HW 2

Math 703
methods of applied mathematics I

Fall 2014
University of Wisconsin, Madison

Nasser M. Abbasi

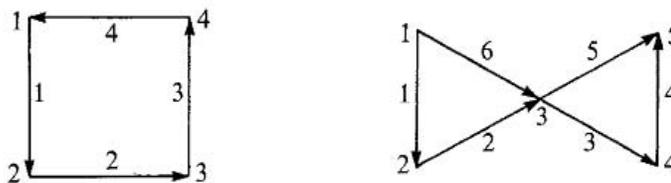January 5, 2020   Compiled on January 5, 2020 at 11:20pm   [public]

Contents

1 Problem 1.6.2

1.6.2 Write down the incidence matrices A_1 and A_2 for the following graphs:

For which right sides does $A_1 x = b$ have a solution? Which vectors are in the nullspace of A_1^T?

Figure 1: the Problem statement

In the incidence matrices, the rows indicate the edges, and the columns are the nodes. We put –1 for the node that the edge leaves and +1 for the node that the edges arrives at. Arrows are used to indicate direction.

$$A_1 = \begin{pmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & +1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & -1 & +1 \\ 0 & 0 & -1 & 0 & +1 \\ -1 & 0 & +1 & 0 & 0 \end{pmatrix}$$

We first note that matrix A_1 rank $r = 3, m = 4, n = 4$.

In $A_1 x = b$, the vectors b have to be in the column space of A_1. These are vectors in $R^m = R^4$, that span space of dimension $r = 3$. Since there is a cycle (starting from node 1 we end up at node 1 again by following the edges), this means that all the potentials at each node must be the same. But if the potential at each node is the same, then there can be no flow of current. Since flow of current represent the edge, it means each edge will have zero value. So b must be all vectors/edges that add up to $[0,0,0,0]$ vector. For the case of A_1^T, we obtain the matrix

$$A_1^T = \begin{pmatrix} -1 & 0 & 0 & 1 \\ +1 & -1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & +1 & -1 \end{pmatrix}$$

$N\left(A_1^T\right)$ in the space of $R^m = R^4$ with vectors that span dimension space $m - r = 4 - 3 = 1$. So a line. So one basis vector is all what is needed.

And now we ask about the nodes of this graph. What values can they have? This is the graph associated with this matrix
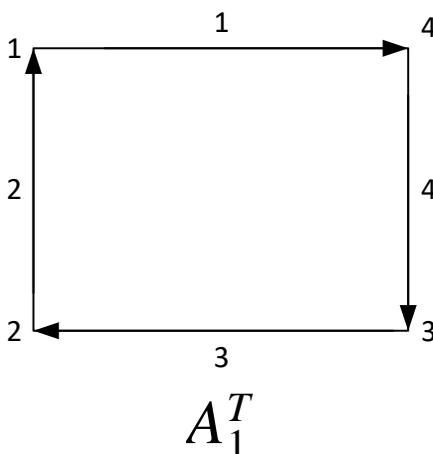

Figure 2: graph associated with this matrix

We now ask, what values should the nodes have in order for the edges to have zero flow in them? It is clear the nodes must all be equal $[1,1,1,1]$ since if the potential is same at each node, then there will be no flow (i.e. zero potential difference) on the edges. Therefore

$$\boxed{N\left(A^T\right) = [1,1,1,1]}$$

We also know from fundamental theory of linear algebra, that $R\left(A\right)$ is orthogonal to $N\left(A^T\right)$.

2 Problem 1.6.3

1.6.3 The previous matrix A_2 should have $n-1$ independent rows; which are they? There should also be $m-n+1$ independent vectors in the nullspace of A_2^T, one from each loop; which are they?

