HW 1

Math 703
methods of applied mathematics I

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Contents

1 Problem 1.2.7

1.2.7 From the multiplication LS show that

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & 0 & 1 \end{bmatrix} \text{ is the inverse of } S = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix}.$$

S subtracts multiples of row 1 and L adds them back.

Figure 1: the Problem statement

Solution

Multiplying LS gives

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $LS = I$ then $L = S^{-1}$ by definition.

2 Problem 1.2.8

1.2.8 Unlike the previous exercise, show that

$$L = \begin{bmatrix} 1 & & \\ l_{21} & 1 & \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ is } \textit{not} \text{ the inverse of } S = \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & -l_{32} & 1 \end{bmatrix}.$$

If S is changed to

$$E = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -l_{21} & 1 & \\ -l_{31} & 0 & 1 \end{bmatrix},$$

show that E is the correct inverse of L. E contains the elimination steps as they are actually done—subtractions of multiples of row 1 followed by subtraction of a multiple of row 2.

Figure 2: the Problem statement

Solution

Multiplying LS gives

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & -l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{21}l_{32} & 0 & 1 \end{pmatrix}$$

Since $LS \neq I$ then L is not the inverse of S. Now let $S = E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ l_{21}l_{32} - l_{31} & -l_{32} & 1 \end{pmatrix}$

and now evaluating LS gives

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ l_{21}l_{32} - l_{31} & -l_{32} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore, with the new S matrix, now L is the inverse of S since $LS = I$

3 Problem 1.2.9

1.2.9 Find examples of 2 by 2 matrices such that

(a) $LU \neq UL$
(b) $A^2 = -I$, with real entries in A
(c) $B^2 = 0$, with no zeros in B
(d) $CD = -DC$, not allowing $CD = 0$.

Figure 3: the Problem statement

Solution

3.1 Part (a)

Take any random 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, By elimination $U = \begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}$. Now LU is found, giving back A as expected

$$\begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

UL is found

$$\begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} = \begin{pmatrix} a + \frac{1}{a}bc & b \\ \frac{1}{a}c\left(d - \frac{1}{a}bc\right) & d - \frac{1}{a}bc \end{pmatrix}$$

Comparing LU and UL above, it can be seen that by setting $b = 0$ the $LU = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ while $UL = \begin{pmatrix} a & 0 \\ \frac{1}{a}cd & d \end{pmatrix}$, which means they will be different as long as $d \neq a$. So picking any A matrix which has $b = 0$ and which $d \neq a$ will work. An example is

$$\boxed{A = \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}}$$

To verify, $U = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, hence $LU = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 2 \end{pmatrix}$ while $UL = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 10 & 2 \end{pmatrix}$. They are different.

3.2 Part (b)

Take any random 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{pmatrix}$ Now solving

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

gives 4 equations for a,b,c,d

$$
\begin{aligned}
a^2+bc&=1\\
ab+bd&=0\\
ac+cd&=0\\
d^2+bc&=1
\end{aligned}
$$

Gives the following solutions

$$
\begin{aligned}
a&=-1,b=0,c=0,d=-1\\
a&=1,b=0,c=0,d=-1\\
a&=-1,b=0,c=0,d=1\\
a&=1,b=0,c=0,d=1
\end{aligned}
$$

Any of the above solutions will satisfy $A^2=I$. For example, using the first one gives

$$
\boxed{A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}}
$$

3.3 Part (c)

As was done above, the following set of equations are solved.

$$
\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & d^2+bc\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}
$$

Hence

$$
\begin{aligned}
a^2+bc&=0\\
ab+bd&=0\\
ac+cd&=0\\
d^2+bc&=0
\end{aligned}
$$

Solution is

```
eq1:=a^2+b*c=0;eq2:=a*b+b*d=0;eq3:=a*c+c*d=0;eq4:=a^2+b*c=0;
solve({eq1,eq2,eq3,eq4},{a,b,c,d});
 {a = a, b = b, c = -a^2/b, d = -a}, {a = 0, b = 0, c = 0, d = d}
```

Since we are looking for non-zero elements in B, then the first solution $\{a=a,b=b,c=-\frac{a^2}{b},d=-a\}$ is used. For example, letting $a=1,b=2,c=-\frac{1}{2},d=-1$ gives

$$
\boxed{B=\begin{pmatrix}1 & 2\\ -\frac{1}{2} & -1\end{pmatrix}}
$$

To verify

$$
\begin{pmatrix}1 & 2\\ -\frac{1}{2} & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ -\frac{1}{2} & -1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}
$$

3.4 Part (d)

Let $C=\begin{pmatrix}a & b\\ c & d\end{pmatrix},D=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$, hence we want $CD=-DC$. To simplify this, let the diagonal be zero in both cases. This reduced the equations to 4 unknowns. Hence Let $C=\begin{pmatrix}0 & b\\ c & 0\end{pmatrix},D=\begin{pmatrix}0 & f\\ g & 0\end{pmatrix}$ and

$$
\begin{aligned}
CD&=\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\begin{pmatrix}0 & f\\ g & 0\end{pmatrix}=\begin{pmatrix}bg & 0\\ 0 & cf\end{pmatrix}\\
DC&=\begin{pmatrix}0 & f\\ g & 0\end{pmatrix}\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}=\begin{pmatrix}cf & 0\\ 0 & bg\end{pmatrix}
\end{aligned}
$$

