HW 6

Math 703
methods of applied mathematics I

Fall 2014
University of Wisconsin, Madison

Nasser M. Abbasi

January 5, 2020 Compiled on January 5, 2020 at 11:28pm [public]

Contents

1 Problem 1 **2**
- 1.1 Part(a) transcritical bifurcation . . . 2
- 1.2 Part (b) pitchfork bifurcation . . . 2

2 Problem 2 **3**

3 Problem 4.4.6 **4**

4 Problem 4.4.7 **4**
- 4.1 Part(a) . . . 4
- 4.2 Part(b) . . . 4
- 4.3 Part(c) . . . 5

5 Problem 4.4.17 **5**
- 5.1 Part (a) . . . 5
- 5.2 Part(b) . . . 6
- 5.3 Part(c) . . . 6

6 Problem 4.4.23 **6**
- 6.1 Part(a) . . . 6

7 Problem 6.1.11 **7**
- 7.1 Part(a) . . . 7
- 7.2 Part(b) . . . 7
- 7.3 Part(c) . . . 7

8 Problem 6.1.12 **8**
- 8.1 Part(a) . . . 8
- 8.2 Part(b) . . . 8
- 8.3 Part(c) . . . 8
- 8.4 Part(d) . . . 8

9 Problem 6.2.2 **9**
- 9.1 Part(1) . . . 9
- 9.2 Part(2) . . . 9
- 9.3 Part(3) . . . 9
- 9.4 Part(4) . . . 9

10 Problem 6.2.12 **9**

11 Problem 6.2.13 **11**

12 Problem 6.2.19 **12**

12.1 Part(a) 12

12.2 Part(b) 12

12.3 Part(d) 12

1 Problem 1

Draw bifurcation diagrams for the normal form of the transcritical bifurcation: $\frac{dx}{dt} = rx - x^2$, and of the pitchfork bifurcation: $\frac{dx}{dt} = rx - x^3$

Solution:

1.1 Part(a) transcritical bifurcation

For transcritical bifurcation $\frac{dx}{dt} = f(r,x) = rx - x^2$. The critical points are $x^* = 0$ and $x^* = r$.

There are 3 cases to consider. $r = 0, r < 0$ and $r > 0$.The the vector field plot is first made, using x as the x-axis, and using x' as the y-axis.

Using Mathematica,a plot of the 3 above cases was generated

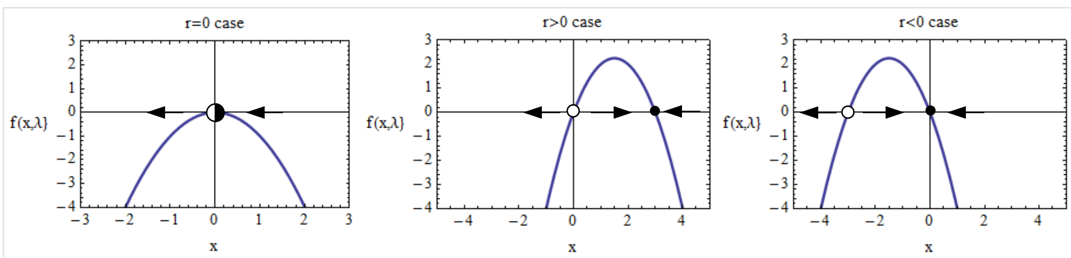

Figure 1: plot for problem 1

To plot the Bifurcation diagram, we have to now use r as the x-axis and use x for the y-axis. This was done by hand similar to what the textbook at page 50 shows.

