HW 5

Math 703
methods of applied mathematics I

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Contents

1 Problem 4.1.1(d)

4.1.1 Find the Fourier series on $-\pi < x < \pi$ for

(a) $f(x) = \sin^3 x$, an odd function

(b) $f(x) = |\sin x|$, an even function

(c) $f(x) = x^2$, integrating either $x^2 \cos kx$ or the sine series for $f = x$

(d) $f(x) = e^x$, using the complex form of the series.

What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

Figure 1: the Problem statement

$$f(x) = e^x = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

Where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-ik)x} dx$$

$$= \frac{1}{2\pi} \left[\frac{e^{(1-ik)x}}{1-ik} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1-ik)} \left[\frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2} \right]$$

But $\frac{e^z}{2} - \frac{e^{-z}}{2} = \sinh(z)$, hence the above reduces to

$$c_k = \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) \tag{2}$$

Substituting (2) into (1) gives

$$e^x = \sum_{k=-\infty}^{\infty} \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) e^{ikx}$$

Here are few terms in the series generated using symbolic software:

```
ClearAll[x, k, n, f, ck]
ck[k_, x_] := 1/(2 Pi) Integrate[Exp[x] Exp[-I k x], {x, -Pi, Pi}]
f[k_, x_] := ck[k, x]*Exp[I k x];
term[n_] := If[n == 0, N@f[0, x], N@Simplify@ComplexExpand[f[-n, x] + f[n, x]]]
tbl = Table[{k, Simplify@TrigToExp@ck[k, x]}, {k, -5, 5, 1}];
Grid[Join[{{"k", "C_k"}}, tbl], Frame -> All]
```

k	C_k
-5	$\frac{(1-5i)e^{-\pi}-(1-5i)e^{\pi}}{52\pi}$
-4	$\frac{\left(\frac{1}{34}-\frac{2i}{17}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
-3	$\frac{(1-3i)e^{-\pi}-(1-3i)e^{\pi}}{20\pi}$
-2	$\frac{\left(\frac{1}{10}-\frac{i}{5}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
-1	$-\frac{\left(\frac{1}{4}-\frac{i}{4}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
0	$-\frac{e^{-\pi}-e^{\pi}}{2\pi}$
1	$\frac{(1+i)e^{-\pi}-(1+i)e^{\pi}}{4\pi}$
2	$\frac{\left(\frac{1}{10}+\frac{i}{5}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
3	$\frac{(1+3i)e^{-\pi}-(1+3i)e^{\pi}}{20\pi}$
4	$\frac{\left(\frac{1}{34}+\frac{2i}{17}\right)e^{-\pi}\left(e^{2\pi}-1\right)}{\pi}$
5	$\frac{(1+5i)e^{-\pi}-(1+5i)e^{\pi}}{52\pi}$

Here is a plot of Fourier series of e^x for k increasing range to compare with e^x. To generate this plot the terms with $c_{-k}+c_k$ were added in order together to obtain a real valued function before plotting. Plotting was done from $x=-\pi\cdots\pi$. We see as more terms are added, the approximation improves. At 20 terms, the approximations became very good. Here is the plot

```
ck = 1/(2 Pi) Integrate[Exp[x] Exp[-I k1 x], {x, -Pi, Pi}]
f[k_] := (ck /. k1 -> k)*Exp[I k x];
fs[n_] := Sum[Simplify[f[-k] + f[k]], {k, 1, n}] + f[0];
tbl = Table[Plot[{fs[n], Exp[x]}, {x, -Pi, Pi}, Frame -> True, Axes -> False,
FrameLabel -> {{"f(x)", None},
{"x", Row[{"Using " <> ToString[n] <> " terms"}]}},
PlotStyle -> {Dashed, Red}], {n, 1, 20, 1}];
Grid[Partition[tbl, 4]]
```
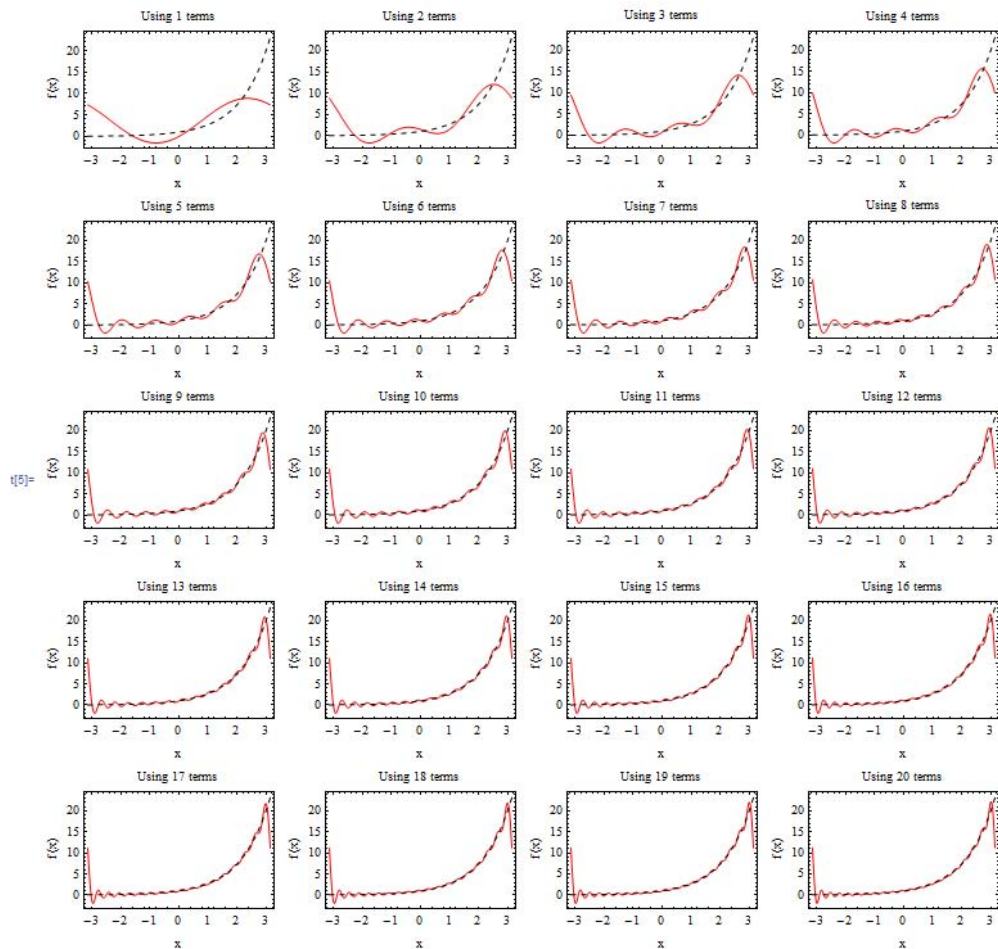

Figure 2: Plot for problem 4.1.1

The even part of e^x are given by $\frac{e^x + e^{-x}}{2} = \cosh x$ and the odd part is $\frac{e^x - e^{-x}}{2} = \sinh x$. For e^{ix}, the even part is $\frac{e^{ix} + e^{-ix}}{2} = \cos x$ and the odd part is $\frac{e^{ix} - e^{-ix}}{2} = i \sin x$

2 Problem 4.1.2

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = +1$ on the right side $0 < x < \pi$.