Hence we want to solve $\begin{pmatrix}bg & 0\\ 0 & cf\end{pmatrix}=-\begin{pmatrix}cf & 0\\ 0 & bg\end{pmatrix}$ Hence this reduces to just solving

$$
bg=-cf
$$

Let $b = n, c = -n, g = n, f = n$ which satisfies the above. I.e. $n \times n = -(-n \times n) \Rightarrow n^2 = n^2$, therefore

$$C = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$$

To verify, $CD = \begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix} = \begin{pmatrix} n^2 & 0 \\ 0 & -n^2 \end{pmatrix}$ and $DC = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} = \begin{pmatrix} -n^2 & 0 \\ 0 & n^2 \end{pmatrix}$ hence $DC = -CD$. Let $n = 2$ for example, then

$$\boxed{C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}}$$

$$\boxed{D = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}}$$

4 Problem 1.3.2

1.3.2 Factor $A = \begin{bmatrix} 3 & 6 \\ 6 & 8 \end{bmatrix}$ into $A = LDL^T$. Is this matrix positive definite? Write x^TAx as a combination of two squares.

Figure 4: the Problem statement

Solution

$$A = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix}$$

Hence $U = \begin{pmatrix} 3 & 6 \\ 0 & -4 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, therefore $D = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$. D has the pivots on its diagonal. The pivots is the diagonal of U. Therefore

$$LDL^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}^T = \begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix} = A$$

Since not all the pivots are positive and the matrix is symmetric, then this is not positive definite (P.D.). This can be confirmed by writing

$$\begin{aligned} x^TAx &= \begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 6 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1(3x_1 + 6x_2) + x_2(6x_1 + 8x_2) \\ &= 3x_1^2 + 12x_1x_2 + 8x_2^2 \end{aligned}$$

We now need to complete the squares.

$$\begin{aligned} x^TAx &= 3(x_1 + ax_2)^2 + cx_2^2 \\ &= 3\left(x_1^2 + a^2x_2^2 + 2ax_1x_2\right) + cx_2^2 \\ &= 3x_1^2 + \left(3a^2 + c\right)x_2^2 + 6ax_1x_2 \end{aligned}$$

Comparing to $3x_1^2 + 12x_1x_2 + 8x_2^2$ we see that $a = 2$ and $c = 8 - 3a^2 = 8 - 12 = -4$, hence

$$x^TAx = 3(x_1 + 2x_2)^2 - 4x_2^2$$

This shows that x^TAx is not positive for all x due to the -4 term. For example, if $x = \{1, -1\}$ then $x^TAx = -1$. Basically, we obtain the same result as before. For a symmetric matrix A, if not all the pivots are positive, then the matrix is not P.D. Using x^TAx is another method to answer the same question. After completing the squares, we look to see if all the coefficients are positive or not.

5 Problem 1.3.6

1.3.6 In the 2 by 2 case, suppose the positive coefficients a and c dominate b in the sense that $a + c > 2b$. Is this enough to guarantee that $ac > b^2$ and the matrix is positive definite? Give a proof or a counterexample.

Figure 5: the Problem statement

Solution

A counter example is $a = 8, b = 2, c = 4$. We see that $a + b > 2c$ but $ac = 16$ and $c^2 = 16$, hence ac is not greater than b^2. So $a + c > 2b$ do not guarantee that $ac > c^2$. Therefore, we also can not guarantee that the matrix is P.D. this comes from the pivots. The pivots of $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are $\{a, c - \frac{b^2}{a}\}$. Since $a > 0$ as given, then we just need to check if $c - \frac{b^2}{a} > 0$. This means $ac - b^2 > 0$. But since we can't guarantee that $ac > b^2$ then this means the second pivot can be negative. Hence the matrix A with such property can not be guaranteed to be P.D.

6 Problem 1.3.7

1.3.7 Decide for or against the positive definiteness of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Write A as $l_1 d_1 l_1^T$ and write A' as LDL^T.

Figure 6: the Problem statement

Solution

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

To show if A is P.D., we need to show that all the pivots are positive. This is the same as showing that $x^T Ax > 0$ for all non-zero x.To obtain the pivots, we generate the U and look at the diagonal values. From the above, we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{U}$$

Hence using $l_1 = 1$ we see that the pivots are not all positive. There are zero pivot. Hence A is not P.D. For

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \rightarrow \overbrace{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}^{U}$$

Hence all the pivots are positive. Therefore A' is P.D. We can write it as LDL^T

$$A' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

7 Problem 1.3.8

1.3.8 If each diagonal entry a_{ii} is larger than the sum of the absolute values $|a_{ij}|$ along the rest of its row, then the symmetric matrix A is positive definite. How large would c have to be in

$$A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$$

for this statement to apply? How large does c actually have to be to assure that A is positive definite? Note that

$$x^TAx = (x_1 + x_2 + x_3)^2 + (c-1)(x_1^2 + x_2^2 + x_3^2);$$

when is this positive?

Figure 7: the Problem statement

Solution

$$A = \begin{pmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{pmatrix}$$

$c > 2$ is enough to guarantee row dominant matrix. For P.D., looking at $x^TAx = (x_1 + x_2 + x_3)^2 + (c-1)\left(x_1^2 + x_2^2 + x_3^2\right)$ shows that $c - 1 > 0$ is the condition for P.D. which implies $c > 1$. Hence it is enough that $c > 1$.

