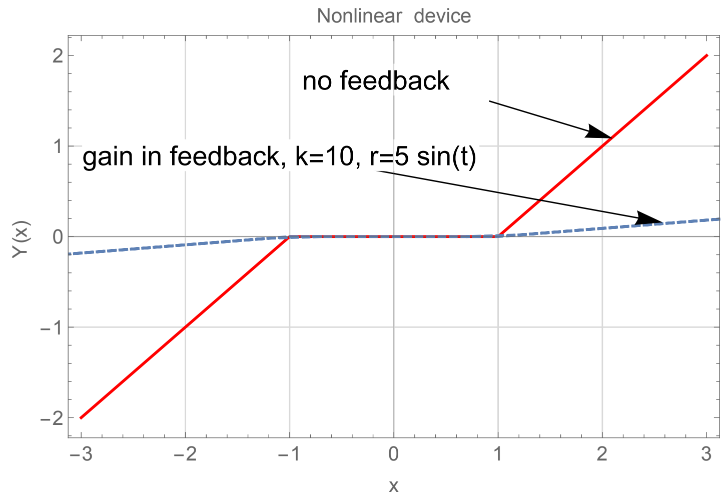
$$x = R - kY$$
$$Y = \frac{(R - x)}{k}$$

Before, when the gain was in the feedforward, $Y = R - \frac{x}{k}$, so now k affects R as well. Using this new x, the above plot was reproduced for the case of $r(t) = 5\sin(t)$.

the following table shows result for $t = -2 \dots 2$.

t	r = 5 sin(t)	(r-x)/10	solution of N(x)=(r-x)/10	Y at solution
-2.	-4.54649	(-4.54649-x)/10	-1.32241	-0.322408
-1.9	-4.7315	(-4.7315-x)/10	-1.33923	-0.339227
-1.8	-4.86924	(-4.86924-x)/10	-1.35175	-0.351749
-1.7	-4.95832	(-4.95832-x)/10	-1.35985	-0.359848
-1.6	-4.99787	(-4.99787-x)/10	-1.36344	-0.363443
-1.5	-4.98747	(-4.98747-x)/10	-1.3625	-0.362498
-1.4	-4.92725	(-4.92725-x)/10	-1.35702	-0.357023
-1.3	-4.81779	(-4.81779-x)/10	-1.34707	-0.347072
-1.2	-4.6602	(-4.6602-x)/10	-1.33275	-0.332745
-1.1	-4.45604	(-4.45604-x)/10	-1.31419	-0.314185
-1.	-4.20735	(-4.20735-x)/10	-1.29158	-0.291578
-0.9	-3.91663	(-3.91663-x)/10	-1.26515	-0.265149
-0.8	-3.58678	(-3.58678-x)/10	-1.23516	-0.235162
-0.7	-3.22109	(-3.22109-x)/10	-1.20192	-0.201917
-0.6	-2.82321	(-2.82321-x)/10	-1.16575	-0.165747
-0.5	-2.39713	(-2.39713-x)/10	-1.12701	-0.127012
-0.4	-1.94709	(-1.94709-x)/10	-1.0861	-0.0860992
-0.3	-1.4776	(-1.4776-x)/10	-1.04342	-0.0434183
-0.2	-0.993347	(-0.993347-x)/10	-0.993347	0
-0.1	-0.499167	(-0.499167-x)/10	-0.499167	0
0.	0.	(0.-x)/10	0	0
0.1	0.499167	(0.499167-x)/10	0.499167	0
0.2	0.993347	(0.993347-x)/10	0.993347	0
0.3	1.4776	(1.4776-x)/10	1.04342	0.0434183
0.4	1.94709	(1.94709-x)/10	1.0861	0.0860992
0.5	2.39713	(2.39713-x)/10	1.12701	0.127012
0.6	2.82321	(2.82321-x)/10	1.16575	0.165747
0.7	3.22109	(3.22109-x)/10	1.20192	0.201917
0.8	3.58678	(3.58678-x)/10	1.23516	0.235162
0.9	3.91663	(3.91663-x)/10	1.26515	0.265149
1.	4.20735	(4.20735-x)/10	1.29158	0.291578
1.1	4.45604	(4.45604-x)/10	1.31419	0.314185
1.2	4.6602	(4.6602-x)/10	1.33275	0.332745
1.3	4.81779	(4.81779-x)/10	1.34707	0.347072
1.4	4.92725	(4.92725-x)/10	1.35702	0.357023
1.5	4.98747	(4.98747-x)/10	1.3625	0.362498
1.6	4.99787	(4.99787-x)/10	1.36344	0.363443
1.7	4.95832	(4.95832-x)/10	1.35985	0.359848
1.8	4.86924	(4.86924-x)/10	1.35175	0.351749
1.9	4.7315	(4.7315-x)/10	1.33923	0.339227
2.	4.54649	(4.54649-x)/10	1.32241	0.322408