(1) Why are all the cosine coefficients $a_k = 0$?

(2) Find the sine series $\Sigma\, b_k \sin kx$ from equation (6).

Figure 3: the Problem statement

2.1 Part (a)

Since $f(-\pi) = -f(-\pi)$ then $f(x)$ is an odd function. For an odd function all the $a_k = 0$ since these go with the even part.

2.2 Part(b)

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} f(x) \sin(kx)\, dx + \int_{0}^{\pi} f(x) \sin(kx)\, dx \right)$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} -\sin(kx)\, dx + \int_{0}^{\pi} \sin(kx)\, dx \right)$$

Changing the limits of integration changes the sign, hence the above can be written as

$$b_k = \frac{1}{\pi} \left(\int_{0}^{\pi} \sin(kx)\, dx + \int_{0}^{\pi} \sin(kx)\, dx \right)$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin(kx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{-\cos kx}{k} \right]_{0}^{\pi}$$

$$= \frac{-2}{\pi k} [\cos kx]_{0}^{\pi}$$

$$= \frac{-2}{\pi k} [\cos k\pi - \cos 0]$$

$$= \frac{2}{\pi k} (1 - \cos k\pi) \qquad k = 1, 2, 3, \cdots$$

Hence

$$b_k = \begin{cases} \frac{4}{\pi k} & k = 1, 3, 5, \cdots \\ 0 & k = 2, 4, 6, \cdots \end{cases}$$

Hence using $f(x) = \sum_{k=1}^{\infty} b_k \sin kx$, we can write the Fourier series of $f(x)$ as

$$f(x) = \sum_{k=1,3,\cdots}^{\infty} \frac{4}{\pi k} \sin kx$$

$$= \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \cdots$$

$$= \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

Here is a plot showing the Fourier series approximation to the square wave from $x = -\pi \cdots \pi$ as more terms are added

```
Clear[f, k, x];
f[x_, k_] := Sum[2/(Pi n) (1 - Cos[n Pi]) Sin[n x], {n, 1, k}];
tbl = Partition[Table[
Plot[{Sign[x], f[x, k]}, {x, -Pi, Pi},
Exclusions -> None, PlotLabel -> Row[{"k=", k}],
PlotStyle -> {Thin, Red}], {k, 1, 20,2}], 3];
Grid[tbl, Frame -> All]
```
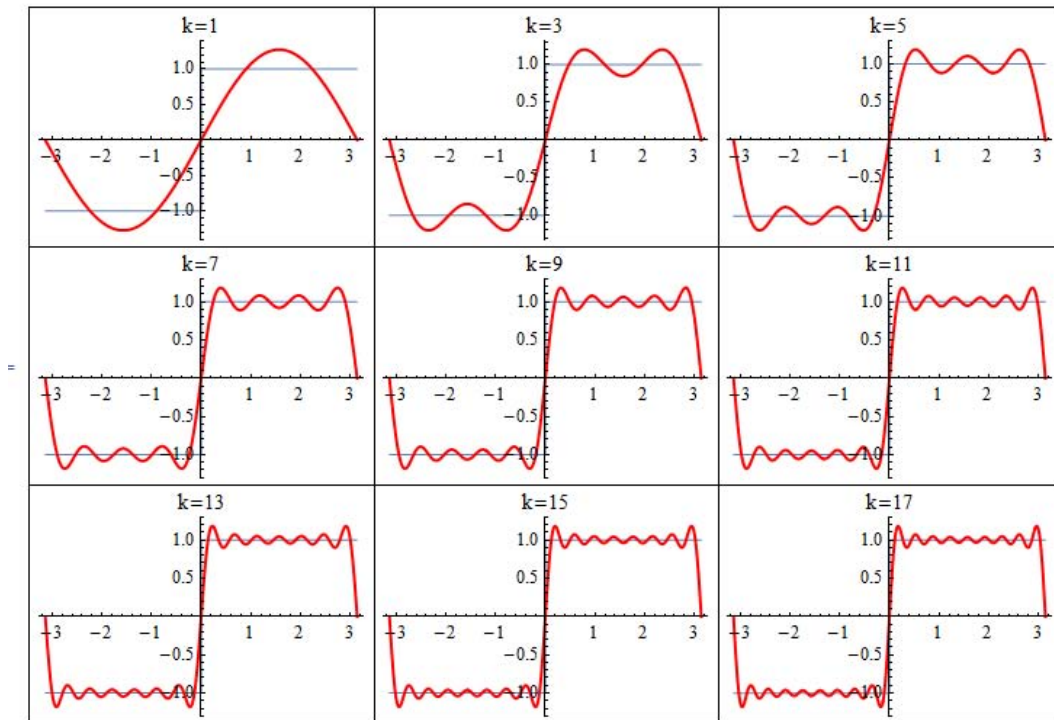

Figure 4: Plot for problem 4.1.2

3 Problem 4.1.3

4.1.3 Find this sine series for the square wave f in another way, by showing that

(a) $df/dx = 2\delta(x) - 2\delta(x + \pi)$ extended periodically

(b) $2\delta(x) - 2\delta(x + \pi) = \dfrac{4}{\pi}(\cos x + \cos 3x + \cdots)$ from (10)

Integrate each term to find the square wave f.

