HW 3

Math 703
methods of applied mathematics I

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Contents

1 Problem 2.2.7

2.2.7 How far is it from the origin $(0,0,0)$ to the plane $y_1 + 2y_2 + 2y_3 = 18$? Write this constraint as $A^T y = 18$, and solve for y in

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix}.$$

Figure 1: the Problem statement

The objective function is $\frac{1}{2}\|d\|^2$ where d is the distance from origin the plane. Hence $Q(y) = \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right)$. The constraint $R = y_1 + 2y_2 + 2y_3 - 18$. Therefore, the Lagrangian is

$$\begin{aligned} L(y,x) &= Q(y) + xR \\ &= \frac{1}{2}\left(y_1^2 + y_2^2 + y_3^2\right) + x\left(y_1 + 2y_2 + 2y_3 - 18\right) \end{aligned}$$

Now we set up the optimization problem

$$\begin{aligned} \frac{\partial L}{\partial y_1} &= y_1 + x = 0 \\ \frac{\partial L}{\partial y_2} &= y_2 + 2x = 0 \\ \frac{\partial L}{\partial y_3} &= y_3 + 2x = 0 \\ \frac{\partial L}{\partial x} &= y_1 + 2y_2 + 2y_3 - 18 \end{aligned}$$

In Matrix form

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}} \tag{1}$$

Comparing the above to the standard form given

$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$$

We see that $A = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. Now we solve (1) using Gaussian elimination

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix}$$

Hence $U = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix}$. Therefore $Lc = x$ or

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}$$

Hence $c_1 = 0, c_2 = 0, c_3 = 0,\ c_4 = 18$. Now solving $Ux = c$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -9 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 18 \end{pmatrix}$$

Hence solution is Solution is:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ -2 \end{pmatrix}$$

So the Lagrangian multiplier is $\boxed{x = -2}$. Now we can calculate the distance

$$\begin{aligned} d &= \sqrt{y_1^2 + y_2^2 + y_3^2} \\ &= \sqrt{2^2 + 4^2 + 4^2} \\ &= 6 \end{aligned}$$

2 Problem 2.2.8

2.2.8 The previous question brings together several parts of mathematics if you answer it more than once:

(i) The vector y to the nearest point on the plane must be on the perpendicular ray. Therefore y must be a multiple of (1, 2, 2). What multiple lies on the plane $y_1 + 2y_2 + 2y_3 = 18$? What is the length of this y?

(ii) Since $A^T = [1\ 2\ 2]$ has length $(1 + 4 + 4)^{1/2} = 3$, the Schwarz inequality for inner products gives

$$A^T y \leq \|A\|\ \|y\| \quad \text{or} \quad 18 \leq 3\|y\|.$$

What is the minimum possible length $\|y\|$? Conclusion: The distance to the plane $A^T y = f$ is $|f|/\|A\|$.

Figure 2: the Problem statement

2.1 Part(i)

Let us assume that

$$y = k \times [1, 2, 2]$$

where k is this multiple. This means $y_1 = k, y_2 = 2k, y_2 = 2k$. In other words, the vector is

$$y = [k, 2k, 2k]$$

But since the constraint is $y_1 + 2y_2 + 2y_3 = 18$ this substituting the values of each y_i in the constraint gives

$$\begin{aligned} k + 2(2k) + 2(2k) &= 18 \\ 9k &= 18 \end{aligned}$$

Hence

$$\boxed{k = 2}$$

Using this k, the vector is

$$\begin{aligned} y &= [k, 2k, 2k] \\ &= [2, 4, 4] \end{aligned}$$

Hence the norm of the vector is

$$\begin{aligned} \|y\| &= \sqrt{y_1^2 + y_2^2 + y_3^2} \\ &= \sqrt{2^2 + 4^2 + 4^2} \\ &= 6 \end{aligned}$$

2.2 Part (ii)

Using

$$18 \leq 3 \left\| y \right\| \qquad (1)$$
$$\left\| y \right\| \geq 6$$

Therefore minimum length of y must be 6.

In (1), $18 = f$ from the equation $A^T y = f$ and $3 = ||A||$. This means the

$$y_m in = \frac{f}{||A||}$$

3 Problem 2.2.9

2.2.9 In the first example of duality—"the minimum distance to points equals the maximum distance to planes"—how do you know immediately that maximum $\leq$ minimum? In other words explain *weak duality*: The distance to any plane through the line is not greater than the distance to any point on the line.