Figure 3: Problem description

The matrix A_2 has rank $r = 4, m = 6, n = 5$. The number of independent rows (edges) is $n-1$ or $5-1 = 4$ which is its rank. These can be read from the graph directly. Any 4 edges, as long as they do not complete a cycle, will qualify. Hence the edges that meet this condition are

$$6,5,4,2$$
$$6,5,4,1$$
$$6,5,3,2$$
$$6,5,3,1$$
$$6,4,3,2$$
$$6,4,3,1$$
$$5,4,2,1$$
$$5,3,2,1$$
$$4,3,2,1$$

Notice that we could not have selected for example $6,5,4,3$ since $5,4,3$ are in one loop.

The $N\left(A_2^T\right)$ has $m-r = 6-4 = 2$ dimensions. Now we take the edges on each loop. Since the loop is the null space. Since there are two loops, this give us the two independent rows. The left loop has

$$\boxed{edge\left(1\right) + edge\left(2\right) - edge\left(6\right) = [1,1,0,0,0,-1]} \tag{1}$$

Second loop has

$$\boxed{edge\left(3\right) + edge\left(4\right) - edge\left(5\right) = [0,0,+1,+1,-1,0]}$$

In other words, we put a 0 for the edge that is not there and put a +1 for the edge the goes one direction and −1 for the edge that goes in the opposite direction. For example, in (1) we put 1 for edge(1) since edge(1) is in the loop. We put 0 for edge (3) since edge (3) is not in the loop at all. We put −1 for edge(6) since it goes in the opposite direction from the others. It is arbitrary which direction is positive and which is negative, as long as one is consistent. Notice the above two basis vectors span $N\left(A_2^T\right)$ and live inside R^6 since $m = 6$ in this case.

3 Problem 1.6.5

1.6.5 If A is the incidence matrix of a connected graph and $Ax = 0$, show that $x_1 = x_2 = \cdots = x_n$. Each row of $Ax = 0$ is an equation $x_j - x_k = 0$; how do you prove that $x_j = x_k$ even when no edge goes from node j to node k?

Figure 4: Problem description

Since $Ax = 0$ then we set up the equations from incidence matrix one for each edge as follows
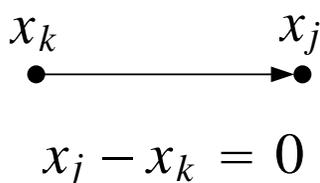
$$x_k \qquad\qquad x_j$$

$$x_j - x_k = 0$$

Figure 5: plot for prob 1.6.5

If we assign one node any arbitrary value, say $x_k = 1$, then $x_j = 1$ as well. But then any node on the other side of x_j, say x_i will now have value 1 as well. By transitivity, all other nodes will end up with the same value assigned to the first node. Hence all nodes have the same value.

For the case of a two nodes not connect. Assume the nodes are x_1 and x_4 and that there is no edge between them. Now assume there is an edge $x_1 x_2$ and edge $x_2 x_3$ and edge $x_3 x_4$. Since $x_2 = x_1$ since $Ax = 0$ then this implies $x_3 = x_2 = x_1$ as well. This also implies $x_4 = x_3 = x_2 = x_1$ or $x_1 = x_4$ even though there is no direct edge.

4 Problem 1.6.6

1.6.6 In a graph with N nodes and N edges show that there must be a loop.

Figure 6: Problem description

Proof by contradiction: Assuming there is no loop. Hence the graph must be a spanning tree. But by definition, a spanning tree with N nodes have $N - 1$ edges. But we are given that number of edges is the same as the number of nodes. Hence the assumption is not valid, and there must be a loop, called the fundamental loop or fundamental cycle.

5 Problem 1.6.7

1.6.7 For electrical networks x represents potentials, Ax represents potential differences, y represents currents, and $A^T y = 0$ is Kirchhoff's current law (Section 2.3). Tellegen's theorem says that Ax is perpendicular to y. How does this follow from the fundamental theorem of linear algebra?