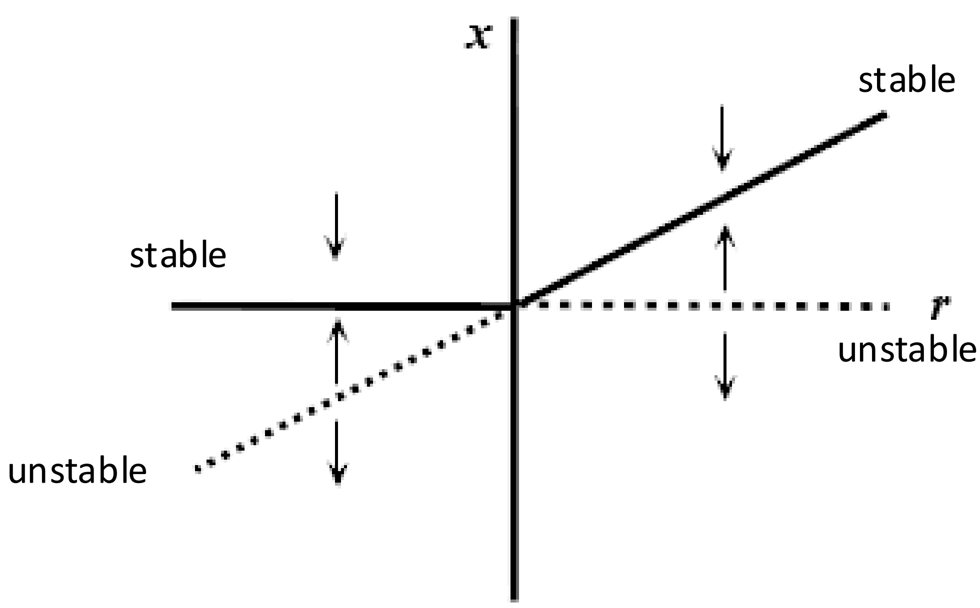

Figure 2: second plot for problem 1

1.2 Part (b) pitchfork bifurcation

$\frac{dx}{dt} = rx - x^3$. The critical points are $x\left(r - x^2\right) = 0$, hence $x^* = 0$ and $x^* = \pm\sqrt{r}$. When $r = 0$ then $x' = -x^3$. So it approaches $x = 0$ from the right and approaches $x = 0$ from the left. Hence $x^* = 0$ is stable in this case. When $r < 0$, then only $x^* = 0$ is fixed point (since we can't have complex values). So this is similar to $r = 0$ case. When $r > 0$ then there are 3 critical points now $x^* = 0, -\sqrt{r}, \sqrt{r}$. The following Bifurcation illustrates these cases (from textbook, Nonlinear Dynamics and Chaos, page 56)

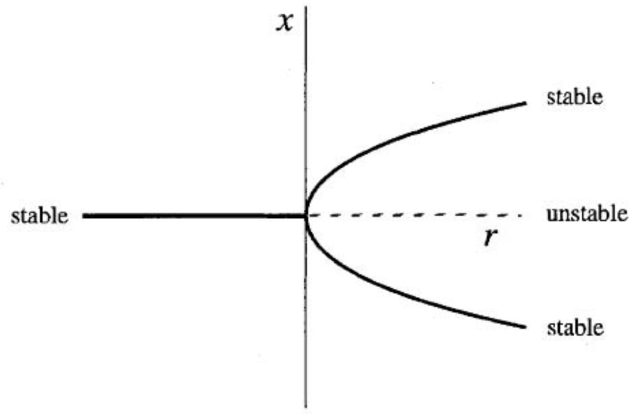

Figure 3: pitchfork bifurcation

2 Problem 2

Find a 2D dynamical system that undergoes Hopf bifurcation, and explain why the Hopf bifurcation occurs.

Solution:

Hopf bifurcation requires a minimum of 2D system to occur. Hopf bifurcation shows up when spiral changes from stable to unstable (or vice versa) with a new periodic solution showing up. So Hopf bifurcation considers when a 2D system with stable fixed point losses the stability at this point when a parameter changes. So changes in the parameters, causes one of the eigenvalues of the Jacobian to become positive, causing instability. An example from the textbook is given by

$$r' = \mu r - r^3$$
$$\theta' = \omega + br^2$$

The phase portrait is shown in figure below from the text book. This shows that when $\mu < 0$, the origin was stable. (spiral in). But when $\mu > 0$, a limit cycle show up with radius $r = \sqrt{\mu}$ and inside this radius, it is spiral out, hence the origin became unstable, moving to the limit cycle, and outside the limit cycle, it is stable and state trajectory moves towards the limit cycle. Here is the diagram from the text