Figure 5: the Problem statement

3.1 Part(a)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x + \pi)$. For $\delta(x)$ we find

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\delta(x)\,dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\cos kx\,dx = \frac{1}{\pi} \qquad (\text{since } \cos 0 = 1)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi}\delta(x)\sin kx\,dx = 0 \qquad (\text{since } \sin 0 = 0)$$

Hence

$$\delta(x) = \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos kx$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} (\cos x + \cos 2x + \cos 3x + \cdots)$$

Now to determine Fourier series for $\delta(x + \pi)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x + \pi) \, dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x + \pi) \cos kx dx = \frac{(-1)^k}{\pi} \qquad (\text{since } \cos(-k\pi) = \cos k\pi = (-1)^k)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \sin kx dx = 0 \qquad (\text{since } \sin(-k\pi) = 0)$$

Hence

$$\delta(x + \pi) = \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \cos kx$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} (-\cos x + \cos 2x - \cos 3x + \cdots)$$

Therefore

$$2\delta(x) - 2\delta(x + \pi) = 2 \left[\frac{1}{2\pi} + \frac{1}{\pi} (\cos x + \cos 2x + \cos 3x + \cdots) \right] - 2 \left[\frac{1}{2\pi} + \frac{1}{\pi} (-\cos x + \cos 2x - \cos 3x + \cdots) \right]$$

$$= \frac{1}{\pi} + \frac{2}{\pi} (\cos x + \cos 2x + \cos 3x + \cdots) - \frac{1}{\pi} + \frac{2}{\pi} (\cos x - \cos 2x + \cos 3x - \cos 5x + \cdots)$$

$$= \frac{2}{\pi} (2 \cos x + 2 \cos 3x + 2 \cos 5x + \cdots)$$

$$= \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \cdots)$$

Hence

$$\frac{df}{dx} = \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \cdots)$$

Hence

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

3.2   Part (b)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x + \pi)$. For $\delta(x)$ we find

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) e^{-ikx} dx = \frac{1}{2\pi}$$

Hence

$$\delta(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} e^{ikx}$$

$$= \frac{1}{2\pi} \left(1 + e^{-ikx} + e^{ikx} + e^{-2ik} + e^{2ik} + \cdots \right)$$

$$= \frac{1}{2\pi} (1 + 2 \cos kx + 2 \cos 2kx + 2 \cos 3kx + \cdots)$$

Now to determine Fourier series for $\delta(x + \pi)$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x + \pi) e^{-ikx} dx = \frac{1}{2\pi} e^{ik\pi} = \frac{1}{2\pi} \cos k\pi = \frac{(-1)^k}{2\pi}$$

Hence

$$\delta(x+\pi) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{2\pi} e^{ikx}$$

$$= \frac{1}{2\pi}\left(1 - e^{-ix} - e^{ix} + e^{-2ix} + e^{2ix} - e^{-3ix} - e^{3ix} + \cdots\right)$$

$$= \frac{1}{2\pi}\left(1 - \left(e^{-ix} + e^{ix}\right) + e^{-2ix} + e^{2ix} - \left(e^{-3ix} + e^{3ix}\right) + \cdots\right)$$

$$= \frac{1}{2\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right)$$

Therefore

$$2\delta(x) - 2\delta(x+\pi) = 2\left[\frac{1}{2\pi}\left(1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right)\right] - 2\left[\frac{1}{2\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right)\right]$$

$$= \frac{1}{\pi}\left(1 + 2\cos x + 2\cos 2x + 2\cos 3x + \cdots\right) - \frac{1}{\pi}\left(1 - 2\cos x + 2\cos 2x - 2\cos 3x + \cdots\right)$$

$$= \frac{1}{\pi}\left(4\cos x + 4\cos 3x + 4\cos 5x + \cdots\right)$$

$$= \frac{4}{\pi}\left(\cos x + \cos 3x + \cos 5x + \cdots\right)$$

Hence

$$\frac{df}{dx} = \frac{4}{\pi}\left(\cos x + \cos 3x + \cos 5x + \cdots\right)$$

Therefore

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Which is the same as above using the a_k, b_k method.

4 Problem 4.1.4

4.1.4 At $x = \pi/2$ the square wave equals 1. From the Fourier series at this point find the alternating sum that equals π:

$$\pi = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} \cdots).$$

Figure 6: the Problem statement

From above we found that the Fourier series for square wave is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

Therefore at $x = \frac{\pi}{2}$, the above becomes

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \cdots\right)$$

Hence

$$\pi = 4\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin 3\frac{\pi}{2} + \frac{1}{5}\sin 5\frac{\pi}{2} + \cdots\right)$$

$$= 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

5 Problem 4.1.5

We found that only the b_k survive for the Fourier series of the wave function. They are

$$b_k = \begin{cases} \dfrac{4}{\pi k} & k = 1,3,5,\cdots \\ 0 & k = 2,4,6,\cdots \end{cases}$$

Applying Parseval's formula leads to

$$\pi \left(b_1^2 + b_3^2 + b_5^2 + \cdots\right) = \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi$$

Where we used only the odd b_k terms since all others are zero. The above becomes

$$\pi \left(\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \cdots \right) = 2\pi$$

$$\pi \left(\frac{1}{\pi^2}4^2 + \frac{1}{\pi^2}\left(\frac{4}{3}\right)^2 + \frac{1}{\pi^2}\left(\frac{4}{5}\right)^2 + \cdots \right) = 2\pi$$

$$\left(4^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{4}{5}\right)^2 + \cdots \right) = 2\pi^2$$

$$\pi^2 = 8 \left(1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \cdots \right)$$

Hence

$$\pi^2 = 8 \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots \right)$$

6 Problem 4.1.8

ps. In the solution below, I was using T when I should be using $\frac{T}{2}$ in all the limits. Need to correct later. Or just let period be $2T$ then the math works ok.

In this problem, the basic idea is to observe that when the period was 2π then

$$f(x) = \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

Now when the period is a general value T we use $\left(\frac{2\pi}{T}k\right)$ in place of just k. So the above becomes

$$f(x) = \sum_{k=0}^{\infty} a_k \cos\left(k\frac{2\pi}{T}x\right) + \sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{T}x\right) \tag{1}$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(\frac{2\pi}{T}k\right)x} \tag{2}$$

We now need to determine a_k, b_k, c_k using (1) and (2) in similar way we did when the period was 2π.

To find a_k we multiply (1) by $\cos\left(m\frac{2\pi}{T}x\right)$ where m is some integer between $1\cdots\infty$, and integrating from $-T$ to T gives

$$\int_{-T}^{T} f(x)\cos\left(m\frac{2\pi}{T}x\right)dx = \int_{-T}^{T}\sum_{k=0}^{\infty} a_k \cos\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx + \int_{-T}^{T}\sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx$$

$$= \sum_{k=0}^{\infty}\int_{-T}^{T} a_k \cos\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx + \sum_{k=1}^{\infty}\int_{-T}^{T} b_k \sin\left(k\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx$$

Due to orthogonality between the $\sin$ and $\cos$, all the product of $\sin\cos$ vanish, and only one term in the product of $\cos\cos$ remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^{T} f(x)\cos\left(m\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} a_m \cos\left(m\frac{2\pi}{T}x\right)\cos\left(m\frac{2\pi}{T}x\right)dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} a_k \cos^2\left(k\frac{2\pi}{T}x\right)dx \tag{3}$$

When $k = 0$ we find

$$\int_{-T}^{T} f(x)\,dx = \int_{-T}^{T} a_0\,dx$$
$$= 2a_0 T$$

Hence

$$a_0 = \frac{1}{2T}\int_{-T}^{T} f(x)\,dx$$

Notice, when $T = \pi$, the above reduces to $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$. Now to find a_k for $k \geq 1$, then from (3)

$$\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} a_k \cos^2\left(k\frac{2\pi}{T}x\right)dx$$
$$= a_k T$$

Hence

$$a_k = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(k\frac{2\pi}{T}x\right)dx$$

Notice that when $T = \pi$ the above reduces to $a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx$ as before.