And the following is the plot



We see the effect of reducing R (since it is now divided by $k > 1$) and the output from the non-linear device is not as good as when the gain was in the feedforward. The dead zone has returned back and the output after the dead zone is much smaller in amplitude than the original open loop output. Putting the gain in the feedback loop does not appear to be a good choice in this case.

0.3 Problem 3

3. The block diagram of a feedback control system is shown in Figure 3

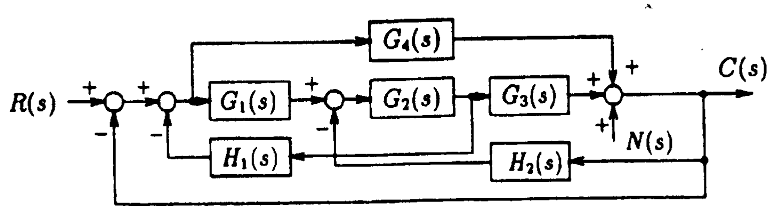


Figure 3: Block Diagram

(a) Apply Mason's gain formula to the block diagram to find the transfer functions

$$\left.\frac{C(s)}{R(s)}\right|_{N=0} \qquad \left.\frac{C(s)}{N(s)}\right|_{R=0}$$

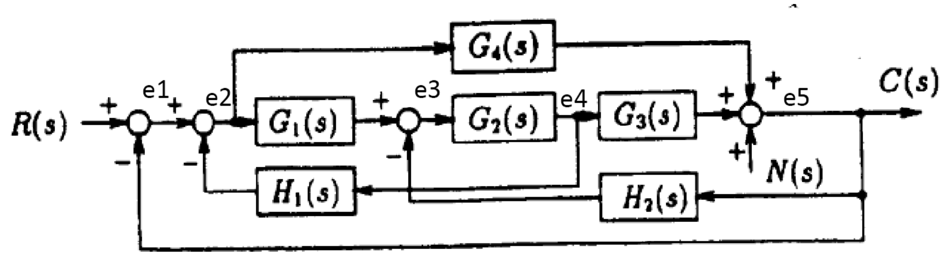
Express $C(s)$ in terms of $R(s)$ and $N(s)$ when both inputs are applied simultaneously.

(b) Find the desired relation among the transfer functions $G_1(s)$, $G_2(s)$, $G_3(s)$, $G_4(s)$, $H_1(s)$ and $H_2(s)$ so that the output $C(s)$ is not affected by the disturbance signal $N(s)$ at all.

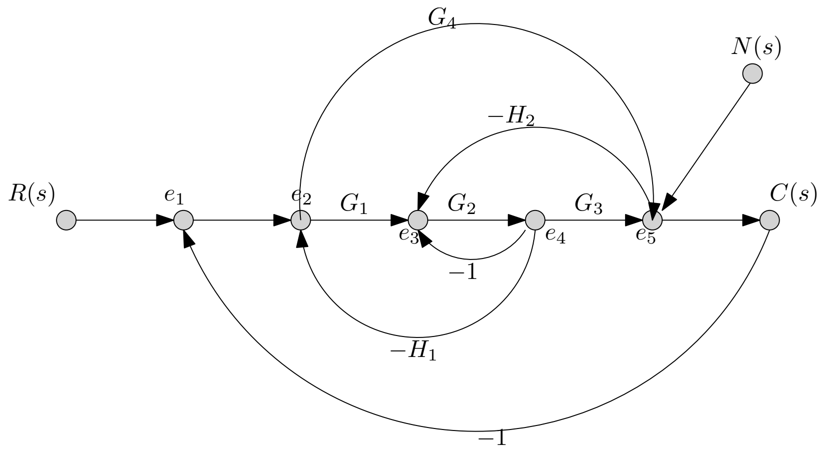
SOLUTION:

0.3.1 Part(a)

The first step is to convert the block diagram to signal flow diagram. By assigning variables as shown below, the following signal diagram we drawn