Figure 3: the Problem statement

The primal problem is minimization of $Q(y)$ over y (unconstrained optimization), and the dual problem is maximization of $-P(x)$ over x. The minimum of $Q(y)$ is the maximum of $-P(x)$. This is the weak duality. In this problem, the point on the line must also be on a point on the plane since the line is constrained to be on the plane.

So the distance to the plane can not be larger than the distance to the line. The distance to the plane is represented by $-P(x)$ and the distance to the the line is represented by $Q(y)$. So this leads to

$$-P(x) \leq Q(y)$$

4 Problem 2.2.10

2.2.10 If $b = (15, 10)$ in the geometry example of Fig. 2.4, what are the optimal Ax and y and what are the lengths in $\|Ax\|^2 + \|y\|^2 = \|b\|^2$?

Figure 4: the Problem statement

The figure mentioned in the problem is

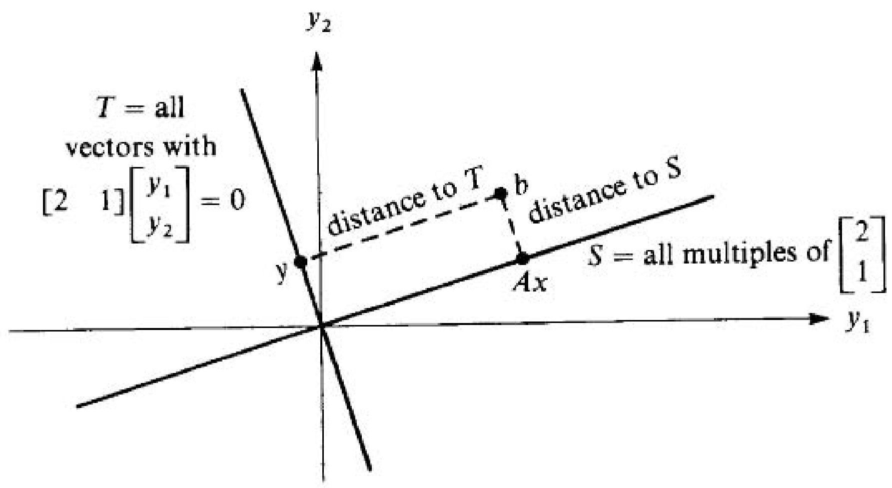

Fig. 2.4. Projection of b onto orthogonal subspaces S and T.

Figure 5: figure mentioned in problem 2.2.10

To find distance to S, we need to solve

$$\begin{aligned}
(\text{distance to S})^2 &= \min_x (Ax-b)^T (Ax-b) \\
&= \min_x \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} x - \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right)^T \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} x - \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right) \\
&= \min_x (x-10)^2 + (2x-15)^2 \\
&= \min_x 5x^2 - 80x + 325
\end{aligned}$$

Hence $\frac{d}{dx}\left(5x^2 - 80x + 325\right) = 10x - 80$ hence $x = \frac{80}{10} = 8$. Therefore

$$Ax = \begin{pmatrix} 16 \\ 8 \end{pmatrix}$$

To find y we need to solve

$$\begin{aligned}
(\text{distance to T})^2 &= \min_{A^T y=0} \|b-y\|^2 = \min_{A^T y=0} y^T y - 2b^T y + b^T b \\
&= \min_{A^T y=0} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - 2 \begin{pmatrix} 15 \\ 10 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 15 \\ 10 \end{pmatrix}^T \begin{pmatrix} 15 \\ 10 \end{pmatrix} \\
&= y_1^2 - 30y_1 + y_2^2 - 20y_2 + 325
\end{aligned}$$

Need to minimize the above subject to $A^T y = 0$ or $\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$, or $2y_1 + y_2 = 0$. Therefore, we setup an optimization problem

$$\begin{aligned}
L &= Q + xR \\
&= y_1^2 - 30y_1 + y_2^2 - 20y_2 + 325 + x\left(2y_1 + y_2\right)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial L}{\partial y_1} &= 2y_1 - 30 + 2x = 0 \\
\frac{\partial L}{\partial y_2} &= 2y_2 - 20 + x = 0 \\
\frac{\partial L}{\partial x} &= 2y_1 + y_2 = 0
\end{aligned}$$