Figure 7: Problem description

The fundamental theorem of linear algebra says that vectors in $R(A)$ are orthogonal to vectors in $N(A^T)$. Ax gives the vectors in $R(A)$ which is the potential difference. While currents y which results in $A^T y = 0$ are in $N(A^T)$. The following diagram illustrates this
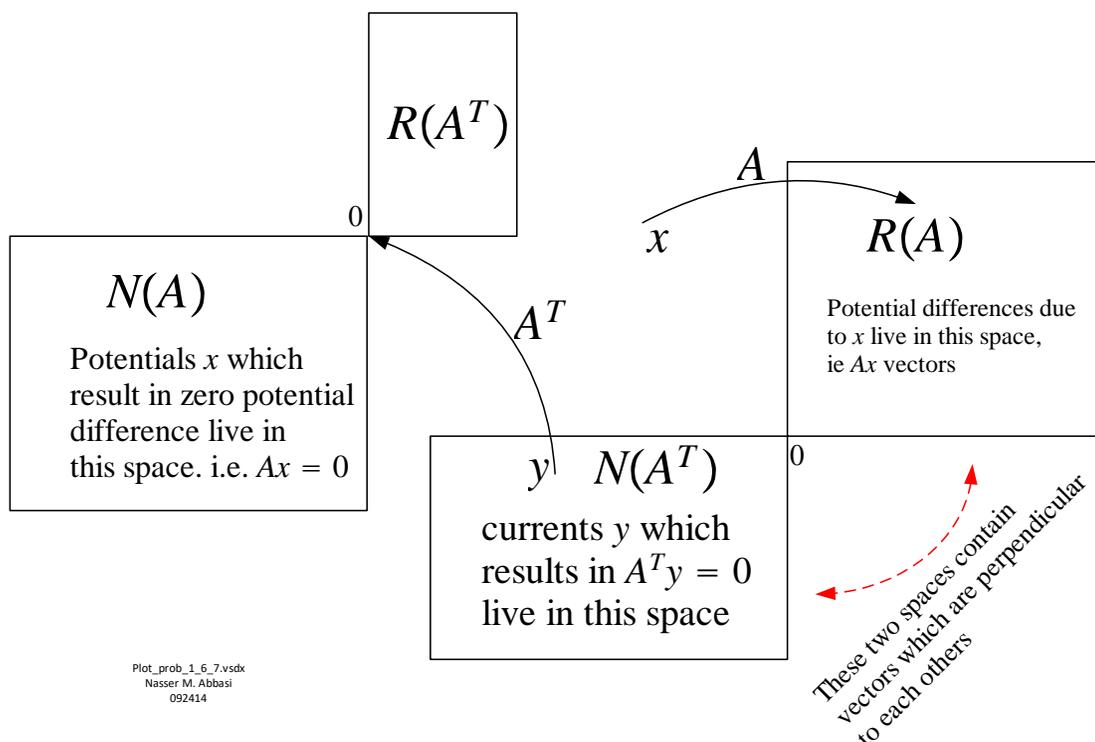
$R(A^T)$

A

0

x

$R(A)$

$N(A)$

Potentials x which result in zero potential difference live in this space. i.e. $Ax = 0$

A^T

Potential differences due to x live in this space, ie Ax vectors

y $N(A^T)$

0

currents y which results in $A^T y = 0$ live in this space

These two spaces contain vectors which are perpendicular to each others

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Nasser M. Abbasi
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Figure 8: Plot for Problem 1.6.7

6 Problem 2.1.2

2.1.2 (a) Compute the 4 by 4 matrices $A_0^T A_0$ and $A_0^T C A_0$ for the network in Fig. 2.1. Notice that like the original A_0, its columns add up to the zero column.

(b) Verify that removing the last row and column of $A_0^T C A_0$ leaves $A^T C A$ in equation (7). What is $A^T A$?

(c) Show that this $A^T A$ is positive definite by applying one of the tests in Chapter 1 (for example, compute the determinants or the pivots).