Converted to signal flow as



For finding $\frac{C(s)}{R(s)}$ then $N(s)$ is set to zero. There are two forward paths from $R(s)$ to $C(s)$ they are

$$M_1 = \{1, 1, G_1, G_2, G_3, 1\} = G_1 G_2 G_3$$
$$M_2 = \{1, 1, G_4, 1\} = G_4$$

The corresponding Mason deltas are

$$\Delta_1 = 1$$
$$\Delta_2 = 1 - (G_2)(-1) = 1 + G_2$$

The loops, one at a time are

$$\begin{aligned} L_1 &= (G_2)(-1), (G_1)(G_2)(-H_1), (G_2)(G_3)(-H_2), (1)(G_1)(G_2)(G_3)(1)(-1), (G_4)(1)(-1) \\ &= -G_2, -G_1 G_2 H_1, -H_2 G_2 G_3, -G_1 G_2 G_3, -G_4 \end{aligned}$$

Two at a time are

$$\begin{aligned} L_2 &= \{(G_2)(-1) \times (G_4)(1)(-1)(1)\} \\ &= G_2 G_4 \end{aligned}$$

Hence

$$\begin{aligned} \Delta &= 1 - \sum (-G_2 - G_1 G_2 H_1 - H_2 G_2 G_3 - G_1 G_2 G_3 - G_4) + \sum G_2 G_4 \\ &= 1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4 \end{aligned}$$

Hence

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{\sum M_i \Delta_i}{\Delta} \\ &= \frac{G_1 G_2 G_3 (1) + G_4 (1 + G_2)}{1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4} \end{aligned}$$

Hence

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_4 + G_4 G_2}{1 + G_2 + G_1 G_2 H_1 + H_2 G_2 G_3 + G_1 G_2 G_3 + G_4 + G_2 G_4}}$$

For finding $\frac{C(s)}{N(s)}$ then $R(s)$ is set to zero. There is now one forward path from $N(s)$ to $C(s)$

$$M_1 = \{1,1\} = 1$$

The corresponding Mason deltas are

$$\Delta_1 = 1 - \{(G_2)(-1) + (G_1)(G_2)(-H_1)\} = 1 + G_2 + G_1G_2H_1$$

The loops remain the same as above. Hence the mason delta do not change. Therefore

$$\begin{aligned}\frac{C(s)}{N(s)} &= \frac{\sum M_i\Delta_i}{\Delta} \\ &= \frac{(1)(1 + G_2 + G_1G_2H_1)}{1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4}\end{aligned}$$

Hence

$$\boxed{\frac{C(s)}{N(s)} = \frac{1+G_2+G_1G_2H_1}{1+G_2+G_1G_2H_1+H_2G_2G_3+G_1G_2G_3+G_4+G_2G_4}}$$

0.3.2 Part (b)

Since

$$C(s) = \frac{1 + G_2 + G_1G_2H_1}{1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4}N(s)$$

Then we want $1 + G_2 + G_1G_2H_1 = 0$ or for the denominator

$$1 + G_2 + G_1G_2H_1 + H_2G_2G_3 + G_1G_2G_3 + G_4 + G_2G_4$$

to be very large. Both of these will cause $C(s)$ to remain zero for any value of $N(s)$. But since the denominator is the same as for $\frac{C(s)}{R(s)}$ then making this very large will also affect $\frac{C(s)}{R(s)}$ which we do not want to. Hence the choice left is

$$1 + G_2 + G_1G_2H_1 = 0$$

0.4 Problem 4

Consider the signal flow graph shown in Figure 3.

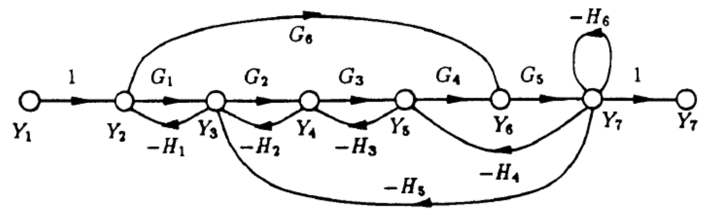


Figure 3: Signal Flow Graph

(a) Identify all the forward paths and their loop gains.