Hence

$$\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 0 \end{pmatrix}$$

Solving gives

$$\begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 16 \end{pmatrix}$$

Hence

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Since now we know the optimal Ax and y, we can find the lengths.

$$\boxed{\|Ax\| = \left\| \begin{pmatrix} 16 \\ 8 \end{pmatrix} \right\| = 8\sqrt{5}}$$

and

$$\boxed{\|y\| = \left\| \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\| = \sqrt{5}}$$

and

$$\boxed{\|b\| = \left\| \begin{pmatrix} 15 \\ 10 \end{pmatrix} \right\| = 5\sqrt{13}}$$

Therefore

$$\left(8\sqrt{5}\right)^2 + \left(\sqrt{5}\right)^2 = \left(5\sqrt{13}\right)^2$$
$$325 = 325$$

OK, verified.

5 Problem 2.2.16

2.2.16. In m dimensions, how far is it from the origin to the hyperplane $x_1 + x_2 + \cdots + x_m = 1$? Which point on the plane is nearest to the origin?

Figure 6: the Problem statement

The constraint is $x_1 + x_2 + \cdots + x_m = 1$ and the objective function is $\frac{1}{2}\|d\|^2 = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_m^2\right)$. Hence

$$L = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_m^2\right) + x\left(x_1 + x_2 + \cdots + x_m - 1\right)$$

Setting up

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_1 + x = 0 \\
\frac{\partial L}{\partial x_2} &= x_2 + x = 0 \\
&\vdots \\
\frac{\partial L}{\partial x_n} &= x_n + x = 0 \\
\frac{\partial L}{\partial x} &= x_1 + x_2 + \cdots + x_m - 1 = 0
\end{aligned}$$

Or in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Solving, for specific m to be able to see the pattern gives for $m = 3$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

So $\boxed{x_i = \frac{1}{m}}$ so the distance is

$$\begin{aligned} \left\| x_1^2 + x_2^2 + \cdots + x_m^2 \right\| &= \sqrt{m\left(\frac{1}{m}\right)^2} \\ &= \sqrt{m} \end{aligned}$$

6 Problem 2.4.1

2.4.1 Write down m, N, r, and n for the three trusses in Fig. 2.10, and establish which is statically determinate, which is statically indeterminate, and which one has a mechanism. Describe the mechanism (the uncontrolled deformation).

Figure 7: the Problem statement

Figure 2.10 is

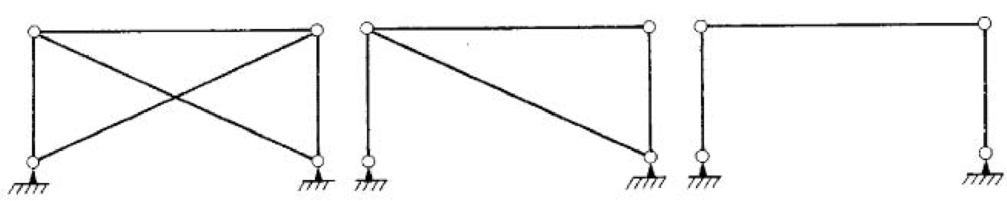

Fig. 2.10. Trusses with $m > n$ (indeterminate), $m = n$ (determinate), $m < n$ (unstable).

Figure 8: Figure 2.10 in book

m is number of bars, and N is number of nodes. Truss is stable if $m \geq 2N - r$ where r is the number of constraints. For determining rigid motion and mechanism, we need to solve $Ax = 0$ and look at the solutions.

	N (nodes)	m(bar)	r	$n = 2N - r$	determinate? $m = n$	indeterminate? $m > n$	stable?
1	4	5	4	4	No	yes	stable
2	4	4	4	4	Yes	No	stable
3	4	3	4	4	No	No	mechanism

For case (3), since it is neither determinate nor indeterminate, we need to look at $Ax = 0$. But it is clear that the truss in (3) will not move as a rigid body, but will deform. It is not stable. The table below summarizes the results.

7 Problem 2.4.4

2.4.4 For the truss in Fig. 2.10c, write down the equations $A^T y = f$ in three unknowns y_1, y_2, y_3 to balance the four external forces $f_H^1, f_H^2, f_V^1, f_V^2$. Under what condition on these forces will the equations have a solution (allowing the truss to avoid collapse)?