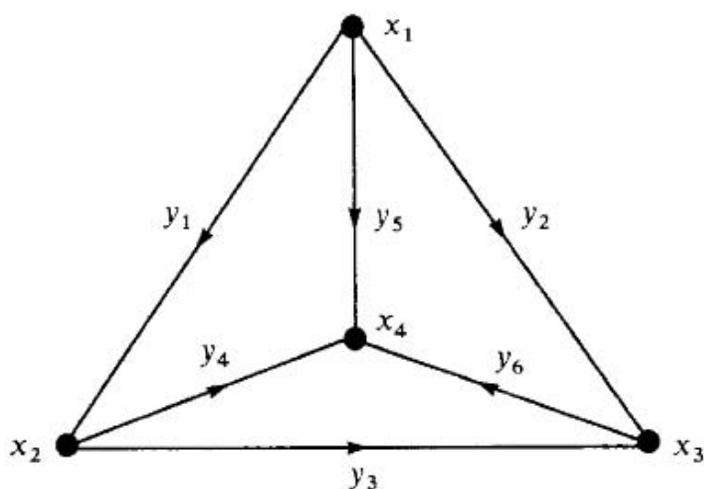Figure 9: Problem description

Figure 2.1 is the following

x_1

y_1 y_5 y_2

x_4

y_4 y_6

x_2 y_3 x_3

Fig. 2.1. Four nodal variables and six edge variables.

Figure 10: Figure 2.1 in book.

The A_o matrix, is the incidence matrix. Since we have 6 edges, the matrix will have 6 rows. Since we have 4 nodes, there will be 4 columns. The matrix is

$$A_o = \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

Hence $A_o^T A_o$ is

$$A_o^T A_o = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$

And the C matrix is $m \times m$ where $m = 6$ since this is the number of rows in A_o. Hence

$$C = \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{pmatrix}$$

Therefore

$$A_o^T C A_o = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_6 \end{pmatrix} \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

Hence

$$A_o^T C A_o = \begin{pmatrix} c_1 + c_2 + c_5 & -c_1 & -c_2 & -c_5 \\ -c_1 & c_1 + c_3 + c_4 & -c_3 & -c_4 \\ -c_2 & -c_3 & c_2 + c_3 + c_6 & -c_6 \\ -c_5 & -c_4 & -c_6 & c_4 + c_5 + c_6 \end{pmatrix}$$

We notice that the diagonal entry on $A_o^T C A_o^T$ matches the sum on the rest of the row.

7 Problem 2.1.3

2.1.3 For the triangular network in Fig. 2.1, let $f_1 = f_2 = f_3 = 1$ and $f_4 = -3$. With $C = I$ and $b = 0$, solve the equilibrium equation $-A^T C A X = f$. (Note that f_4 and x_4 do not enter, because $x_4 = 0$ and the last column of A_0 was removed.) Solve also for y, and describe the flows through the network.

Figure 11: Problem description

From problem 2.1.2, we found

$$A_o = \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 \end{pmatrix}$$

We first start by removing the last column, hence $A = \begin{pmatrix} -1 & +1 & 0 \\ -1 & 0 & +1 \\ 0 & -1 & +1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ which means x is 3×1 vector now.

We are given that $C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, hence we need to solve the equilibrium

equation $-A^T C A x = f - A^T C b$, but $b = 0$, hence this becomes

$$-A^T C A x = f$$

$$-\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We now solve the above by Gaussian elimination which gives

$$x = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

To solve for y we use the first equation of the equilibrium equation after elimination, which is given on page 92 of the textbook as

$$\begin{pmatrix} C^{-1} & A \\ 0 & -A^T C A \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f - A^T C b \end{pmatrix}$$