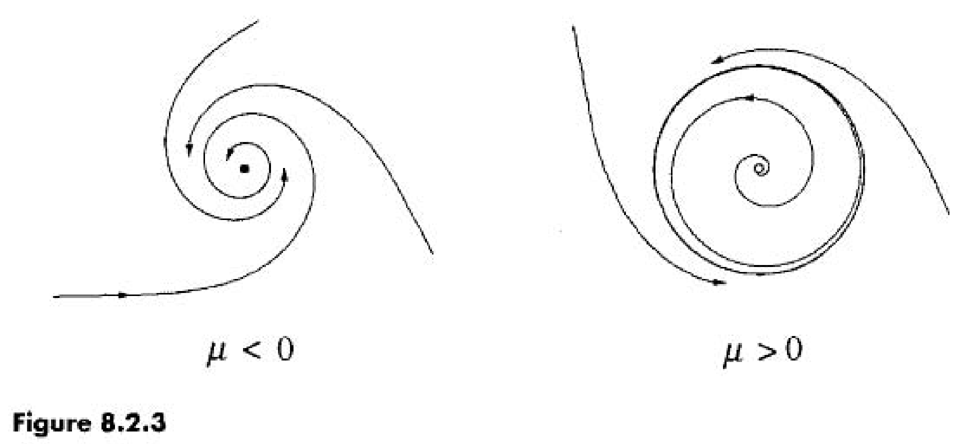

Figure 4: phase portrait

The eigenvalues of the Jacobian, evaluated at the origin (critical point) is shown to be $\lambda = \mu \pm i\omega$. So as μ changes from negative to positive, the system moves from being stable to unstable.

3 Problem 4.4.6

4.4.6 The derivative df/dz of an analytic function is also analytic; it still depends on the combination $z = x + iy$. Find df/dz if $f = 1 + z + z^2 + \cdots$ or $f = z^{1/2}$ (away from $z = 0$).

Figure 5: Problem description

Using the form $f(z) = z^{\frac{1}{2}}$, taking derivative w.r.t. gives $f'(z) = \frac{1}{2}\frac{1}{z^{\frac{1}{2}}}$. But $z = x + iy$, hence

$$f'(z) = \frac{1}{2}\frac{1}{\sqrt{(x+iy)}} = \frac{1}{2}\frac{\sqrt{(x-iy)}}{\sqrt{(x+iy)}\sqrt{(x-iy)}} = \frac{1}{2}\frac{\sqrt{(x-iy)}}{\sqrt{x^2+y^2}} = \frac{1}{2}\frac{1}{|z|}\sqrt{(x-iy)}$$

But $\sqrt{(x-iy)} = \bar{z}^{\frac{1}{2}}$ where $\bar{z}$ is complex conjugate of z. Hence

$$f'(z) = \frac{1}{2|z|}\bar{z}^{\frac{1}{2}}$$

4 Problem 4.4.7

4.4.7 Are the following functions analytic?

(a) $f = |z|^2 = x^2 + y^2$
(b) $f = \text{Re } z = x$
(c) $f = \sin z = \sin x \cosh y + i \cos x \sinh y$.

Can a function satisfy Laplace's equation without being analytic?

Figure 6: the Problem statement

A function $f(z)$ is analytic if it satisfies conditions as given in 4P, page 334

4P A function $f(z)$ is ***analytic*** at $z = a$ if in a neighborhood of that point
(1) it depends on the combination $z = x + iy$ and satisfies $i\partial f/\partial x = \partial f/\partial y$
(2) its real and imaginary parts are connected by the Cauchy-Riemann equations $u_x = s_y$ and $u_y = -s_x$
(3) it is the sum of a convergent power series $c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$

Figure 7: Problem description

4.1 Part(a)

$$f = |z|^2 = x^2 + y^2$$

Using 4P part(1), then $i\frac{\partial f}{\partial x} = i2x$ and $\frac{\partial f}{\partial y} = 2y$. Hence they are not the same. Therefore not analytic.

4.2 Part(b)

$$f = \text{Re}(z) = x$$

$i\frac{\partial f}{\partial x} = i$ and $\frac{\partial f}{\partial y} = 0$, hence not analytic.