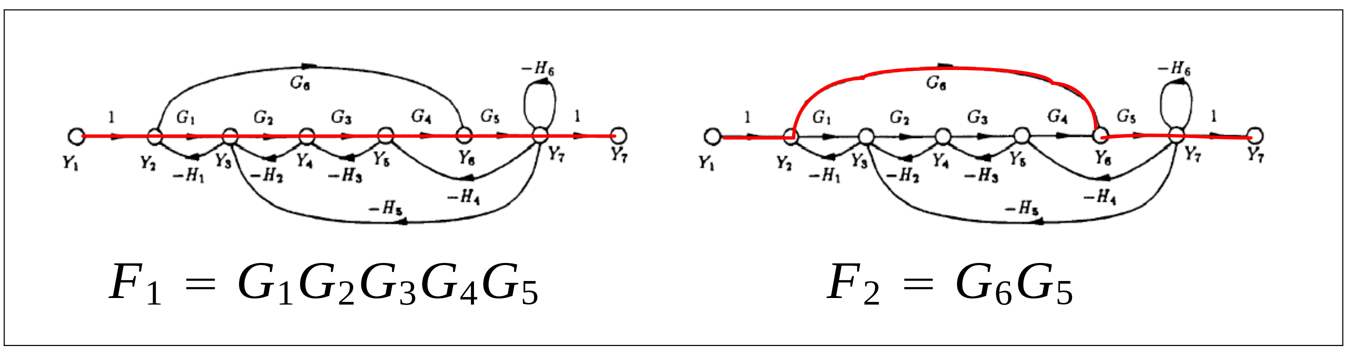
(b) Identify all the loops.

(c) Find the transfer function from Y_1 to Y_7 and from Y_1 to Y_2 using Mason's rule.

SOLUTION:

0.4.1 Part (a)

For the $\frac{Y_7}{Y_1}$, There are two forward paths. The following diagrams shows them with the gain on each.



$$F_1 = G_1G_2G_3G_4G_5$$
$$F_2 = G_6G_5$$

Now Δ_k is found for each forward loop. Δ_k is the Mason Δ but with F_k removed from the graph. Removing F_1 removes all the loops, hence

$$\Delta_1 = 1$$

When removing F_2 what remains is L_2 and L_3, hence

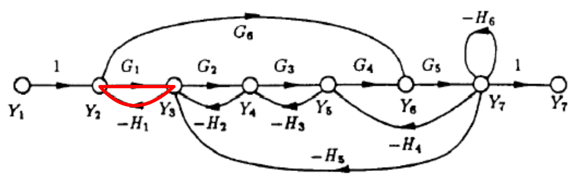
$$\begin{aligned}\Delta_2 &= 1 - (L_2 + L_3)\\ &= 1 - (-H_2G_2 - H_3G_3)\\ &= 1 + (H_2G_2 + H_3G_3)\end{aligned}$$

For the $\frac{Y2}{Y_1}$, there is one forward path $F_1 = 1$, the associated Δ_1 is

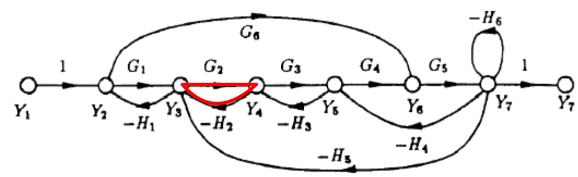
$$\begin{aligned}\Delta_1 &= 1 - \sum -G_2H_2 - G_3H_3 - G_4G_5H_4 - H_6 - G_2G_3G_4G_5H_5\\ &\quad + \sum (-G_2H_2)(-G_4G_5H_4) + (-G_2H_2)(-H_6) + (-G_3H_3)(-H_6)\\ &= 1 + \overbrace{G_2H_2 + G_3H_3 + G_4G_5H_4 + H_6 + G_2G_3G_4G_5H_5}^{\text{one at a time}} + \overbrace{G_2H_2G_4G_5H_4 + G_2H_2H_6 + G_3H_3H_6}^{\text{two at a time}}\end{aligned}$$

0.4.2 Part (b)

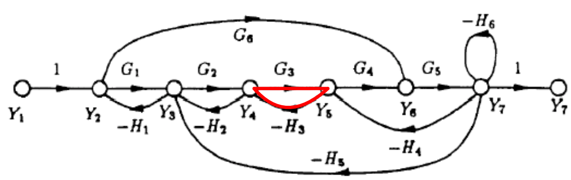
There are 8 loops. The following diagrams shows the loops with the gains