The first equation gives

$$C^{-1} y + A x = b$$

And for $b = 0$ this becomes

$$y = -C A x$$

$$= -\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

or

$$y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

y now is the edges, it is the flow. Hence the above says that in figure 2.1 network, shown again below
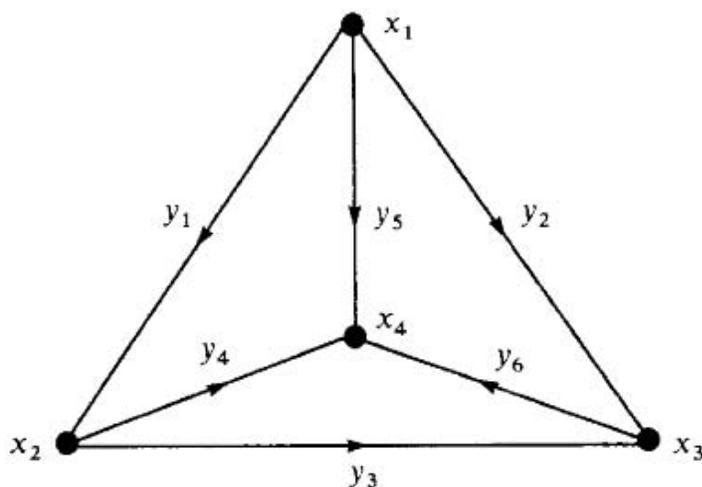

Fig. 2.1. Four nodal variables and six edge variables.

Figure 12: plot for 2.1.2

That there is now flow over edges $1, 2, 3$ (the outer cycle) and flow is only on the inner edges $4, 5, 6$ in opposite direction shown.

8   Problem 2.1.6

2.1.6 Suppose a network has N nodes and every pair is connected by an edge. Find m, the number of edges.

Figure 13: Problem description

The first node needs $N - 1$ edges to connect to the other N. The second node needs $N - 2$ edges to connect to the other nodes. We do not count the first one since it is already connected by now. The third node needs $N - 3$ edges, and so on. The last node needs no edges, since by the time it is reach, it already has an edge from all the others to it. Hence

$$m = (N - 1) + (N - 2) + \cdots + (N - N)$$
$$= \sum_{i=1}^{N} (N - i)$$
$$= \sum_{i=1}^{N} N - \sum_{i=1}^{N} i$$
$$= N^2 - \frac{1}{2} N (N + 1)$$
$$= N^2 - \frac{1}{2} N^2 - \frac{1}{2} N$$
$$= \frac{1}{2} N^2 - \frac{1}{2} N$$

Hence

$$m = \frac{1}{2}N(N-1)$$

9 Problem 2.1.12

2.1.12 Draw a network with no loops (a *tree*). Check that with one node grounded the incidence matrix A is square, and find A^{-1}. All entries of the inverse are 1, -1, or 0.

Figure 14: Problem description

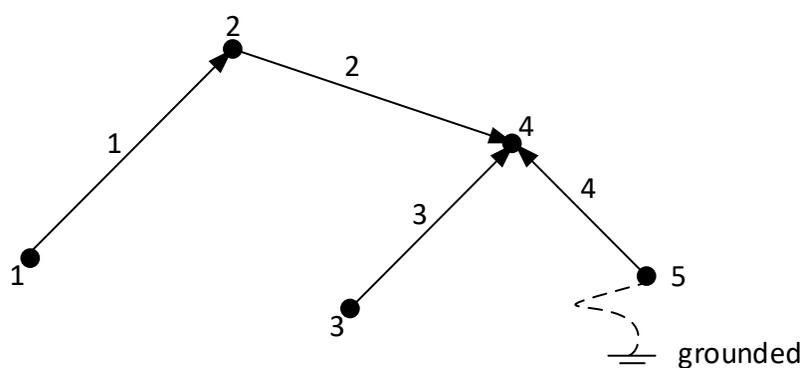A tree is drawn with arbitrary directions

Figure 15: Tree for problem 2.1.12

Before grounded node 5 the A matrix is

$$A = \begin{pmatrix} -1 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & +1 & -1 \end{pmatrix}$$

When node 5 is grounded, then column 5 is removed, now the matrix becomes

$$A = \begin{pmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

And its inverse is

$$A^{-1} = \begin{pmatrix} -1 & -1 & 0 & +1 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

10 Problem 2.2.1

2.2.1 Minimize $Q = \frac{1}{2}(y_1^2 + \frac{1}{3}y_2^2)$ subject to $y_1 + y_2 = 8$ in two ways:

(a) Solve $\partial L/\partial y = 0$, $\partial L/\partial x = 0$ for the Lagrangian $L = Q + x_1(y_1 + y_2 - 8)$.
(b) Solve the equilibrium equations (with $b = 0$) for x and y.
What is the optimal y, and what is the minimum of Q? What is the dual quadratic $-P(x)$, and where is it maximized?