Now we find b_k similarly. We multiply (1) by $\sin\left(m\frac{2\pi}{T}x\right)$ where m is some integer between $1\cdots\infty$, and integrating from $-T$ to T gives

$$\int_{-T}^{T} f(x)\sin\left(m\frac{2\pi}{T}x\right)dx = \sum_{k=0}^{\infty}\int_{-T}^{T} a_k \cos\left(k\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx + \sum_{k=1}^{\infty}\int_{-T}^{T} b_k \sin\left(k\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx$$

Due to orthogonality between the $\sin$ and $\cos$, all the products of $\sin\cos$ vanish, and only one term in the product of $\sin\sin$ remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^{T} f(x)\sin\left(m\frac{2\pi}{T}x\right)dx = \int_{-T}^{T} b_m \sin\left(m\frac{2\pi}{T}x\right)\sin\left(m\frac{2\pi}{T}x\right)dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\int_{-T}^{T} f(x) \sin\left(k\frac{2\pi}{T}x\right) dx = \int_{-T}^{T} b_k \sin^2\left(k\frac{2\pi}{T}x\right) dx$$

$$= b_k T$$

Hence

$$b_k = \frac{1}{T}\int T^T f(x) \sin\left(k\frac{2\pi}{T}x\right) dx$$

Notice that when $T = \pi$ the above reduces to $b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(kx) dx$ as before. We now find c_k.

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(k\frac{2\pi}{T}\right)x}$$

Multiplying both side by $e^{-i\left(m\frac{2\pi}{T}\right)x}$ and integrating over the period

$$\int_{-T}^{T} f(x) e^{-i\left(m\frac{2\pi}{T}\right)x} dx = \sum_{k=-\infty}^{\infty} \int_{-T}^{T} c_k e^{i\left(k\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x} dx$$

All terms other than ones which $k = m$ remain. Hence the above becomes

$$\int_{-T}^{T} f(x) e^{-i\left(m\frac{2\pi}{T}\right)x} dx = \int_{-T}^{T} c_m e^{i\left(m\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x} dx$$

$$= \int_{-T}^{T} c_m dx$$

Therefore, since m is now arbitrary, we rename it back to k and simplifying

$$\int_{-T}^{T} f(x) e^{-i\left(k\frac{2\pi}{T}\right)x} dx = 2T c_k$$

$$c_k = \frac{1}{2T}\int_{-T}^{T} f(x) e^{-i\left(k\frac{2\pi}{T}\right)x} dx$$

7 Problem 4.1.10

4.1.10 What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

Figure 9: the Problem statement

The a_0 term in the Fourier series of $\cos^2 x$ is the constant term. Hence it is the constant that is closest to $\cos^2 x$ in the square sense. Therefore

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$= \frac{1}{2}$$

To find the multiple of $\cos x$ which is closest to $\cos^3 x$, we find a_1 term in the Fourier series of $\cos^3 x$ since that is the term which has $a_1 \cos x$ in it. Hence

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^3 x \cos x \, dx$$

$$= \frac{1}{\pi}\left(\frac{3\pi}{4}\right)$$

$$= \frac{3}{4}$$

8 Problem 4.1.11

4.1.11 Sketch the graph and find the Fourier series of the even function $f = 1 - |x|/\pi$ (extended periodically) in either of two ways: integrate the square wave or compute (with $a_0 = \frac{1}{2}$)

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx = \frac{2}{\pi} \int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right) \cos kx \, dx.$$

Figure 10: the Problem statement

The function we are approximating using Fourier series is

```
f[x_] := Piecewise[{{1 + x/Pi, x < 0}, {1 - x/Pi, x >= 0}}];
Plot[f[x], {x, -Pi, Pi}]
```
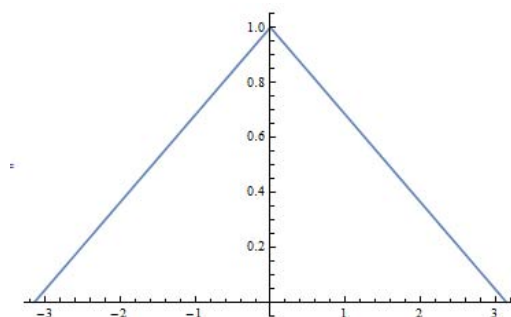

Figure 11: Plot for problem 4.1.11

Since it is even, we only need to determine a_k

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{2}{\pi} \int_{0}^{\pi} \left(1 - \frac{x}{\pi}\right) \cos kx dx$$

$$= \frac{2}{\pi} \left(\frac{1 - \cos k\pi}{k^2 \pi}\right)$$

Hence

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\frac{1 - \cos \pi}{\pi}\right)\cos x + \frac{2}{\pi}\left(\frac{1 - \cos 2\pi}{4\pi}\right)\cos 2x + \frac{2}{\pi}\left(\frac{1 - \cos 3\pi}{9\pi}\right)\cos 3x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left(\frac{2}{\pi}\right)\cos x + \frac{2}{\pi}\left(\frac{2}{9\pi}\right)\cos 3x + \frac{2}{\pi}\left(\frac{2}{25\pi}\right)\cos 5x + \cdots$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\cos x + \frac{4}{9\pi^2}\cos 3x + \frac{4}{25\pi}\cos 5x + \cdots$$