Figure 9: the Problem statement

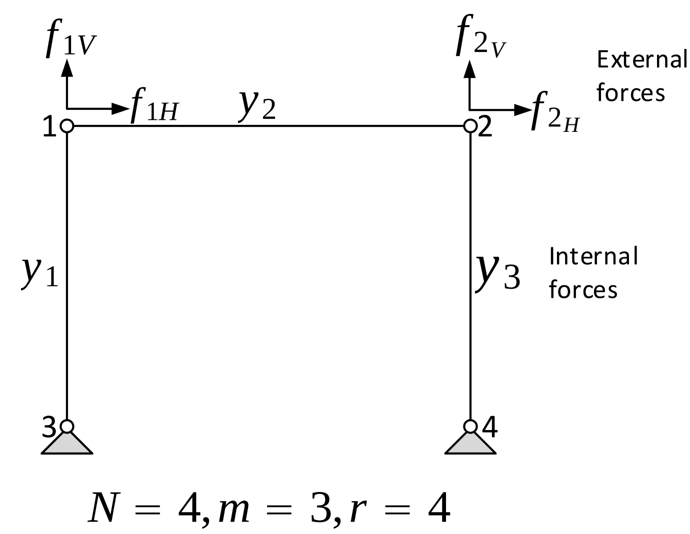

Figure 10: Figure for problem 2.4.4

The A matrix is found from $A^T y = f$. where f is a column vector of length 4 since there are 2 nodal forces, and each has 2 components. This represents a force at each node. So we first find A^T. To do this, we resolve internal forces y to balance the external nodal forces f. We assume there are nodal forces only on nodes 1, 2 in the above diagram and that $f_3 = f_4 = 0$.

Clearly $f_{1V} = y_1$ to make forces balance in the vertical direction at node 1 and that $f_{2V} = y_3$ for similar reason on node 2. On node 1, assuming y_2 is in positive, so in tension, then $-f_{1H} = y_2$ and $+f_{2H} = y_2$. If we had assumed y_2 is in the negative direction then we will get same result but signs reversed.

Therefore

$$\begin{aligned} f_{1V} &= y_1 \\ f_{2V} &= y_3 \\ f_{1H} &= -y_2 \\ f_{2H} &= y_2 \end{aligned}$$

Hence $A^T y = f$ becomes

$$\boxed{\overbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{A^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_{1H} \\ f_{1V} \\ f_{2H} \\ f_{2V} \end{pmatrix}}$$

Hence

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix A has rank 3 and the same for A^T. For $A^T y = f$ to have solution, then f must be in the column space of A^T. For solution, (equilibrium) we need $\sum f_{iH} = 0$ and $\sum f_{iV} = 0$ and moments about a point zero.

8 Problem 2.4.10

2.4.10 If we create a new node in Fig. 2.10a where the diagonals cross, is the resulting truss statically determinate or indeterminate?

Figure 11: the Problem statement

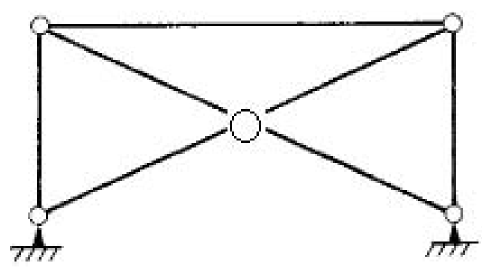

Figure 12: Figure for problem 2.4.10

With the new truss as above, the number of bars $m = 7$, and the number of nodes is $N = 5$. The number of constraints $r = 4$ (two from each support). Hence

$$
\begin{aligned}
n &= 2N - r \\
&= 10 - 4 \\
&= 6
\end{aligned}
$$

Therefore $m > n$ and A is not square. Hence not statically determinate.

9 Problem 2.4.11

2.4.11 In continuum mechanics, work is the product of stress and strain integrated over the structure: $W = \int \sigma \varepsilon \, dV$. If a bar has uniform stress $\sigma = y/A$ and uniform strain $\varepsilon = e/L$, show by integrating over the volume of the bar that $W = ye$. Then the sum over all bars is $W_{\text{total}} = y^T e$; show that this equals $f^T x$.