Figure 16: Problem description

10.1   part(a)

$Q = \frac{1}{2}\left(y_1^2 + \frac{1}{3}y_2^2\right)$ and constraint $r = y_1 + y_2 - 8 = 0$ hence $L = Q + xr$ where x here is the Lagrange multiplier. Hence

$$\boxed{L = \frac{1}{2}\left(y_1^2 + \frac{1}{3}y_2^2\right) + x\left(y_1 + y_2 - 8\right)}$$

Therefore

$$\frac{\partial L}{\partial y_1} = y_1 + x$$

$$\frac{\partial L}{\partial y_2} = \frac{1}{3}y_2 + x$$

$$\frac{\partial L}{\partial x} = \left(y_1 + y_2 - 8\right)$$

In matrix form it becomes

$$\nabla L = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \frac{1}{3} & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} \tag{1}$$

Solving by Gaussian elimination gives

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \\ x \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix}}$$

10.2   Part(b)

We now compare (1) above to the equilibrium matrix equation given by

$$\begin{pmatrix} C^{-1} & A \\ 0 & A^T C A \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} b \\ f - A^T C b \end{pmatrix}$$

Which for $b = 0$ becomes

$$\begin{pmatrix} C^{-1} & A \\ 0 & A^T C A \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

From the above, and comparing to (1) we see that $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, f = 8$. Hence we first solve for x

$$A^T C A x = f$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}x = 8$$

$$4x = 8$$

$$x = 2$$

Now the first equation is used to solve for y

$$C^{-1}y + Ax = 0$$

$$y = CAx$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}x$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}2$$

Hence the optimal y is

$$\boxed{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}}$$

Which is the same as in part(a). At this point, Q is now evaluated

$$Q_{min} = \frac{1}{2}\left(y_1^2 + \frac{1}{3}y_2^2\right)$$

$$= \frac{1}{2}\left(2^2 + \frac{1}{3}6^2\right)$$

$$= 8$$

The dual quadratic is given on page 101 of the text

$$-P(x) = -\frac{1}{2}(Ax - b)^T C(Ax - b) - x^T f$$

And for $b = 0$ it becomes

$$-P(x) = -\frac{1}{2}x^T A^T C A x - x^T f$$

But from above, $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, f = 8$ hence

$$-P(x) = -\frac{1}{2}x^T \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} x - 8x^T$$

$$= -\frac{1}{2}x^T 4x - 8x^T$$

$$= -2x^T x - 8x^T$$

But $x^T x = x^2$ so the above can be written as

$$-P(x) = -2x^2 - 8x$$

$$P(x) = x(2x + 8)$$

To find where it is maximum, since $\frac{dP}{dx} = 0 = 4x + 8$ hence $\boxed{x = -2}$. Therefore, $-P(x)$ is maximized at same x where $Q(x)$ is minimized.

11   Problem 2.2.2

2.2.2   Find the nearest point to the origin on the plane $y_1 + y_2 + \dots + y_m = 1$ by solving for y_1, substituting into $Q = \frac{1}{2}(y_1^2 + \dots + y_m^2)$, and minimizing with respect to the other y's. Then solve the same problem with Lagrange multipliers.