Here is a plot showing the approximation as more terms are added. The label of each plot show the number of terms used. The more terms we use, the better the approximation

```
ck = (2/Pi) Integrate[(1 - x/Pi) Cos[k x], {x, 0, Pi}];
upTo[n_, x_] := (1/2) + Sum[(ck /. k -> m)* Cos[m x], {m, 1, n}];
tbl = Table[Plot[upTo[m, x], {x, -Pi, Pi},
PlotLabel -> Row[{"terms used =", m}]], {m, 0, 18, 2}];
Grid[Partition[tbl, 3], Frame -> All]
```
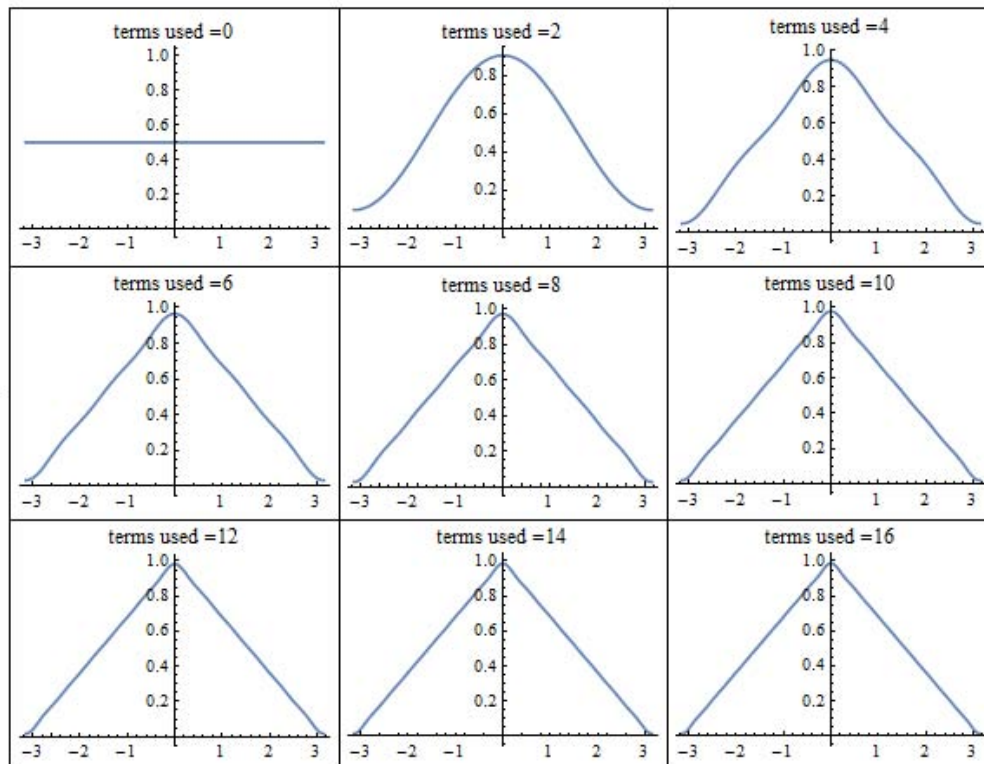
Figure 12: Plot for problem 4.1.11 part 2

9 Problem 4.1.16

4.1.16 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin?

Figure 13: the Problem statement

The first step is to obtain the a_k, b_k coefficients by expanding the boundary value of the solution using Fourier series. On the boundary

$$u_0 = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases}$$

Hence

$$a_0 = \frac{1}{2\pi} \int_0^\pi d\theta = \frac{1}{2}$$

And

$$a_k = \frac{1}{\pi} \int_0^\pi \cos k\theta d\theta = \frac{1}{k\pi} [\sin k\theta]_0^\pi = 0$$

And

$$b_k = \frac{1}{\pi} \int_0^\pi \sin k\theta d\theta = \frac{1}{k\pi} [-\cos k\theta]_0^\pi = 0 = \frac{-1}{k\pi} [\cos k\pi - \cos 0]$$

$$= \left\{ \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \cdots \right\}$$

Only odd values of k survive. Now that we found the Fourier coefficient, we use them in the solution given in equation (22), page 276 on the book

$$u(r, \theta) = a_0 + b_1 r \sin \theta + b_3 r^3 \sin 3\theta + b_5 r^5 \sin^5 \theta + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi} \left(r \sin \theta + \frac{1}{3} r^3 \sin 3\theta + \frac{1}{5} r^5 \sin^5 \theta + \cdots \right)$$

At the origin, let $r = 0$

$$u(0, \theta) = \frac{1}{2}$$

10 Problem 4.1.19

4.1.19 A plucked string goes linearly from $f(0) = 0$ to $f(p) = 1$ and back to $f(\pi) = 0$. The linear part $f = x/p$ reaches to $x = p$, followed by $f = (\pi - x)/(\pi - p)$ to $x = \pi$. Sketch f as an

odd function and find a plucking point p for which the second harmonic $\sin 2x$ will not be heard ($b_2 = 0$).

Figure 14: the Problem statement

A sketch of the function (string) is below.

```
Clear[x, f, p];
f[x_, p_] := Piecewise[{{(-x - Pi)/(Pi - p), x < -p},
{(x + p)/p - 1, -p < x < 0}, {x/p, 0 < x < p},
{(x - Pi)/(p - Pi), p < x < Pi}}]
Plot[f[x, .8 Pi], {x, -Pi, Pi}, Frame -> True,
FrameLabel -> {{"f(x)", None}, {x, "problem 4.1.19"}}]
```
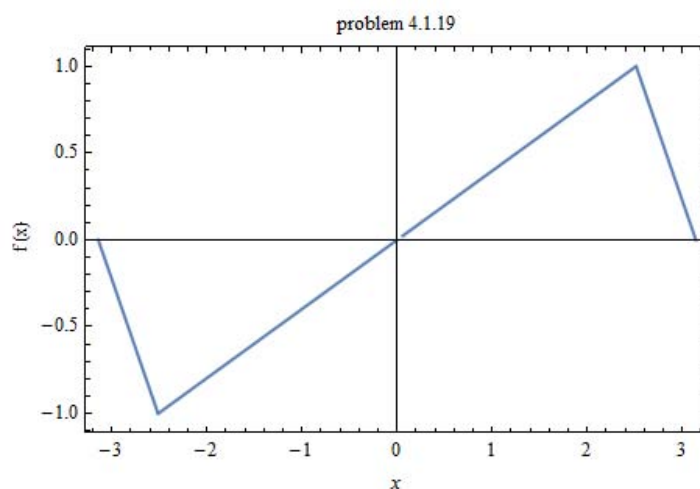