Figure 13: the Problem statement

Work over the first bar, of say length L_1 is

$$
\begin{aligned}
W_i &= \int \sigma_1 \epsilon_1 dV \\
&= \int \frac{y_1}{A_1} \frac{e_1}{L_1} A_1 dL \\
&= y_1 \frac{e_1}{L_1} \int dL \\
&= y_1 \frac{e_1}{L_1} L_1 \\
&= y_1 e_1
\end{aligned}
$$

Therefore, the sum all the truss is $y_1 e_1 + y_2 e_2 + \cdots + y_m e_m$ or

$$
\begin{aligned}
W_{total} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} \\
&= y^T e \qquad (1)
\end{aligned}
$$

But

$$A^T y = f$$
$$\left(A^T y\right)^T = f^T$$
$$y^T A = f^T$$
$$y^T = f^T A^{-1} \tag{2}$$

Substituting (2) into (1) gives

$$W_{total} = f^T A^{-1} e \tag{3}$$

But

$$e = Ax$$

Hence (3) becomes

$$\begin{aligned} W_{total} &= f^T A^{-1} A x \\ &= f^T x \end{aligned}$$

This is an expression of the work done by external forces at nodes. So this says the internal work equals the external work.

10 Problem 2.4.12

2.4.12 At the equilibrium $x = K^{-1}f$, show that the strain energy U (the quadratic term in P) equals $-P_{\min}$, and therefore $U = Q_{\min}$.

Figure 14: the Problem statement

The potential energy is $P(x) = \frac{1}{2}x^T A^T C A x - f^T x$. This is minimum at $A^T C A x = f$. Hence

$$\begin{aligned} P_{\min}(x) &= \frac{1}{2}x^T A^T C A x - \left(A^T C A x\right)^T x \\ &= \frac{1}{2}x^T A^T C A x - x^T A^T C^T A x \end{aligned}$$

But $C = C^T$ since diagonal matrix, then

$$\begin{aligned} P_{\min}(x) &= -\frac{1}{2}x^T A^T C A x \\ -P_{\min}(x) &= \frac{1}{2}x^T A^T C A x \end{aligned}$$

But strain energy is the quadratic term in $P(x)$, which is $\frac{1}{2}x^T A^T C A x$. Hence they are the same, which is what we are asked to show.

11 Problem 2.4.17

2.4.17 For *networks*, a typical row of $A_0^T C A_0$ (say row 1) is described on page 92: The diagonal entry is Σc_i, including all edges into node 1, and each $-c_i$ appears along the row. It is in column k if edge i connects nodes 1 and k. ($A^T C A$ is the same with the grounded row and column removed.) The problem is to describe $A_0^T C A_0$ for *trusses*, and the idea is to put together the special $A_0^T C A_0$ found in the previous exercise (*a 4 by 4 matrix for each bar*).

(a) Suppose bar i goes at angle θ_i from node 1 to node k. By assembling the $A_0^T C A_0$ for each bar, show how the 2 by 2 upper left corner of $A_0^T C A_0$ contains

$$\begin{bmatrix} \Sigma c_i \cos^2\theta_i & \Sigma c_i \cos\theta_i \sin\theta_i \\ \Sigma c_i \cos\theta_i \sin\theta_i & \Sigma c_i \sin^2\theta_i \end{bmatrix}$$

(b) Where do those terms appear (with minus signs) in the first two rows? All rows of $A_0^T C A_0$ add to zero.

Figure 15: the Problem statement

If we have a bar 1, then the elongation is due to total motion of bar two nodes due to motion of all bar attached as was shown on page 124 of the text, which is

$$e_1 = x_1 \cos\theta_1 - x_3 \cos\theta_1 + x_2 \sin\theta_1 - x_4 \sin\theta_1$$

The second bar 2 which could have one joint common with the bar 1, say (x_3, x_4) displacement, will then add to these when bar 2 itself deforms. Hence for bar 2 we have

$$e_2 = x_5 \cos\theta_2 - x_3 \cos\theta_2 + x_6 \sin\theta_2 - x_4 \sin\theta_2$$

Where in the above x_3, x_4 are kept the same as bar 1 since the joint is common. Now if bar 3 had joint (x_1, x_2) common with bar 1, it will have

$$e_3 = x_1 \cos\theta_3 - x_7 \cos\theta_3 + x_2 \sin\theta_3 - x_8 \sin\theta_3$$

When assembling the Ax matrix the pattern given should result using trigonometric relations.