4.3 Part(c)

$$
\begin{aligned}
f &= \sin x \cosh y + i \cos x \sinh y \\
&= u\left(x,y\right) + iv\left(x,y\right)
\end{aligned}
$$

Since

$$
i\frac{\partial f}{\partial x} = i\left(\cos x \cosh y - i \sin x \sinh y\right) = i \cos x \cosh y + \sin x \sinh y \tag{1}
$$

And

$$
\frac{\partial f}{\partial y} = \sin x \sinh y + i \cos x \cosh y \tag{2}
$$

We see that (1) and (2) are the same. Hence analytic.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}
\end{aligned}
$$

5 Problem 4.4.17

4.4.17 For the map $w = \frac{1}{2}(z + z^{-1})$ in Fig. 4.15, what happens to points $z = x > 1$ on the real axis? What happens to points $0 < x < 1$? What happens to the imaginary axis $z = iy$?

Figure 8: Problem description

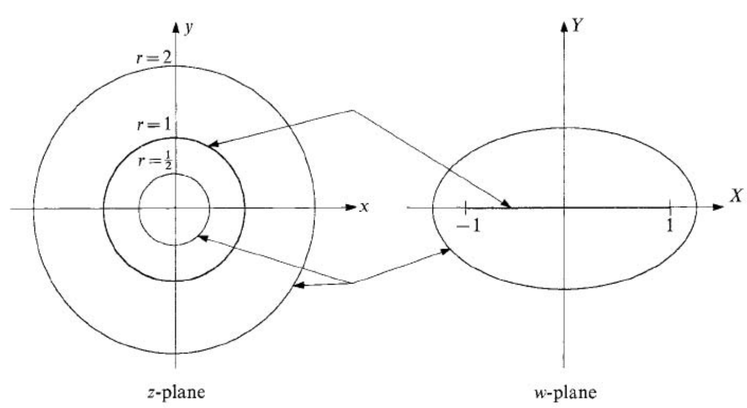

Fig. 4.15. The map from z to $w = \frac{1}{2}(z + z^{-1})$.

Figure 9: Problem description

5.1 Part (a)

The mapping $w = \frac{1}{2}\left(z + z^{-1}\right)$ is

$$
\begin{aligned}
w &= \frac{1}{2}\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) \\
&= \frac{1}{2}\left(r\left(\cos\theta + i\sin\theta\right) + \frac{1}{r}\left(\cos\theta - i\sin\theta\right)\right) \\
&= \frac{1}{2}\left(\left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\right) \\
&= \frac{1}{2}\left(\frac{r^2+1}{r}\right)\cos\theta + i\frac{1}{2}\left(\frac{r^2-1}{r}\right)\sin\theta
\end{aligned}
$$

For example, for unit circle, $r = 1$ and $w = \cos\theta$. Hence all points on unit circle map to $X = \cos\theta$. i.e. the link between $X = -1 \cdots 1$. To answer the question, it might be easier to write

$$\begin{aligned} w &= \frac{1}{2}\left(\left(x+iy\right)+\frac{1}{x+iy}\right) \\ &= \frac{1}{2}\left(\left(x+iy\right)+\frac{x-iy}{\left(x+iy\right)\left(x-iy\right)}\right) \\ &= \frac{1}{2}\left(\left(x+iy\right)+\frac{x-iy}{\left(x^2+y^2\right)}\right) \\ &= \frac{1}{2}\left(x+iy+\frac{x}{\left(x^2+y^2\right)}-i\frac{y}{\left(x^2+y^2\right)}\right) \end{aligned}$$

Write as $w = X + iY$

$$w = \frac{1}{2}\left(x+\frac{x}{\left(x^2+y^2\right)}\right)+i\left(\frac{1}{2}\left(y-\frac{y}{\left(x^2+y^2\right)}\right)\right)$$

Hence for point $(x, 0)$ it maps to $w = \frac{1}{2}\left(x+\frac{1}{x}\right)+i0 = \frac{1}{2}\left(\frac{x^2+1}{x}\right)$. Since $x > 1$ then $\frac{1}{2}\left(\frac{x^2+1}{x}\right)$ maps to all point on X that are larger than $X = 1$

5.2 Part(b)

For $0 < x < 1$, then from $w = \frac{1}{2}\left(x+\frac{1}{x}\right)$, we see that for example, of $x = 1/3$ then $X = \frac{1}{2}\left(\frac{1}{3}+3\right) > 1$.