Figure 15: Plot for problem 4.1.19

Since $f(x)$ is odd, we only need to determine b_k

$$b_k = \frac{2}{\pi}\int_0^\pi f(x)\sin kx\, dx$$

$$= \frac{2}{\pi}\left(\int_0^p \frac{x}{p}\sin kx\, dx + \int_p^\pi \frac{x - \pi}{p - \pi}\sin kx\, dx\right)$$

$$= \frac{2}{\pi}\left(\frac{\sin kp - kp\cos kp}{k^2 p} + \frac{k(\pi - p)\cos kp + \sin kp - \sin k\pi}{k^2(\pi - p)}\right)$$

$$= \frac{2(\pi\sin kp - p\sin k\pi)}{k^2 p\pi(\pi - p)}$$

For $k = 2$

$$b_2 = \frac{(\pi\sin 2p - p\sin 2\pi)}{2p\pi(\pi - p)}$$

$$= \frac{\pi\sin 2p}{2p\pi(\pi - p)}$$

For zero, we need

$$0 = \pi \sin 2p$$
$$\sin 2p = 0$$

Hence

$$p = \frac{\pi}{2}$$

11 Problem 4.1.20

4.1.20 Show that $P_2 = x^2 - \frac{1}{3}$ is orthogonal to $P_0 = 1$ and $P_1 = x$ over the interval $-1 \le x \le 1$. Can you find the next Legendre polynomial by choosing c to make $x^3 - cx$ orthogonal to P_0, P_1, and P_2?

Figure 16: the Problem statement

Two functions f, g are if the inner product is zero $\int_{-1}^{1} f(x) g(x) \, dx = 0$. Hence

$$\int_{-1}^{1} P_2 P_0 dx = \int_{-1}^{1} \left(x^2 - 1 \right) dx = \left(\frac{x^3}{3} - x \right)_{-1}^{1} = 0$$

And

$$\int_{-1}^{1} P_2 P_1 dx = \int_{-1}^{1} \left(x^2 - 1 \right) x dx = \left(\frac{x^4}{4} - \frac{x^2}{2} \right)_{-1}^{1} = 0$$

Now let $P_3 = x^3 - cx$, we want this to be orthogonal to P_0, P_1, P_2. Hence

$$\int_{-1}^{1} P_3 P_0 dx = \int_{-1}^{1} x^3 - cx \, dx = \left(\frac{x^4}{4} - c\frac{x^2}{2} \right)_{-1}^{1} = \left(\frac{1}{4} - c\frac{1}{2} \right) - \left(\frac{1}{4} - c\frac{1}{2} \right)$$
$$0 = 0$$

This equation did not help us find c. We try the next one

$$\int_{-1}^{1} P_3 P_1 dx = \int_{-1}^{1} \left(x^3 - cx \right) x dx = \left(\frac{x^5}{5} - c\frac{x^3}{3} \right)_{-1}^{1} = \left(\frac{1}{5} - c\frac{1}{3} \right) - \left(-\frac{1}{5} + c\frac{1}{3} \right) = \frac{2}{5} - \frac{2}{3}c$$
$$\frac{2}{5} - \frac{2}{3}c = 0$$
$$c = \frac{2}{5}\frac{3}{2}$$
$$= \frac{3}{5}$$

Hence

$$P_3 = x^3 - \frac{3}{5}x$$

12 Problem 4.1.26

4.1.26 If f has the double sine series $\Sigma\Sigma b_{kl} \sin kx \sin ly$, show that Poisson's equation $-u_{xx} - u_{yy} = f$ is solved by the double sine series $u = \Sigma\Sigma b_{kl} \sin kx \sin ly/(k^2 + l^2)$. This is the solution with $u = 0$ on the boundary of the square $-\pi < x, y < \pi$.

Figure 17: the Problem statement

The proposed solution is

$$u(x,y) = \sum\sum \frac{b_{kl} \sin kx \sin ly}{\left(k^2 + l^2 \right)} \tag{1}$$

To see if this solves

$$-u_{xx} - u_{yy} = f = \sum\sum b_{kl} \sin kx \sin ly \tag{1A}$$

we will take (1) and substitute in the LHS of Poisson equation (1A) and see if we get the RHS of (1A) which is f.

$$\frac{\partial u}{\partial x} = \sum\sum \frac{b_{kl}k \cos kx \sin ly}{\left(k^2 + l^2\right)}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum\sum \frac{-b_{kl}k^2 \sin kx \sin ly}{\left(k^2 + l^2\right)} \tag{2}$$

And

$$\frac{\partial u}{\partial y} = \sum\sum \frac{b_{kl} \sin(kx) l \cos ly}{\left(k^2 + l^2\right)}$$

$$\frac{\partial^2 u}{\partial y^2} = \sum\sum \frac{-b_{kl} \sin(kx) l^2 \sin ly}{\left(k^2 + l^2\right)} \tag{3}$$

Substituting (2) and (3) in the LHS of (1A) gives

$$-u_{xx} - u_{yy} = \sum\sum \frac{b_{kl}k^2 \sin kx \sin ly}{\left(k^2 + l^2\right)} + \sum\sum \frac{b_{kl} \sin(kx) l^2 \sin ly}{\left(k^2 + l^2\right)}$$

$$= \sum\sum \frac{b_{kl}k^2 \sin kx \sin ly + b_{kl} \sin(kx) l^2 \sin ly}{\left(k^2 + l^2\right)}$$

$$= \sum\sum \frac{\left(b_{kl} \sin kx \sin ly\right)\left(k^2 + l^2\right)}{\left(k^2 + l^2\right)}$$

$$= \sum\sum b_{kl} \sin kx \sin ly$$

Which is f. Hence $u(x,y) = \sum\sum \frac{b_{kl} \sin kx \sin ly}{\left(k^2 + l^2\right)}$ is the solution verified.

13 Problem 4.3.3

4.3.3 Find the inverse transforms of

(a) $\hat{f}(k) = \delta(k)$ (b) $\hat{f}(k) = e^{-|k|}$ (separate $k < 0$ from $k > 0$).

Figure 18: the Problem statement

13.1 Part(a)

$$f(x) = \frac{1}{2\pi} \int_{k=-\infty}^{\infty} \delta(k) e^{ikx} dk$$

$$= \frac{1}{2\pi} \left[e^{ikx} \right]_{k=0} = \frac{1}{2\pi}$$

13.2 Part(b)

$$f(x) = \frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{-|k|} e^{ikx} dk$$

$$= \frac{1}{2\pi} \left(\int_{k=-\infty}^{0} e^{k} e^{ikx} dk + \int_{0}^{\infty} e^{-k} e^{ikx} dk \right)$$

$$= \frac{1}{2\pi} \left(\int_{k=-\infty}^{0} e^{k(1+ix)} dk + \int_{0}^{\infty} e^{k(-1+ix)} dk \right)$$

$$= \frac{1}{2\pi} \left(\left[\frac{e^{k(1+ix)}}{1+ix} \right]_{-\infty}^{0} + \left[\frac{e^{k(-1+ix)}}{-1+ix} \right]_{0}^{\infty} \right) \tag{1}$$

Looking at the first integral result

$$\left[\frac{e^{k(1+ix)}}{1+ix}\right]_{-\infty}^{0} = \frac{1}{1+ix} - \frac{e^{-\infty(1+ix)}}{1+ix} = \frac{1}{1+ix}$$

Where we looked at real part of $e^{-\infty(1+ix)} = 0$ so that we can make $e^{-\infty(1+ix)}$ to be zero.