8 Problem 1.3.11

1.3.11 A function $F(x, y)$ has a local minimum at any point where its first derivatives $\partial F/\partial x$ and $\partial F/\partial y$ are zero and the matrix of second derivatives

$$A = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x^2} & \dfrac{\partial^2 F}{\partial x \partial y} \\ \dfrac{\partial^2 F}{\partial x \partial y} & \dfrac{\partial^2 F}{\partial y^2} \end{bmatrix}$$

is positive definite. Is this true for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ and $F_2 = \cos x \cos y$ at $x = y = 0$? Does F_1 have a global minimum or can it approach $-\infty$?

Figure 8: the Problem statement

Solution

For $F_1 = x^2 - x^2y^2 + y^2 + y^3$, we find $\frac{\partial F_1}{\partial y} = -2x^2y + 2y + 3y^2 = 0$ at $x = 0, y = 0$. And $\frac{\partial F_1}{\partial x} = 2x - 2xy^2 = 0$

at $x = 0, y = 0$. Now we need to look at the P.D. of

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 - 2y^2 & -4xy \\ -4xy & -2x^2 + 2 + 6y \end{pmatrix}$$

At $x = 0, y = 0$ the above becomes

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

This is already in U form. Since the diagonal is all positive, then this is P.D., which means it is true for $F_1\left(x, y\right)$. Now we check $F_2\left(x, y\right)$

$F_2\left(x, y\right) = \cos x \cos y$. Hence $\frac{\partial F_1}{\partial y} = -\sin y \cos x = 0$ at $x = 0, y = 0$. And $\frac{\partial F_1}{\partial x} = -\sin x \cos y = 0$ at $x = 0, y = 0$. Now we need to look at the P.D. of

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\cos x \cos y & \sin y \sin x \\ \sin y \sin x & -\cos y \cos x \end{pmatrix}$$

And at $x = 0, y = 0$ the above becomes

$$\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence this is not P.D, since the pivots are negative. To answer the part about F_1 having global minimum. The point $x = 0, y = 0$ is local minimum for $F_1 = x^2 - x^2y^2 + y^2 + y^3$ since $\begin{pmatrix} \frac{\partial^2 F_1}{\partial x^2} & \frac{\partial^2 F_1}{\partial x \partial y} \\ \frac{\partial^2 F_1}{\partial x \partial y} & \frac{\partial^2 F_1}{\partial y^2} \end{pmatrix}$ was found to be P.D. at $x = 0, y = 0$. But this is not global minimum. Only when the function can be written as quadratic form $x^T A x$ will the local minumum be global minumum. In this case, F_1 can approach $-\infty$, hence this is the global minimum.

Taking the limit $\lim_{x_1 \to -\infty} F_1 = \left(1 - y^2\right) \infty$. Taking the limit of this as $y \to \infty$ gives $-\infty$. Here is a plot of F_1 around $x = 0, y = 0$ showing it is a local minimum

```
F1 = x^2 - x^2 y^2 + y^2 + y^3
Plot3D[F1, {x, -3, 3}, {y, -3, 3},
PlotLabel -> "F1 function", AxesLabel -> {x, y}]
```

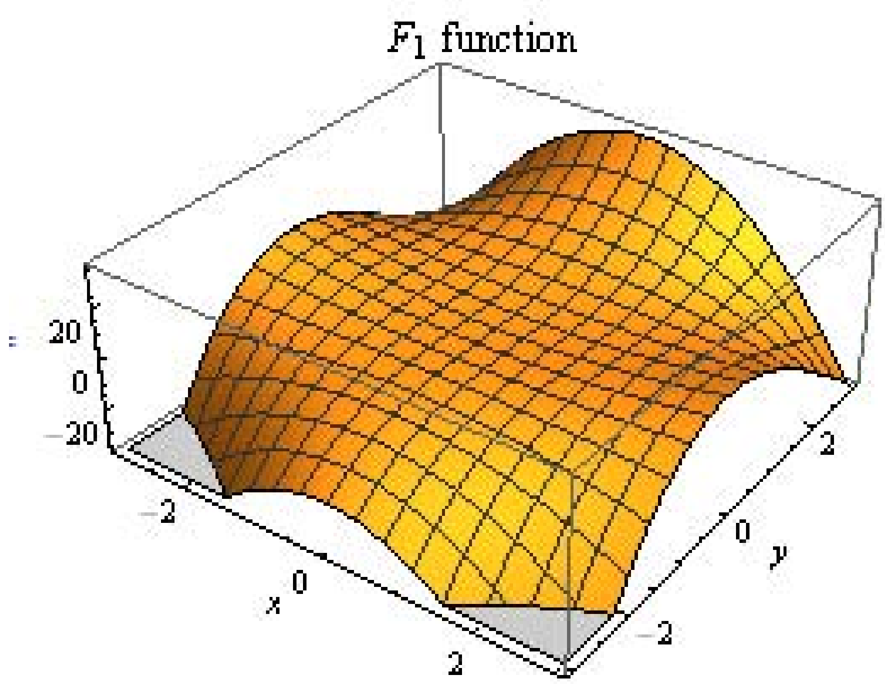


Figure 9: plot of the above

9 Problem 1.4.5

1.4.5 The best fit to b_1, b_2, b_3, b_4 by a horizontal line (a constant function $y = C$) is their average $C = (b_1 + b_2 + b_3 + b_4)/4$. Confirm this by least squares solution of

$$Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [C] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

From calculus, which C minimizes the error $E = (b_1 - C)^2 + \cdots + (b_4 - c)^2$?