Figure 17: Problem description

$$Q = \frac{1}{2}\left(y_1^2 + y_2^2 + \dots + y_m^2\right)$$

Constraints in $y_1 + y_2 + \dots + y_m = 1$. Solving for y_1 from the constrains and substitute the result in Q. Hence

$$y_1 = 1 - \left(y_2 + y_3 + \dots + y_m\right)$$

And Q becomes

$$Q = \frac{1}{2}\left(\left[1 - \left(y_2 + y_3 + \dots + y_m\right)\right]^2 + \left(y_2^2 + \dots + y_m^2\right)\right)$$

$$= \frac{1}{2}\left(1 + \left(y_2 + y_3 + \dots + y_m\right)^2 - 2\left(y_2 + y_3 + \dots + y_m\right) + \left(y_2^2 + \dots + y_m^2\right)\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(y_2 + y_3 + \dots + y_m\right)^2 - \left(y_2 + y_3 + \dots + y_m\right) + \frac{1}{2}\left(y_2^2 + \dots + y_m^2\right)$$

Hence

$$\frac{\partial Q}{\partial y_2} = \left(y_2 + y_3 + \dots + y_m\right) - 1 + y_2 = 0$$

$$\frac{\partial Q}{\partial y_3} = \left(y_2 + y_3 + \dots + y_m\right) - 1 + y_3 = 0$$

$$\vdots$$

$$\frac{\partial Q}{\partial y_m} = \left(y_2 + y_3 + \dots + y_m\right) - 1 + y_m = 0$$

The above can be written as

$$2y_2 + y_3 + \cdots + y_m = 1$$
$$y_2 + 2y_3 + \cdots + y_m = 1$$
$$\vdots$$
$$y_2 + y_3 + \cdots + 2y_m = 1$$

In matrix form,

$$
\begin{pmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 2 & 1 & \cdots & 1 \\
1 & 1 & 2 & \cdots & 1 \\
1 & 1 & 1 & \ddots & 1 \\
1 & 1 & 1 & \cdots & 2
\end{pmatrix}
\begin{pmatrix}
y_2 \\
y_3 \\
y_4 \\
\vdots \\
y_m
\end{pmatrix}
=
\begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
$$

Solving this gives

$$y_2 = y_3 = \cdots = y_m = \frac{1}{m}$$

This was done by solving for $m = 3, 4, 5 \cdots$ on the computer and seeing the result is always $\frac{1}{m}$. Now we solve for y_1. Since

$$y_1 = 1 - \left(y_2 + y_3 + \cdots + y_m \right)$$

Then

$$y_1 = 1 - \left(\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m} \right)$$
$$= 1 - (m-1)\frac{1}{m}$$
$$= 1 - \left(1 - \frac{1}{m} \right)$$
$$= \frac{1}{m}$$

Therefore, all y_i have the value $\frac{1}{m}$

Now the last part is solved, which asks to solve the same problem using Lagrange multiplier. Since there is one constraint, then $n = 1$ and since there are m number of y variables, there will be $n + m$ or $m + 1$ equations.

$$L = Q + xR$$

Where R is the contraints. The above becomes

$$L = \frac{1}{2} \left(y_1^2 + y_2^2 + \cdots + y_m^2 \right) + x \left(y_1 + y_2 + \cdots + y_m - 1 \right)$$

Now we take the derivatives and set up the system of equations

$$\frac{\partial L}{\partial y_1} = y_1 + x = 0$$
$$\frac{\partial L}{\partial y_2} = y_2 + x = 0$$
$$\vdots$$
$$\frac{\partial L}{\partial y_m} = y_m + x = 0$$
$$\frac{\partial L}{\partial x} = \left(y_1 + y_2 + \cdots + y_m - 1 \right) = 0$$

In matrix form the above is

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 1 \\
0 & 1 & 0 & \cdots & 1 \\
0 & 0 & 1 & \cdots & 1 \\
0 & 0 & 0 & \ddots & 1 \\
1 & 1 & 1 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
x
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
1
\end{pmatrix}
$$

Solving this also gives the same answer as above, which is

$$y_i = \frac{1}{m}$$

and the Lagrange multipler is found, using any of the above equation, such as $y_1 + x = 0$ to be

$$\boxed{x = -\frac{1}{m}}$$

12 Problem 2.2.4

2.2.4 Find the rectangle with corners at points $(\pm y_1, \pm y_2)$ on the ellipse $y_1^2 + 4y_2^2 = 1$, such that the perimeter $4y_1 + 4y_2$ is as large as possible.