5.3 Part(c)

For $z = iy$, then $x = 0$, and the mapping becomes

$$w = i\left(\frac{1}{2}\left(y-\frac{1}{y}\right)\right)$$

Hence

$$Y = \frac{1}{2}\left(y-\frac{1}{y}\right)$$

So

$$\begin{aligned} y &= 0 \rightarrow Y = \infty \\ y &= 1 \rightarrow Y = 0 \\ y &= -1 \rightarrow Y = 0 \\ y &> 1 \rightarrow 0 < Y < 1 \end{aligned}$$

6 Problem 4.4.23

4.4.23 Solve Laplace's equation in the 45° wedge if the boundary condition is $u = 0$ on both sides $y = 0$ and $y = x$.

(a) Where does $F(z) = z^4$ map the wedge?
(b) Find a solution with zero boundary conditions other than $u \equiv 0$.

Figure 10: the Problem statement

6.1 Part(a)

We need to transform to XY plane using conformal mapping to be able to solve it in the standard Cartesian system instead of on the quarter circle. Since the angle is 45^0 we need to map it to the full 180^0. So this mapping will work $w^4 = e^{4i\theta}$. So a point on e^{i45^0} will map to e^{i180^0} and point at e^{i0^0} will map to e^{i0^0}, hence the top half plane is where the new XY coordinates is. So we need to solve

$$U_{XX} + U_{YY} = 0 \tag{1}$$

In the upper half plane, then transform the solution back to (x, y) space. Solution to (1) is $U = aX + bY$. Since $U_{XX} = 0$ and $U_{YY} = 0$, hence this solution satisfies (1). We now need to figure how to map this back to (x, y). Using

$$\begin{aligned} w &= (x + iy) \\ w^4 &= (x + iy)^4 = x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4 \\ &= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3) \end{aligned}$$

Hence $X = (x^4 - 6x^2y^2 + y^4)$ and $Y = 4x^3y - 4xy^3$. So the solution is

$$U = aX + bY = a(x^4 - 6x^2y^2 + y^4) + b(4x^3y - 4xy^3)$$

Where a, b are constant found from boundary conditions.

7 Problem 6.1.11

6.1.11 Find the solution with arbitrary constants C and D to

(a) $u'' - 9u = 0$ (b) $u'' - 5u' + 4u = 0$ (c) $u'' + 2u' + 5u = 0$

Figure 11: the Problem statement

7.1 Part(a)

$$u'' - 9u = 0$$

This is constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 - 9 = 0$, hence $\lambda = \pm 3$, therefore the solution is

$$u(t) = De^{3t} + Ce^{-3t}$$

We notice this is not stable ode.

7.2 Part(b)

$$u'' - 5u' + 4u = 0$$

This is also constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 - 5\lambda + 4 = 0$. Solution is $\lambda = \{4, 1\}$, therefore the solution is

$$u(t) = De^{4t} + Ce^{t}$$

This is also not stable ode.

7.3 Part(c)

$$u'' + 2u' + 5u = 0$$

This is also constant coefficients second order ODE. It can solved by finding the zeros of its characteristic equation $\lambda^2 + 2\lambda + 5 = 0$, Solution is: $\lambda = \{-1 + 2i, -1 - 2i\}$, therefore the solution is

$$
\begin{aligned}
u(t) &= De^{(-1+2i)t} + Ce^{(-1-2i)t} \\
&= e^{-t}\left(De^{2it} + C^{-2it}\right)
\end{aligned}
$$

Which can be written as

$$u(t) = e^{-t}(d\cos 2t + c\sin 2t)$$

8 Problem 6.1.12

6.1.12 Find an equation $u'' + pu' + qu = 0$ whose solutions are

(a) e^t, e^{-t} (b) $\sin 2t, \cos 2t$ (c) $1, t$ (d) $e^{-t}\sin t, e^{-t}\cos t$