Looking at the second integral result

$$\left[\frac{e^{k(-1+ix)}}{-1+ix}\right]_{0}^{\infty} = \frac{e^{\infty(-1+ix)}}{-1+ix} - \frac{1}{-1+ix} = -\frac{1}{-1+ix}$$

Where we looked at real part of $e^{\infty(-1+ix)} = 0$ so that we can make $e^{\infty(-1+ix)}$ to be zero. Hence, using the above two results in (1) gives

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\left(\frac{1}{1+ix} - \frac{1}{-1+ix}\right) \\
&= \frac{1}{2\pi}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right) \\
&= \frac{1}{2\pi}\left(\frac{(1-ix)+(1+ix)}{(1+ix)(1-ix)}\right) \\
&= \frac{1}{2\pi}\left(\frac{2}{1+x^2}\right) \\
&= \frac{1}{\pi}\frac{1}{1+x^2}
\end{aligned}$$

14   Problem 4.3.5

4.3.5   Verify Plancherel's energy equation for $f = \delta$ and $f = e^{-x^2/2}$. Infinite energy is allowed.

Figure 19: the Problem statement

14.1   Part(a)

For $f(x) = \delta(x)$

$$2\pi \int_{-\infty}^{\infty} \delta^2(x)\,dx = 2\pi \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta(x)\,g_n(x)\,dx$$

Where $g_n(x)$ is sequence of Gaussian functions. The RHS above becomes

$$2\pi \int_{-\infty}^{\infty} \delta^2(x)\,dx = 2\pi \lim_{n\to\infty} g_n(0)$$

But $\lim_{n\to\infty} g_n(0) = \infty$ hence

$$2\pi \int_{-\infty}^{\infty} \delta^2(x)\,dx = \infty$$

Now $\hat{f}(k) = 1$ for the Dirac delta. Hence

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} (1)\,e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} e^{-ikx}dx \\
&= \left[\frac{e^{-ikx}}{-ik}\right]_{-\infty}^{\infty} = \frac{1}{-ik}\left(e^{-ik\infty} - e^{+ik\infty}\right) = \frac{1}{-ik}(0-\infty) = \infty
\end{aligned}$$

Hence verified for δ OK.

14.2 Part(b)

For $f(x) = e^{-\frac{x^2}{2}}$ then

$$2\pi \int\limits_{-\infty}^{\infty} |f(x)|^2 \, dx = 2\pi \int\limits_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} \right|^2 dx$$

$$= 2\pi \int\limits_{0}^{\infty} e^{-x^2} dx$$

$$= 2\pi \left(\frac{\sqrt{\pi}}{2} \right)$$

$$= \pi^{\frac{3}{2}}$$

Now $\hat{f}(k)$ for the above function is

$$\hat{f}(k) = \int\limits_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx$$

$$= e^{-\frac{k^2}{2}} \sqrt{2\pi}$$

Hence

$$\int\limits_{-\infty}^{\infty} |\hat{f}(k)|^2 \, dk = \int\limits_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \sqrt{2\pi} \right|^2 dk$$

$$= 2\pi \int\limits_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \right|^2 dk$$

$$= 2\pi \int\limits_{0}^{\infty} e^{-k^2} dk$$

$$= 2\pi \left(\sqrt{\frac{\pi}{2}} \right)$$

$$= \pi^{\frac{3}{2}}$$

Which is the same as before. Hence verified.

15 Problem 4.3.6

4.3.6 What are the half-widths W_x and W_k of the bell-shaped function $f = e^{-x^2/2}$ and its transform? Show that equality holds in the uncertainty principle.

Figure 20: the Problem statement

For $f(x) = e^{-\frac{x^2}{2}}$

$$W_x^2 = \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 \, dx}$$

$$= \frac{\int_{-\infty}^{\infty} x^2 \left| e^{-\frac{x^2}{2}} \right|^2 dx}{\int_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} \right|^2 dx}$$

$$= \frac{\int_{0}^{\infty} x^2 e^{-x^2} dx}{\int_{0}^{\infty} e^{-x^2} dx}$$

$$= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2}$$

Now $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx = e^{-\frac{k^2}{2}} \sqrt{2\pi}$, hence

$$W_k^2 = \frac{\int_{-\infty}^{\infty} k^2 \left|\hat{f}(k)\right|^2 dx}{\int_{-\infty}^{\infty} \left|\hat{f}(k)\right|^2 dx}$$

$$= \frac{\int_{-\infty}^{\infty} k^2 \left|e^{-\frac{k^2}{2}} \sqrt{2\pi}\right|^2 dx}{\int_{-\infty}^{\infty} \left|e^{-\frac{k^2}{2}} \sqrt{2\pi}\right|^2 dx}$$

$$= \frac{2\pi \int_0^{\infty} k^2 e^{-k^2} dx}{2\pi \int_0^{\infty} e^{-k^2} dx}$$

$$= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2}$$

Hence

$$W_x W_k = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} = \frac{1}{2}$$

But uncertainty principle says that $W_x W_k \geq \frac{1}{2}$. Hence verified OK.

16 Problem 4.3.7

4.3.7 What is the transform of $xe^{-x^2/2}$? What about $x^2 e^{-x^2/2}$, using **4L**?

Figure 21: the Problem statement

16.1 Part(a)

Using 4L(1), let $f(x) = e^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx = \sqrt{2\pi} e^{-\frac{k^2}{2}}$, hence $\frac{d}{dx} f(x)$ will have the transform $ik\hat{f}(k)$, therefore,

$$\mathscr{F}\left(\frac{d}{dx} f(x)\right) = \mathscr{F}\left(-xe^{-\frac{x^2}{2}}\right) = ik\sqrt{2\pi} e^{-\frac{k^2}{2}}$$

Therefore $xe^{-\frac{x^2}{2}}$ has the transform $-ik\sqrt{2\pi} e^{-\frac{k^2}{2}}$