Figure 18: Problem description

We want to maximize $Q = 4y_1 + 4y_2$ subject to $y_1^2 + 4y_2^2 = 1$. Hence

$$L = Q + x\left(y_1^2 + 4y_2^2 - 1\right)$$
$$= 4y_1 + 4y_2 + x\left(y_1^2 + 4y_2^2 - 1\right)$$

And

$$\frac{\partial L}{\partial y_1} = 4 + 2xy_1 = 0 \tag{1}$$

$$\frac{\partial L}{\partial y_2} = 4 + 8xy_2 = 0 \tag{2}$$

$$\frac{\partial L}{\partial x} = y_1^2 + 4y_2^2 - 1 = 0 \tag{3}$$

Or

$$y_1 = \frac{-2}{x} \tag{1}$$

$$y_2 = \frac{-1}{2x} \tag{2}$$

$$y_1^2 + 4y_2^2 = 1 \tag{3}$$

From (1),(2) we see that $y_1 = 4y_2$. Substituting in (3) gives

$$\left(4y_2\right)^2 + 4y_2^2 = 1$$
$$16y_2^2 + 4y_2^2 = 1$$
$$y_2 = \pm\sqrt{\frac{1}{20}}$$

Hence

$$y_1 = \pm\sqrt{\frac{16}{20}} = \pm\sqrt{\frac{4}{5}}$$

So the corners are $\left(\pm\sqrt{\frac{4}{5}}, \pm\sqrt{\frac{1}{20}}\right)$. Here is a plot of the ellipse showing the 4 corners given by the above solution to verify

```
a = 1;
b = (1/2);
y1 = Sqrt[4/5]; y2 = Sqrt[1/20];
Graphics[
{
Circle[{0, 0}, {a, b}],
{EdgeForm[Thick], LightGray, Rectangle[{-y1, -y2}, {y1, y2}]}
},
Axes -> True]
```
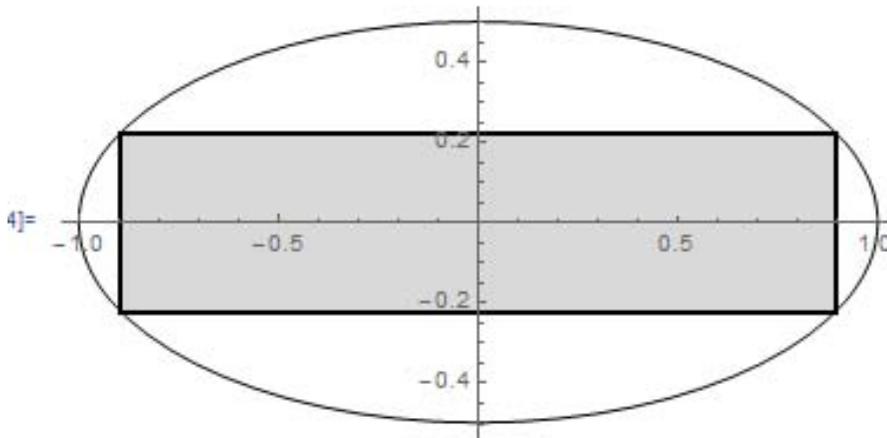
Figure 19: plot for prob 2.2.4

13   Problem 2.2.6

2.2.6   The minimum distance to the surface $A^T y = f$ equals the maximum distance to the hyperplanes which _____

Complete this statement of duality.

Figure 20: Problem description

From the duality statement on page 100 of the text, we can complete this sentence similarly by saying

The minimum distance to the surface $A^T y = f$ equals the maximum distance to the hyperplanes which go through those hyperplanes.