Figure 12: Problem description

8.1 Part(a)

From the solutions, we see that roots of the characteristic equation are $\{1, -1\}$, which means the characteristic equation is

$$p(\lambda) = (\lambda - 1)(\lambda + 1) = \lambda^2 - 1$$

Which implies the ODE is $u'' - u = 0$

8.2 Part(b)

Since the solution contains no damping (no e^{-t} term), and only contain oscillation, then it means the ode much contain only friction term, hence the ode is of the form

$$u'' + qu = 0$$

Since oscillation frequency is 2, then $\lambda_1 = 2i, \lambda_2 = -2i$ so to be able to contain the sin/cos shown as the solutions. Hence

$$p(\lambda) = (\lambda - 2i)(\lambda + 2i) = \lambda^2 + 4$$

Therefore

$$u'' + 4u = 0$$

8.3 Part(c)

Let

$$u(t) = Au_1 + Bu_2$$

Where A, B are constants of integration. Then $u(t) = A + Bt$ or $u' = B$ or $u'' = 0$

8.4 Part(d)

Since the solution contains damping (has e^{-t} term), and since oscillation oscillation exist, then the solution must be of form

$$u'' + pu' + qu = 0$$

The roots of the characteristic equation are therefore $\lambda_1 = -1 + i, \lambda_2 = -1 - i$. Hence

$$p(\lambda) = (\lambda - (-1 + i))(\lambda - (-1 - i)) = \lambda^2 + 2\lambda + 2$$

Therefore the ODE is

$$u'' + 2u' + 2u = 0$$

9 Problem 6.2.2

6.2.2 What types of critical points can $u' = Au$ have if
(1) A is symmetric positive definite
(2) A is symmetric negative definite
(3) A is skew-symmetric
(4) A is negative definite plus skew-symmetric (choose example).

Figure 13: the Problem statement

9.1 Part(1)

Since eigenvalues of A are real and positive, then not stable

9.2 Part(2)

Since eigenvalues of A are real and negative, then stable

9.3 Part(3)

(real) skew symmetric matrix always have pure imaginary eigenvalues. Hence phase plane is circles. This is called marginally stable.

9.4 Part(4)

And example of negative definite is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and skew symmetric is $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, hence $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$

The eigenvalues are found from $\begin{vmatrix} \lambda + 1 & -2 \\ 2 & \lambda + 1 \end{vmatrix} = 0$ or $(\lambda + 1)^2 + 4 = 0$, hence $(\lambda + 1)^2 = -4$ or $\lambda + 1 = \pm 2i$, therefore $\lambda = -1 \pm 2i$

Hence the eigenvalues have negative real part and imaginary parts. This is stable, and spiral due to the sin/cos which will result in the solution. It will spiral in, since the real part is negative.

10 Problem 6.2.12

6.2.12 With internal competition the predator-prey system might be

$$u_1' = u_1 - u_1^2 - bu_1u_2, \qquad u_2' = u_2 - u_2^2 + cu_1u_2$$

Find all equilibrium points and their stability (for $c < 1$ and $c > 1$). Which points make sense biologically?

Figure 14: the Problem statement

$$
\begin{aligned}
u_1' &= u_1 - u_1^2 - bu_1u_2 = F_1(u_1, u_2) \\
u_2' &= u_2 - u_2^2 + cu_1u_2 = F_2(u_1, u_2)
\end{aligned}
$$

We first need to find critical points by solving $F_1(u_1, u_2) = 0$ and $F_2(u_1, u_2) = 0$

From $F_1(u_1, u_2) = 0$ we obtain

$$
u_1(1 - u_1 - bu_2) = 0
$$

Hence $u_1 = 0$ or $u_1 = 1 - bu_2$. looking at the second equation $F_2(u_1, u_2) = 0$ which gives

$$
u_2(1 - u_2 + cu_1) = 0
$$

Hence $u_2 = 0$ or $u_2 = 1 + cu_1$.