16.2 Part(b)

Let $f(x) = xe^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = -ik\sqrt{2\pi} e^{-\frac{k^2}{2}}$ from part(a). But $\frac{d}{dx} f(x) = e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}}$. Hence the transform of $\frac{d}{dx} f(x) = ik\hat{f}(k)$. Therefore

$$\mathscr{F}\left(e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}}\right) = ik\left(-ik\sqrt{2\pi} e^{-\frac{k^2}{2}}\right)$$

$$\mathscr{F}\left(e^{-\frac{x^2}{2}}\right) - \mathscr{F}\left(x^2 e^{-\frac{x^2}{2}}\right) = k^2 \sqrt{2\pi} e^{-\frac{k^2}{2}}$$

But $\mathscr{F}\left(e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi} e^{-\frac{k^2}{2}}$, hence

$$\mathscr{F}\left(x^2 e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi} e^{-\frac{k^2}{2}} - k^2 \sqrt{2\pi} e^{-\frac{k^2}{2}}$$

$$\mathscr{F}\left(x^2 e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi} e^{-\frac{k^2}{2}} \left(1 - k^2\right)$$

Therefore

$$\mathscr{F}\left(x^2 e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi} e^{-\frac{k^2}{2}} \left(1 - k^2\right)$$

17 Problem 4.3.10

4.3.10 Solve the differential equation

$$\frac{du}{dx} + au = \delta(x)$$

by taking Fourier transforms to find $\hat{u}(k)$. What is the solution u (the Green's function for this equation)?

Figure 22: the Problem statement

Let $\hat{u}(k)$ be the Fourier transform of $u(x)$. Using $\mathscr{F}\left(\frac{du}{dx}\right) = ik\hat{u}(k)$ and $\mathscr{F}(\delta) = 1$, then applying Fourier transform on the ODE gives

$$ik\hat{u}(k) + a\hat{u}(k) = 1$$

Solving for $\hat{u}(k)$

$$\hat{u}(k)(a + ik) = 1$$

$$\hat{u}(k) = \frac{1}{a + ik}$$

Hence, from page 310 in text book, it gives the inverse Fourier transform for the above as

$$u(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$$

18 Problem 4.3.21

4.3.21 Apply Fourier transforms to $\int_{-\infty}^{\infty} e^{-|x-y|} u(y) dy - 2u(x) = f(x)$ to show that the solution is $u = -\frac{1}{2}f + \frac{1}{2}g$, where g comes from integrating f twice. (Its transform is $\hat{g} = \hat{f}/(i\omega)^2$.) If $f = e^{-|x|}$ find u and verify that it solves the integral equation.

Figure 23: the Problem statement

Comparing the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-y|} u\left(y\right) dy - 2u(x) = f(x) \tag{1}$$

with the one in the textbook, page 322 in example one, where the Fourier transform of

$$\int_{-\infty}^{\infty} e^{-|x-y|} u\left(y\right) dy = f(x)$$

Is given as

$$\frac{2}{1 + \omega^2} \hat{u}(\omega) = \hat{f}(\omega)$$

The only difference is that in this problem we have an extra $-2u(x)$ term, whose Fourier transform is $-2\hat{u}(\omega)$. Hence the Fourier transform for (1) becomes

$$\frac{2}{1 + \omega^2} \hat{u}(\omega) - 2\hat{u}(\omega) = \hat{f}(\omega)$$

Solving for $\hat{u}(\omega)$

$$\hat{u}(\omega)\left(\frac{2}{1 + \omega^2} - 2\right) = \hat{f}(\omega)$$

$$\hat{u}(\omega)\left(\frac{2 - 2\left(1 + \omega^2\right)}{1 + \omega^2}\right) = \hat{f}(\omega)$$

$$\hat{u}(\omega) = \frac{1 + \omega^2}{-2\omega^2}\hat{f}(\omega)$$

We need to write the above as $\hat{u}(\omega) = \frac{-1}{2}f + \frac{1}{2}g$. Hence

$$\hat{u}(\omega) = \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \tag{2}$$

Let $f(x) = e^{-|x|}$, then

$$\begin{aligned}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= \int_{-\infty}^{\infty} e^{-|x|}e^{-i\omega x}dx \\
&= \int_{-\infty}^{0} e^{x}e^{-i\omega x}dx + \int_{0}^{\infty} e^{-x}e^{-i\omega x}dx \\
&= \left[\frac{e^{x(1-i\omega)}}{1-i\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-x(1+i\omega)}}{1+i\omega}\right]_{0}^{\infty} \\
&= \frac{1}{1-i\omega} - \frac{1}{1+i\omega} \\
&= \frac{(1+i\omega)-(1-i\omega)}{(1-i\omega)(1+i\omega)} \\
&= \frac{2}{1+\omega^2}
\end{aligned}$$

Hence using (2)

$$\begin{aligned}
\hat{u}(\omega) &= \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \\
&= \frac{-1}{2}\frac{2}{1+\omega^2} + \frac{1}{-2\omega^2}\frac{2}{1+\omega^2} \\
&= -\frac{1}{\omega^2}
\end{aligned}$$

Hence

$$u(x) = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2}e^{i\omega x}d\omega$$

Using tables $u(x) = \frac{-1}{2}|x|$.

19 Problem 4.3.27

4.3.27 Take Fourier transforms in the equation $d^4G/dx^4 - 2a^2 d^2G/dx^2 + a^4 G = \delta$ to find the transform $\hat{G}$ of the fundamental solution. How would it be possible to find G?

Figure 24: the Problem statement

The equation is

$$\frac{d^4G(x)}{dx^4} - 2a^2\frac{d^2G(x)}{dx^2} + a^4 G(x) = \delta$$

Taking Fourier transform, and using $\frac{d^nG}{dx^n} \implies (ik)^n \hat{g}(k)$, hence $G'(x) \implies ik\hat{g}(k)$, $G''(x) \implies -k^2\hat{g}(k)$, $G'''(x) \implies (ik)^4 \hat{g}(k) = k^4\hat{g}(k)$. Therefore the Fourier transform of the above differential equation is

$$k^4\hat{g}(k) + 2a^2k^2\hat{g}(x) + a^4\hat{g}(k) = 1$$

Solving for $\hat{g}(k)$

$$\hat{g}(k)\left(k^4 + 2a^2k^2 + a^4\right) = 1$$

$$\begin{aligned}
\hat{g}(k) &= \frac{1}{k^4 + 2a^2k^2 + a^4} \\
&= \frac{1}{\left(k^2 + a^2\right)^2}
\end{aligned}$$

To find $G(x)$ we need to find the inverse Fourier transform.

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\left(k^2 + a^2\right)^2}e^{ikx}dk$$

With the help of computer, I obtained the following result

$$G(x) = \frac{(1 + a|x|)}{4a^3} e^{-a|x|}$$