Figure 10: the Problem statement

Solution

The equation of the line is $y = C$, hence we obtain 4 equations.

$$b_1 = C$$
$$b_2 = C$$
$$b_3 = C$$
$$b_4 = C$$

or

$$\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{A} C = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Hence now we set $A^T A x = A^T b$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$4C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

$$4C = b_1 + b_2 + b_3 + b_4$$

Hence $C = \frac{b_1+b_2+b_3+b_4}{4}$, Which is the average. Using calculus, to minimize $E = (b_1 - C)^2 + (b_2 - C)^2 + (b_3 - C)^2 + (b_4 - C)^2$

$$\begin{aligned} \frac{dE}{dC} &= -2(b_1 - C) - 2(b_2 - C) - 2(b_3 - C) - 2(b_4 - C) \\ 0 &= 8C - 2b_1 - 2b_2 - 2b_3 - 2b_4 \\ 8C &= 2b_1 + 2b_2 + 2b_3 + 2b_4 \\ C &= \frac{b_1 + b_2 + b_3 + b_4}{4} \end{aligned}$$

Which is the same found using $A^T A x = A^T b$ solution.

10 Problem 1.4.7

1.4.7 For the three measurements $b = 0, 3, 12$ at times $t = 0, 1, 2$, find

(i) the best horizontal line $y = C$
(ii) the best straight line $y = C + Dt$
(iii) the best parabola $y = C + Dt + Et^2$.

Figure 11: the Problem statement

Solution

10.1 Part (a)

For $y = C$ we obtain the following equations

$$\begin{aligned} b_1 &= C \\ b_2 &= C \\ b_3 &= C \end{aligned}$$

Hence

$$\overbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}^{A} C = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Applying $A^T Ax = A^T b$ gives

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} C &= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ 3C &= b_1 + b_2 + b_3 \\ C &= \frac{b_1 + b_2 + b_3}{3} \end{aligned}$$

Therefore $y = C = \frac{b_1+b_2+b_3}{3} = \frac{0+3+12}{3} = 5$, or

$$y = 5$$

10.2 Part (b)

For $y = C + Dt$ we obtain the following equations

$$\begin{aligned} b_1 &= C + Dt \\ b_2 &= C + Dt \\ b_3 &= C + Dt \end{aligned}$$

Applying the numerical values gives results in

$$\begin{aligned} 0 &= C \\ 3 &= C + D \\ 12 &= C + D(2) \end{aligned}$$

Hence

$$\overbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}}^{A} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$$

Applying $A^T A x = A^T b$ gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 15 \\ 27 \end{pmatrix}$$

Now we solve this using Gaussian elimination. First U is found

$$\begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix}$$

Hence $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $Lc = b$, then $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 27 \end{pmatrix}$ now c is found by forward substitution, giving $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Now we solve $Ux = c$ or $\begin{pmatrix} 3 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$ by backward substitution, the result is

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

Hence the line is

$$\boxed{y = -1 + 6t}$$

Here is a plot of the fit found above

```
b = {0, 3, 12}; t = {0, 1, 2};
p1 = ListPlot[Transpose[{t, b}], PlotStyle -> Red];
p2 = Plot[-1 + 6 t, {t, -.5, 3}, PlotTheme -> "Detailed",
FrameLabel -> {{"y(t)", None}, {"t", "Fit by least squares"}}];
Show[p2, p1]
```

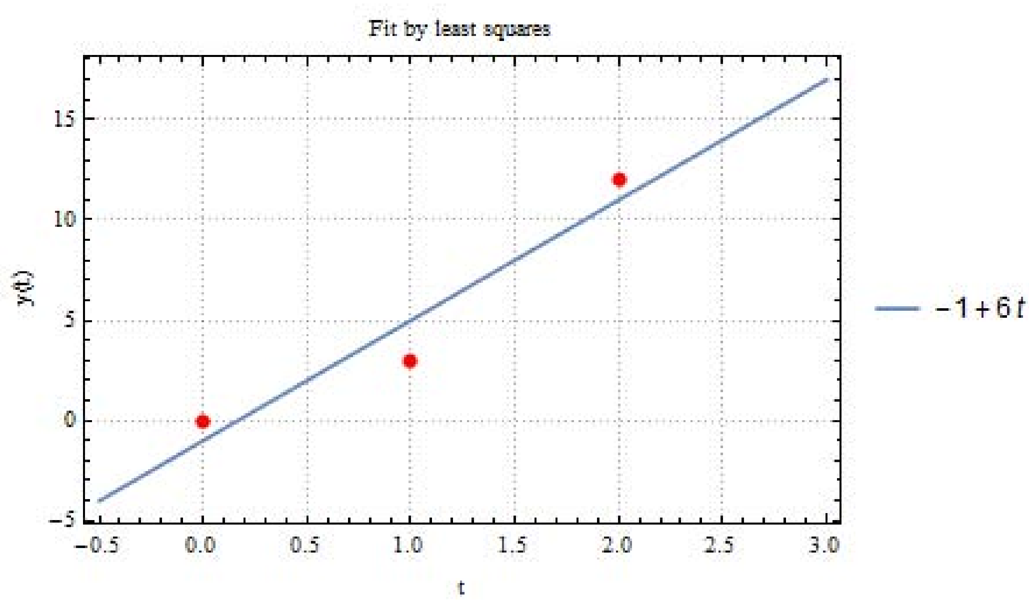


Figure 12: Plot of the above

10.3 Part c

For $y = C + Dt + Et^2$ we obtain the following equations

$$\begin{aligned} b_1 &= C + Dt + Et^2 \\ b_2 &= C + Dt + Et^2 \\ b_3 &= C + Dt + Et^2 \end{aligned}$$