Considering the case of $u_1 = 0$, then $u_2 = 1$, and when $u_1 = 1 - bu_2$, then

$$
\begin{aligned}
u_2 &= 1 + c(1 - bu_2) \\
&= 1 + c - cbu_2 \\
u_2 + cbu_2 &= 1 + c \\
u_2 &= \frac{1 + c}{1 + cb}
\end{aligned}
$$

And when $u_2 = 0$ then $u_1 = 1$ and when $u_2 = \frac{1+c}{1+cb}$ then $u_1 = 1 - bu_2 = 1 - b\frac{1+c}{1+cb}$. Hence the critical points are

$$
\begin{aligned}
&u_1 = 0, u_2 = 0 \\
&u_1 = 0, u_2 = 1 \\
&u_1 = 1, u_2 = 0 \\
&u_1 = -\frac{(b-1)}{bc+1}, u_2 = \frac{1+c}{1+cb}
\end{aligned}
$$

To find stability, we evaluate the Jacobian at each of the critical points. The Jacobian is

$$
J = \begin{bmatrix} \frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} \\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial\left(u_1 - u_1^2 - bu_1u_2\right)}{\partial u_1} & \frac{\partial\left(u_1 - u_1^2 - bu_1u_2\right)}{\partial u_2} \\ \frac{\partial\left(u_2 - u_2^2 + cu_1u_2\right)}{\partial u_1} & \frac{\partial\left(u_2 - u_2^2 + cu_1u_2\right)}{\partial u_2} \end{bmatrix} = \begin{bmatrix} (1 - 2u_1 - bu_2) & -bu_1 \\ cu_2 & 1 - 2u_2 + cu_1 \end{bmatrix}
$$

At point $u_1 = 0, u_2 = 0$ we obtain $J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ this has eigenvalues $\lambda = 1$ (double). Hence not stable node.

At point $u_1 = 0, u_2 = 1$ we obtain $J = \begin{bmatrix} 1-b & 0 \\ c & -1 \end{bmatrix}$ which has eigenvalues: $\{1 - b, -1\}$. Hence if $b > 1$ then both are stable. (negative), hence stable node. But if $b < 1$ then one is stable and the other is not. Which means unstable saddle point.

At point $u_1 = 1, u_2 = 0$ we obtain $J = \begin{bmatrix} -1 & -b \\ 0 & 1+c \end{bmatrix}$, eigenvalues: $c + 1, -1$. Hence if $c < -1$ then both are stable, and we have stable node. If $c > -1$ then one is stable and the other is not, so we have unstable saddle.

11 Problem 6.2.13

6.2.13 According to Braun, reptiles, mammals, and plants on the island of Komodo have populations governed by

$$u_r' = -au_r - bu_ru_m + cu_ru_p$$
$$u_m' = -du_m + eu_ru_m$$
$$u_p' = fu_p - gu_p^2 - hu_ru_p.$$

Who is eating whom? Find all equilibrium solutions u^*.

Figure 15: the Problem statement

$$u_r' = -au_r - bu_ru_m + cu_ru_p = F_1(u_r, u_m, u_p)$$
$$u_m' = -du_m + eu_ru_m = F_2(u_r, u_m, u_p)$$
$$u_p' = fu_p - gu_p^2 - hu_ru_p = F_3(u_r, u_m, u_p)$$

We first need to find critical points by solving $F_1(u_r, u_m, u_p) = 0$ and $F_2(u_r, u_m, u_p) = 0$ and $F_3(u_r, u_m, u_p) = 0$. Solving using computer algebra gives

```
eq1:=-a*u[r]-b*u[r]*u[m]+c*u[r]*u[p]=0;
eq2:=-d*u[m]-e*u[r]*u[m]=0;
eq3:=f*u[p]-g*(u[p])^2-h*u[r]*u[p]=0;
solve({eq1,eq2,eq3},{u[r],u[p],u[m]});
```

$$u_m = 0, u_p = 0, u_r = 0$$
$$u_m = -\frac{a}{b}, u_p = 0, u_r = -\frac{d}{e}$$
$$u_m = 0, u_p = \frac{f}{g}, u_r = 0$$
$$u_m = -\frac{aeg - dch - cfe}{ebg}, u_p = \frac{hd + fe}{eg}, u_r = -\frac{d}{e}$$
$$u_m = 0, u_p = \frac{a}{c}, u_r = -\frac{ag - cf}{ch}$$