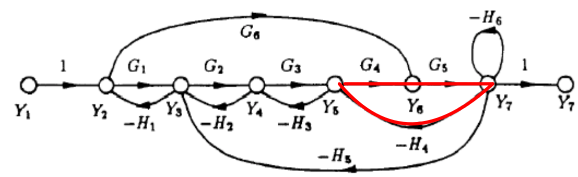
$L_1 = (G_1)(-H_1)$



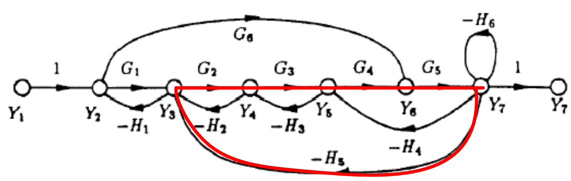
$L_2 = (G_2)(-H_2)$



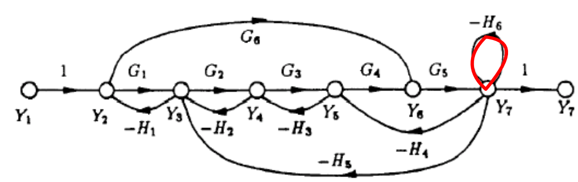
$L_3 = (G_3)(-H_3)$



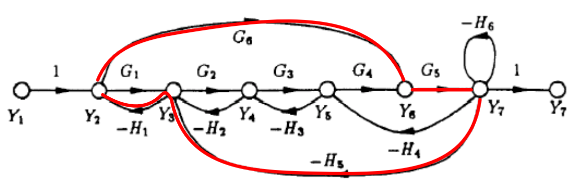
$L_4 = (G_4G_5)(-H_4)$



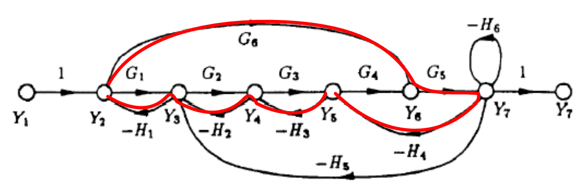
$L_5 = (G_2G_3G_4G_5)(-H_5)$



$L_6 = -H_6$



$L_7 = G_6G_5H_5H_1$



$L_8 = G_6G_5H_4H_3H_2H_1$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_1L_3 + L_1L_4 + L_1L_6 + L_2L_4 + L_2L_6 + L_3L_6 + L_3L_7) - L_1L_3L_6$$

Therefore

$$\Delta = 1 + \overbrace{H_1G_1 + H_2G_2 + H_3G_3 + H_4G_4G_5 + H_5G_2G_3G_4G_5 + H_6 - G_5G_6H_1H_5 - G_6G_5H_4H_3H_2H_1}^{\text{one at a time}} \quad (1)$$

$$+ \overbrace{(H_1G_1H_3G_3 + H_1G_1H_4G_4G_5 + H_1H_6G_1 + H_2G_2H_4G_4G_5 + H_2G_2H_6 + H_3G_3H_6 - G_3H_3G_6G_5H_5H_1)}^{\text{two at time}}$$

$$+ \overbrace{H_1G_1H_3G_3H_6}^{\text{three at time}}$$

0.4.3 Part (c)

For $G(s) = \frac{Y_7}{Y_1}$, and using result found above in part (a) and part (b)

$$\begin{aligned} G(s) &= \frac{Y_7}{Y_1} \\ &= \frac{\Delta_1 F_1 + \Delta_2 F_2}{\Delta} \\ &= \frac{(G_1 G_2 G_3 G_4 G_5) + G_6 G_5 (1 + H_2 G_2 + H_3 G_3)}{\Delta} \end{aligned}$$

Where Δ is given in (1) found in part(b). To obtain $\frac{Y_2}{Y_1}$

$$\begin{aligned} \frac{Y_2}{Y_1} &= \frac{\Delta_1 F_1}{\Delta} \\ &= \frac{1 + \overbrace{G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5}^{\text{one at a time}} + \overbrace{G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6}^{\text{two at a time}}}{\Delta} \\ &= \frac{1 + G_2 H_2 + G_3 H_3 + G_4 G_5 H_4 + H_6 + G_2 G_3 G_4 G_5 H_5 + G_2 H_2 G_4 G_5 H_4 + G_2 H_2 H_6 + G_3 H_3 H_6}{\Delta} \end{aligned}$$