Applying the numerical values gives results in

$$\begin{aligned} 0 &= C \\ 3 &= C + D + E \\ 12 &= C + 2D + 4E \end{aligned}$$

Hence

$$\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}}^{A} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$$

Now we solve this using Gaussian elimination. We do not need to use $A^T A$ least squares since the number of rows is the same as number of columns. First U is found

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Hence $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ and $Lc = b$, then $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 12 \end{pmatrix}$ now c is found by forward substitution, giving $\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$

Now we solve $Ux = c$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ by backward substitution, giving

$$\begin{pmatrix} C \\ D \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Hence the solution is

$$y = 3t^2$$

Here is a plot of the fit

```
b = {0, 3, 12}; t = {0, 1, 2};
p1 = ListPlot[Transpose[{t, b}], PlotStyle -> Red];
p2 = Plot[3 t^2, {t, -.5, 3}, PlotTheme -> "Detailed",
FrameLabel -> {{"y(t)", None}, {"t", "Fit by least squares"}}];
Show[p2, p1]
```

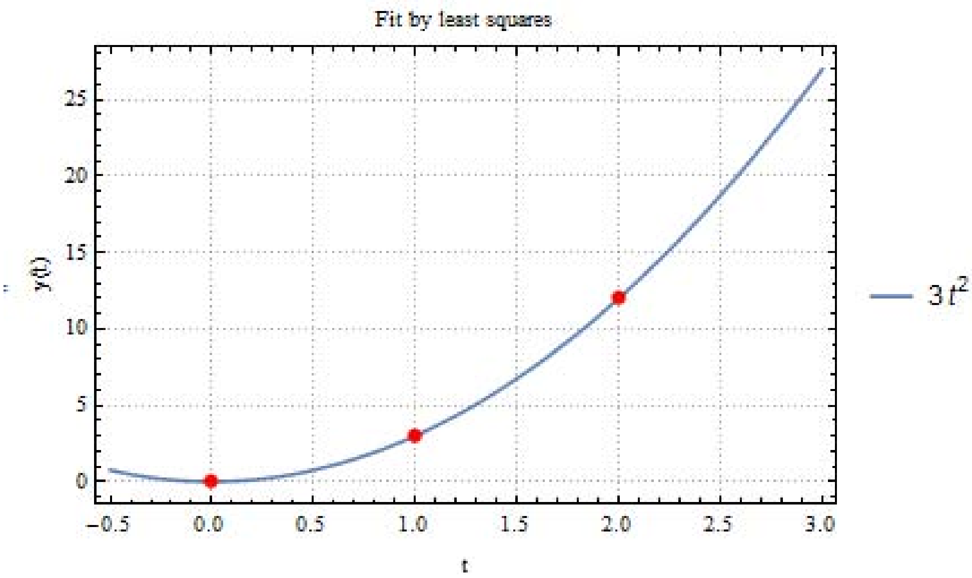
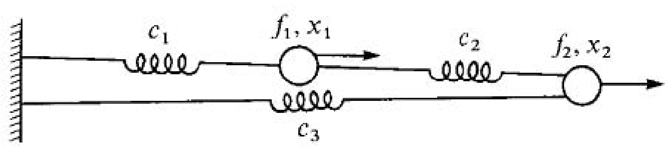


Figure 13: Plot of the above

We can see this is an exact fit since no least squares was used.

11 Problem 1.4.10



1.4.10 In a system with three springs and two forces and displacements write out the equations $e = Ax$, $y = Ce$, and $A^T y = f$. For unit forces and spring constants, what are the displacements?

Figure 14: Problem description

Solution

From page 40 in textbook, y is force in spring, e is the elongation of spring from equilibrium and f external force at each mass. Hence for $Ax = e$, we see that $e_1 = x_1, e_2 = x_2 - x_1$ and $e_3 = x_2$. Therefore

$$\overbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}}^{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

For $y = Ce$, here y is the internal force in spring. Hence $y_1 = c_1 e_1, y_2 = c_2 e_2, y_3 = c_3 e_3$, therefore

$$\overbrace{\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}}^{A} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

For $A^T y = f$, we need to find the external forces at each node first. From diagram we see that $f_1 = y_1 - y_2$ and $f_2 = y_2 + y_3$, therefore

$$\overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}}^{A^T} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

12 Problem 1.4.11

1.4.11 Suppose the lowest spring in Fig. 1.7 is removed, leaving masses m_1, m_2, m_3 hanging from the three remaining springs. The equation $e = Ax$ becomes

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Find the corresponding equations $y = Ce$ and $A^T y = f$, and solve the last equation for y. This is the ***determinate*** case, with square matrices, when the factors in A^TCA can be inverted separately and y can be found before x.

Figure 15: Problem description

Solution

To find $y = Ce$. In this equation, e is the elongation of the spring and y is the internal force. Hence from figure 1.7 we obtain

$$\begin{aligned} y_1 &= c_1 e_1 \\ y_2 &= c_2 e_2 \\ y_3 &= c_3 e_3 \end{aligned}$$

Hence in matrix form

$$\overbrace{\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}}^{C} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

In the question $A^T y = f$, f is the external force. Hence by balance of force at each mass, we obtain

$$\begin{aligned} f_1 &= y_1 - y_2 \\ f_2 &= y_2 - y_3 \\ f_3 &= y_3 \end{aligned}$$

or in matrix form

$$\overbrace{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}^{A} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

To solve, since already in U form, we will just need to do backward substitution. Hence

$$\begin{aligned} y_3 &= f_3 \\ y_2 &= f_2 + f_3 \\ y_1 &= f_1 + f_2 + f_3 \end{aligned}$$

13 Problem 1.4.12

1.4.12 For the same 3 by 3 problem find $K = A^TCA$ and A^{-1} and K^{-1}. If the forces f_1, f_2, f_3 are all positive, acting in the same direction, how do you know that the displacements x_1, x_2, x_3 are also positive?