We now need to find the Jacobian and evaluate it at each of the above points to determine the type of stability.

```
jac:=Matrix([[diff(eq1,u[r]),diff(eq1,u[m]),diff(eq1,u[p])],
[diff(eq2,u[r]),diff(eq2,u[m]),diff(eq2,u[p])],
[diff(eq3,u[r]),diff(eq3,u[m]),diff(eq3,u[p])]]);
```

Which gives

$$J = \begin{bmatrix} -bu_m + cu_p - a & -bu_r & cu_r \\ -eu_m & -eu_r - d & 0 \\ -hu_p & 0 & -2gu_p - hu_r + f \end{bmatrix}$$

At point $u_m = 0, u_p = 0, u_r = 0$, $J = \begin{bmatrix} -a & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & f \end{bmatrix}$ so assuming all a, d, f are positive, this shows this point is not stable. It is $\boxed{\text{unstable spiral}}$ since one of the eigenvalues is positive.

At point $u_m = -\frac{a}{b}, u_p = 0, u_r = -\frac{d}{e}$, $J = \begin{bmatrix} -b\frac{a}{b} - a & b\frac{d}{e} & -c\frac{d}{e} \\ e\frac{a}{b} & e\frac{d}{e} - d & 0 \\ 0 & 0 & h\frac{d}{e} + f \end{bmatrix} = \begin{bmatrix} -2a & b\frac{d}{e} & -c\frac{d}{e} \\ e\frac{a}{b} & 0 & 0 \\ 0 & 0 & h\frac{d}{e} + f \end{bmatrix}$, eigenvalues are $\left\{-a-\sqrt{a(a+d)}, \sqrt{a(a+d)}-a, \frac{1}{e}\left(fe+dh\right)\right\}$. So for positive parameters $\sqrt{a(a+d)}-a>0$, hence not stable .

At $u_m = 0, u_p = \frac{f}{g}, u_r = 0, J = \begin{bmatrix} c\frac{f}{g}-a & 0 & 0 \\ 0 & -d & 0 \\ -h\frac{f}{g} & 0 & -2g\frac{f}{g}+f \end{bmatrix}$, eigenvalues: $-d, -f, -\frac{1}{g}\left(ag-cf\right)$. Therefore, for positive parameters, this is stable node.

12 Problem 6.2.19

6.2.19 (Epidemic theory). Suppose $u(t)$ people are healthy at time t and $v(t)$ are infected. If the latter become dead or otherwise immune at rate b and infection occurs at rate a, then $u' = -auv, \quad v' = auv - bv.$

(a) Show that $v' > 0$ if $u > b/a$, so the epidemic spreads.
(b) Show that $v' < 0$ if $u < b/a$, so the epidemic slows down. (It never starts if $u_0 < b/a$.)
(c) Show that $E = u + v - (b/a)\log u$ is constant during the epidemic.
(d) What is $v_{\max}$ (when $u = b/a$) in terms of u_0?

Figure 16: the Problem statement

$$\begin{aligned} u' &= -auv = F_1(u,v) \\ v' &= auv - bv = F_2(u,v) \end{aligned}$$

The critical points are $u = any, v = 0$.

12.1 Part (a)

If $u > \frac{b}{a}$, then we write $u = \frac{b+\varepsilon}{a}$ for $\varepsilon > 0$. Substituting in $v' = auv - bv$ results in

$$\begin{aligned} v' &= a\frac{b+\varepsilon}{a}v - bv \\ &= bv + \varepsilon v - bv \\ &= \varepsilon v \end{aligned}$$

Hence $v' > 0$ and the epidemic spreads.

12.2 Part (b)

If $u < \frac{b}{a}$, then we write $u = \frac{b-\varepsilon}{a}$ for $\varepsilon > 0$, Substituting in $v' = auv - bv$ results in

$$\begin{aligned} v' &= a\frac{b-\varepsilon}{a}v - bv \\ &= bv - \varepsilon v - bv \\ &= -\varepsilon v \end{aligned}$$

Hence $v' < 0$ and the epidemic slows down.

12.3 Part (d)

From second equation, $v(t) = Ae^{\int (u(t)a-b)dt}$, hence when $u(t) = \frac{b}{a}$, then $v(t) = k$. A constant u_0.