Figure 16: Problem description

Solution

$K = A^T C A$, but $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ given in problem 1.4.11, and $C = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$. Hence

$$K = A^T C A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^T \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{pmatrix}$$

And

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

And

$$\begin{aligned} K^{-1} &= \left(A^T C A\right)^{-1} \\ &= A^{-1} C^{-1} \left(A^T\right)^{-1} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c_1} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & \frac{1}{c_3} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{c_1} & \frac{1}{c_1} & \frac{1}{c_1} \\ \frac{1}{c_1} & \frac{1}{c_1} + \frac{1}{c_2} & \frac{1}{c_1} + \frac{1}{c_2} \\ \frac{1}{c_1} & \frac{1}{c_1} + \frac{1}{c_2} & \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} \end{pmatrix} \end{aligned}$$

Since $f = Kx$ then $x = K^{-1} f$. Since we are told f_1, f_2, f_3 are all positive, and so the sign of x the displacement, is determined by the sign of K^{-1}. But K^{-1} has positive entries only, since c_i is positive by definition. Therefore all displacements x must be positive.

14 Problem 1.5.6

1.5.6 Solve the second-order system

$$\frac{d^2 u}{dt^2} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} u = 0 \quad \text{with} \quad u_0 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad u_0' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

These initial conditions do not activate the zero eigenvalue (see the following exercises).

Figure 17: Problem description

Solution

$$\begin{pmatrix} u_1'' \\ u_2'' \\ u_3'' \end{pmatrix} + \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \left(a_1 \cos\sqrt{\lambda_1} t + b_1 \sin\sqrt{\lambda_1} t\right) \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} + \left(a_2 \cos\sqrt{\lambda_2} t + b_2 \sin\sqrt{\lambda_2} t\right) \begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} + \left(a_3 \cos\sqrt{\lambda_3} t + b_3 \sin\sqrt{\lambda_3} t\right) \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix}$$

Where λ_i are the eigenvalues and v_i are the corresponding eigenvectors of A. The constants are found from initial conditions.

For the matrix A, the eigenvalues are found by solving

$$|A-\lambda I|=0$$

Solving for eigenvalues gives $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$ and the corresponding eigenvectors are $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-2\\1\end{pmatrix}$ hence the solution becomes

$$\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix} = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + (a_2\cos t + b_2\sin t)\begin{pmatrix}-1\\0\\1\end{pmatrix} + \left(a_3\cos\sqrt{3}t + b_3\sin\sqrt{3}t\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

At $t=0$

$$\begin{pmatrix}2\\-1\\-1\end{pmatrix} = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + a_2\begin{pmatrix}-1\\0\\1\end{pmatrix} + a_3\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{1}$$

And taking derivative of the solution gives

$$\begin{pmatrix}u_1'(t)\\u_2'(t)\\u_3'(t)\end{pmatrix} = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + (-a_2\sin t + b_2\cos t)\begin{pmatrix}-1\\0\\1\end{pmatrix} + \left(-a_3\sqrt{3}\sin\sqrt{3}t + b_3\sqrt{3}\cos\sqrt{3}t\right)\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

At $t=0$ the above becomes

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = a_1\begin{pmatrix}1\\1\\1\end{pmatrix} + b_2\begin{pmatrix}-1\\0\\1\end{pmatrix} + b_3\sqrt{3}\begin{pmatrix}1\\-2\\1\end{pmatrix} \tag{2}$$

Now (1),(2) needs to be solved for the constants. From (1)

$$\begin{pmatrix}1&-1&1\\1&0&-2\\1&1&1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

This is solved using Gaussian elimination. $\begin{pmatrix}1&-1&1\\1&0&-2\\1&1&1\end{pmatrix} \to \begin{pmatrix}1&-1&1\\0&1&-3\\0&2&0\end{pmatrix} \to \begin{pmatrix}1&-1&1\\0&1&-3\\0&0&6\end{pmatrix}$

Hence $U = \begin{pmatrix}1&-1&1\\0&1&-3\\0&0&6\end{pmatrix}, L = \begin{pmatrix}1&0&0\\1&1&0\\1&2&1\end{pmatrix}$ and hence $Lc=b$ is solved first for c using forward substitution

$$\begin{pmatrix}1&0&0\\1&1&0\\1&2&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

Which gives $c_1 = 2, c_2 = -3, c_3 = 3$, hence now we solved for x from $Ux = c$

$$\begin{pmatrix}1&-1&1\\0&1&-3\\0&0&6\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}2\\-3\\3\end{pmatrix}$$

Giving $a_3 = \frac{1}{2}, a_2 = -\frac{3}{2}, a_1 = 0$.

Now we solve for the rest of the constant in same way. From (2)

$$\begin{pmatrix}1&-1&\sqrt{3}\\1&0&-2\sqrt{3}\\1&1&\sqrt{3}\end{pmatrix}\begin{pmatrix}a_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

This is solved using Gaussian elimination. $\begin{pmatrix}1&-1&\sqrt{3}\\1&0&-2\sqrt{3}\\1&1&\sqrt{3}\end{pmatrix} \to \begin{pmatrix}1&-1&\sqrt{3}\\0&1&-3\sqrt{3}\\0&2&0\end{pmatrix} \to \begin{pmatrix}1&-1&\sqrt{3}\\0&1&-3\sqrt{3}\\0&0&6\sqrt{3}\end{pmatrix}$

Hence $U=\begin{pmatrix}1 & -1 & \sqrt{3}\\ 0 & 1 & -3\sqrt{3}\\ 0 & 0 & 6\sqrt{3}\end{pmatrix}, L=\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix}$ and $Lc=b$ is solved first for c using forward substitution

$$\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$c_1=0, c_2=0, c_3=0$, therefore now we solved for x from $Ux=c$

$$\begin{pmatrix}1 & -1 & \sqrt{3}\\ 0 & 1 & -3\sqrt{3}\\ 0 & 0 & 6\sqrt{3}\end{pmatrix}\begin{pmatrix}a_1\\ b_2\\ b_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Which gives $b_3=0, b_2=0, a_1=0$. Now that all constants are found the final solution is

$$\begin{pmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{pmatrix}=-\frac{1}{2}\cos t\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}+\frac{1}{2}\cos\sqrt{3}t\begin{pmatrix}1\\ -2\\ 1\end{pmatrix}$$

Hence

$$\begin{aligned}u_1(t)&=\frac{3}{2}\cos t+\frac{1}{2}\cos\sqrt{3}t\\ u_2(t)&=-\cos\sqrt{3}t\\ u_3(t)&=-\frac{3}{2}\cos t+\frac{1}{2}\cos\sqrt{3}t\end{aligned}$$

15 Problem 1.5.7

1.5.7 Suppose each column of A adds to zero, as in

$$A=\begin{bmatrix}3 & -1 & 0\\ -2 & 2 & -1\\ -1 & -1 & 1\end{bmatrix}.$$

(a) Prove that zero is an eigenvalue and A is singular, by showing that the vector of ones is an eigenvector of A^T. (A and A^T have the same eigenvalues, but not the same eigenvectors.)
(b) Find the other eigenvalues of this matrix A, and all three eigenvectors.

Figure 18: Problem description

Solution

15.1 Part (a)

$$A=\begin{pmatrix}3 & -1 & 0\\ -2 & 2 & -1\\ -1 & -1 & 1\end{pmatrix}$$

$$A^T=\begin{pmatrix}3 & -2 & -1\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}$$

For eigenvector v of ones, we write

$$A^Tv=\lambda v$$

Hence

$$\begin{pmatrix}3 & -2 & -1\\ -1 & 2 & -1\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Which implies $\lambda = 0$. Since A^T has same eigenvalues of A then A has zero eigenvalue. But the determinant of A is the products of its eigenvalues. Since one eigenvalue is zero, then $|A| = 0$, which means A is singular.

15.2 Part (b)

To find all three eigenvalues of A we solve $|\lambda I - A| = 0$. Hence

$$\begin{vmatrix}\lambda-3 & 1 & 0\\ 2 & \lambda-2 & 1\\ 1 & 1 & \lambda-1\end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 8\lambda = 0$$

$$\lambda\left(\lambda^2 - 6\lambda + 8\right) = 0$$

Hence $\lambda = 0, \lambda = 2, \lambda = 4$.To find the eigenvectors, we solve $Av_i = \lambda_i v_i$ for each eigenvalue. This means solving $(\lambda_i I - A)\,v_i = 0$ for each eigenvalue. For $\lambda = 0$

$$-\begin{pmatrix}3 & -1 & 0\\ -2 & 2 & -1\\ -1 & -1 & 1\end{pmatrix}\begin{pmatrix}v_{11}\\v_{21}\\v_{31}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

We always set $v_{1j} = 1$ and then go to find v_{2j}, v_{3j} in finding eigenvectors. Hence we solve

$$\begin{pmatrix}-3 & 1 & 0\\ 2 & -2 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}1\\v_{21}\\v_{31}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving gives $v_1 = \begin{pmatrix}1\\3\\4\end{pmatrix}$ For the second eigenvalue $\lambda = 2$ we obtain

$$\left(\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} - \begin{pmatrix}3 & -1 & 0\\ -2 & 2 & -1\\ -1 & -1 & 1\end{pmatrix}\right)\begin{pmatrix}v_{12}\\v_{22}\\v_{32}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1 & 0\\ 2 & 0 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\v_{22}\\v_{32}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving gives $v_2 = \begin{pmatrix}-1\\-1\\2\end{pmatrix}$. For the last eigenvalue $\lambda = 4$ we obtain

$$\left(\begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix} - \begin{pmatrix}3 & -1 & 0\\ -2 & 2 & -1\\ -1 & -1 & 1\end{pmatrix}\right)\begin{pmatrix}v_{13}\\v_{23}\\v_{33}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1 & 0\\ 2 & 2 & 1\\ 1 & 1 & 3\end{pmatrix}\begin{pmatrix}1\\v_{23}\\v_{33}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving gives $v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.Summary. The eigenvalues are $\{0,2,4\}$ and the eigenvectors are

$$\boxed{\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

16 Problem 1.5.11

1.5.11 Why is the sum of entries on the diagonal of AB equal to the sum along the diagonal of BA? In other words, what terms contribute to the trace of AB?

Figure 19: Problem description

Solution

The elements of the diagonal of AB come from multiplying row i in A with column i in B. Therefore, looking at the diagonal elements only, we can write, using a_{ij} as element in A and using b_{ij} as element in B

$$AB = \begin{pmatrix} \sum_i a_{1i}b_{i1} & & & \\ & \sum_i a_{2i}b_{i2} & & \\ & & \ddots & \\ & & & \sum_i a_{ni}b_{in} \end{pmatrix}$$

Hence the trace of AB is

$$tr(AB) = \sum_i a_{1i}b_{i1} + \sum_i a_{2i}b_{i2} + \cdots + \sum_i a_{ni}b_{in}$$

But the above can be combined as

$$tr(AB) = \sum_k \sum_i a_{ki}b_{ik} \tag{1}$$

Now if we consider BA, then the result comes from multiplying row i in B with column i in A

$$BA = \begin{pmatrix} \sum_i b_{1i}a_{i1} & & & \\ & \sum_i b_{2i}a_{i2} & & \\ & & \ddots & \\ & & & \sum_i b_{ni}a_{in} \end{pmatrix}$$

Hence the trace of BA is

$$tr(BA) = \sum_i b_{1i}a_{i1} + \sum_i b_{2i}a_{i2} + \cdots + \sum_i b_{ni}a_{in}$$

But the above can be combined as

$$tr(BA) = \sum_k \sum_i b_{ki}a_{ik} \tag{2}$$

Looking at (1) and (2) above we can see that both traces contain the same elements, but arranged differently. The indices can be changes in the sum without changing the value of the sum. This can be seen more directly by looking at specific example of 2×2 case. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, hence the elements on the diagonal of AB are $\begin{pmatrix} ae+bg & \\ & cf+dh \end{pmatrix}$ while for BA the result is $\begin{pmatrix} ae+cf & \\ & bg+dh \end{pmatrix}$. We see that the trace is the same.

17 Problem 1.5.12

1.5.12 Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda), \qquad (*)$$

and making a clever choice of λ.

Figure 20: Problem description

Solution

$\det(A - \lambda I)$ is a polynomial in λ. Hence it can be factored in its roots as

$$\det(A - \lambda I) = P(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda)$$

Assuming there is n eigenvalues. When $\lambda = 0$ (which is the independent variable now, and not any specific eigenvalue, then (1) becomes

$$\det(A) = P(0) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$$

Hence

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n$$

Which is what we are asked to show.

18 Problem 1.5.13

1.5.13 Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of (*). Next, look for all the terms in

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

which involve $(-\lambda)^{n-1}$. Explain why they all come from the main diagonal, and find the coefficient of $(-\lambda)^{n-1}$ on the left side of (*). Compare.

Figure 21: Problem description

Solution

$$\det(A - \lambda I) = P(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \cdots (\lambda_n - \lambda) \qquad (*)$$

Let look at the case of $n = 2$

$$\begin{aligned} P(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \\ &= \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2 \end{aligned}$$

Hence the coefficient of $(-\lambda)^{n-1}$ which is $-\lambda$ is $(\lambda_1 + \lambda_2)$ which is the sum of the eigenvalues. Lets look at $n = 3$

$$\begin{aligned} P(\lambda) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) \\ &= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + \lambda_1\lambda_2\lambda_3 \end{aligned}$$

So the pattern is now clear. The coefficient of $(-\lambda)^{n-1}$ is the sum of all the eigenvalues of A.

For $\det(A-\lambda I)$, looking at $n=2$ we write

$$\det(A-\lambda I)=\det\begin{pmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{pmatrix}=(a_{11}-\lambda)(a_{22}-\lambda)-a_{21}a_{12}$$
$$=\lambda^2-\lambda(a_{11}+a_{22})+(a_{11}a_{22}-a_{21}a_{12})$$

We see in this case that the coefficient of $(-\lambda)^{n-1}=-\lambda$ is the trace of A. Lets look at $n=3$

$$\det(A-\lambda I)=\det\begin{pmatrix}a_{11}-\lambda & a_{12} & a_{13}\\ a_{21} & a_{22}-\lambda & a_{23}\\ a_{31} & a_{32} & a_{33}-\lambda\end{pmatrix}$$
$$=(a_{11}-\lambda)\det\begin{pmatrix}a_{22}-\lambda & a_{23}\\ a_{32} & a_{33}-\lambda\end{pmatrix}-a_{12}\det\begin{pmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}-\lambda\end{pmatrix}+a_{13}\det\begin{pmatrix}a_{21} & a_{22}-\lambda\\ a_{31} & a_{32}\end{pmatrix}$$
$$=(a_{11}-\lambda)\left(\lambda^2-\lambda(a_{22}+a_{33})+(a_{22}a_{33}-a_{23}a_{32})\right)$$
$$-a_{12}(a_{21}a_{33}-\lambda a_{21}-a_{31}a_{23})+a_{13}(\lambda a_{31}+a_{21}a_{32}-a_{22}a_{31})$$
$$=-\lambda^3+\lambda^2(a_{11}+a_{22}+a_{33})-\lambda(a_{11}a_{22}-a_{12}a_{21}+a_{11}a_{33}-a_{13}a_{31}+a_{22}a_{33}-a_{23}a_{32})$$
$$+(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}+a_{12}a_{31}a_{23}+a_{21}a_{13}a_{32}-a_{13}a_{22}a_{31})$$

We see again that the coefficient of $(-\lambda)^{n-1}=\lambda^2$ is the trace of A. So by construction we can show that coefficient of $(-\lambda)^{n-1}$ is the trace of A. But we showed above that coefficient of $(-\lambda)^{n-1}$ is the sum of all the eigenvalues of A. Hence the sum of all the eigenvalues of $A=tr(A)$